CHAPTER - 4

ON LEBESGUE FUNCTIONS AND SUMMABILITY OF GENERAL ORTHOGONAL SERIES

Let $\{\phi_n(x)\}$ $(n = o, 1, 2, \ldots)$ be an orthonormal system (ONS) of L^2 - integrable functions defined in the closed interval $[a, b]$. We consider the orthogonal series

$$(4.1.1) \qquad \sum_{n=o}^{\infty} C_n \phi_n(x)$$

with real coefficients C_n's.

Let $S_n(x)$ denote the n^{th} partial sum and

$$\bar{T}_n(x) = \frac{1}{P_n} \sum_{v=o}^{n} p_v S_v(x),$$

$$T_n(x) = \frac{1}{2^n} \sum_{v=o}^{n} \binom{n}{v} S_v(x),$$

where $P_n = p_o + p_1 + \cdots + p_n$, $p_o > o$, $p_n \geqslant o$ denote the $(\bar{N}, p_n)$ means and Euler means of (4.1.1), respectively.

Define

$$k_n(t, x) = \sum_{k=o}^{n} \phi_k(t) \phi_k(x),$$

$$L_n(x) = \int_a^b |k_n(t, x)| \, dt,$$

$$\bar{N}_n(t, x) = \frac{1}{P_n} \sum_{v=o}^{n} p_v k_v(t, x),$$

$$\bar{Q}_n(t, x) = \int_a^b |\bar{N}_n(t, x)| \, \rho(t) \, dt,$$

$$N_n(t, x) = \frac{1}{P_n} \sum_{v=o}^{n} p_{n-v} k_v(t, x),$$

$$\bar{Z}_n(x) = \int_a^b |N_n(t, x)| \, \rho(t) \, dt,$$

the n^{th} kernel of the ONS $\{\phi_n(x)\}$, n^{th} Lebesgue function, n^{th} $(\bar{N}, p_n)$ kernel, n^{th} Lebsgue $(\bar{N}, p_n)$ function, n^{th} (N, p_n) kernel and n^{th} Lebesgue (N, p_n) function respectively.

Convergence of orthogonal series may be affected by Lebesgue functions. Kolmogoroff - Seliverstoff[1) have given the idea, how Lebesgue functions effect the convergence of Fourier trigonometric series. This result was, then extended to orthogonal series for the convergence and Cesàro summability by Kaczmarz[2) and Tandori[3) similarly,

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1) Kolmogoroff and Seliverstoff ([59],[60])
2) Kaczmarz [51]
3) Tandori [127]

Meder[1] and Kantawala[2] have worked in this direction for Riesz means, Logarithmic means and Nörlund means respectively. Alexits G.[3] has discussed the influence of Lebesgue functions on the Cesàro summability of orthogonal series. He has proved the following theorem.

Theorem A :-

If the Lebesgue functions

$$(4.1.2) \qquad L_{2^n}(x) = \int_a^b \left| \sum_{k=o}^{2^n} \phi_k(t)\, \phi_k(x) \right| dt$$

of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a, b]$, then the condition

$$(4.1.3) \qquad \sum_{n=o}^{\infty} C_n^2 < \infty$$

implies the $(C, \alpha > o)$ summability of the orthogonal series (4.1.1) almost everywhere on E.

In this chapter we extend the above results to $(\bar{N}, p_n)$ summability and Euler summability as follows :

Theorem 1 :-

If the Lebesgue functions (4.1.2) of ONS

1) Meder [77]
2) Kantawala [50]
3) Alexits G. ([5], p.128)

$\{\phi_n(x)\}$ are uniformly bounded in the set $E \subset [a, b]$ then the relation (4.1.3) implies the estimate

$$T_n(x) = o_x(n)$$

holds almost everywhere on E.

Theorem 2 :-

If $\{p_n\} \in M^{\alpha}$ and the Lebesgue functions (4.1.2) of an ONS $\{\phi_n(x)\}$ are uniformly bounded on the set $E \subset [a, b]$, then the orthogonal series (4.1.1) is $(\bar{N}, p_n)$ summable almost everywhere under the condition (4.1.3).

Moreover, in this chapter we have also discussed the order of Magnitude of various Lebesgue functions.

Alexits[1)] has estimated the order of magnitude of Lebesgue functions of an ONS $\{\phi_n(x)\}$. He has proved the following theorem :

Theorem A :-

If $\{\lambda_n\}$ is a monotonic increasing sequence of numbers (positive) satisfying the condition

$$\sum_{n=o}^{\infty} \frac{1}{\lambda_n} < \infty \text{ then,}$$

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1) Alexits [5]

the order of magnitude of Lebesgue functions of the ONS $\{\phi_n(x)\}$ can apart at most from a set of measure zero be obtained by the relation

$$L_n(x) = o_x(\sqrt{\lambda_n})$$

If however, $\{\phi_n(x)\}$ satisfies, in the set E, the condition

$$\sum_{k=o}^{n} \phi_k^2(x) = O(n),$$

then at the points $x \in E$ this order of magnitude can be pushed down to

$$L_n(x) = O(\sqrt{n})$$

In this chapter we have estimated the order of magnitude of Nörlund Lebesgue function and $(\bar{N}, p_n)$ Lebesgue functions. Our theorems are as follows .

Theorem 3 :-

If $\{\lambda_n\}$ is a monotone increasing sequence of numbers, satisfying the condition

$$\sum_{n=o}^{\infty} \lambda_n^{-1} < \infty$$

then the order of magnitude of Lebesgue functions of the orthonormal system $\{\phi_n(x)\}$ can apart at most from a set of measure zero be estimated by the relation

$$\bar{Q}_n(x) = o_x(\sqrt{\lambda_n}).$$

If however $\{\phi_n(x)\}$ satisfies in the set E the condition

$$\sum_{k=o}^{n} \phi_k^2(x) = O(n)$$

then at the point $x \in E$ this order of magnitude can be pushed down to

$$\bar{Q}_n(x) = O(\sqrt{n})$$

Theorem 4 :-

Under the same condition as of theorem 3, we estimat,

$$\bar{Z}_n(x) = o_x(\sqrt{\lambda_n})$$

and also

$$\bar{Z}_n(x) = O(\sqrt{n})$$

For proving these theorems we need the following Lemmas :

Lemma 1 :-[1)]

Under the condition (4.1.3) the relation

$S_{v_n}(x) - \sigma_{v_n}(x) = o_x(1)$ is valid almost everywhere

1) Alexits ([5], p. 118)

for every index sequence $\{v_n\}$ with

$$\frac{v_{n+1}}{v_n} \geq q > 1 .$$

Lemma 2 :-[1]

If $\{\lambda_n\}$ is a positive, nondecreasing sequence for which the relation

$$L_{v_n}(x) = O(\lambda_{v_n}) \qquad (v_1 < v_2 < \ldots .)$$

holds in a set $E \subset [a, b]$ then for the partial sums $\{S_{v_n}(x)\}$ of the orthogonal series (4.1.1) under the condition (4.1.3), the estimate

$$S_{v_n}(x) = O_x(\lambda_{v_n}^{\frac{1}{2}})$$

holds almost everywhere on E.

Lemma 3 :-[2]

If $\{p_n\} \in M^{\alpha}$, $\alpha > \frac{1}{2}$ then

$$\lim_{n \to \infty} \frac{n}{P_n^2} \sum_{k=0}^{n} \frac{P_k^2}{(k+1)^2} = \frac{1}{2\alpha-1} .$$

Lemma 4 :-[3]

If $\{\lambda_n\}$ denotes a monotone increasing

1) Alexits ([5], p. 118)
2) Meder [78]
3) Alexits G.([5], p.38)

sequence of numbers for which

$$\sum_{n=o}^{\infty} \lambda_n^{-1} < \infty ,$$

then the estimate

$$\sum_{k=o}^{n} \phi_k^2 (x) = o_x (\lambda_n)$$

is valid for every orthonormal system $\{\phi_n (x)\}$ almost everywhere.

Lemma 5 :-

Let $\sum_{n=o}^{\infty} U_n$ be a given series with $\{\sigma_n\}$ as its sequence of n^{th} (C, 1) sum. If $\{m_n\}$ is a convex null sequence and $\sigma_n = O(1)$ then

$$T_n (m) = o(n)$$

holds, where $T_n(m)$ denotes the (E, 1) means of the series

$$\sum_{n=o}^{\infty} U_n m_n .$$

Proof :-

Here,

$$T_n (m) = \frac{1}{2^n} \sum_{k=o}^{n} U_k m_k \sum_{v=k}^{n} \binom{n}{v}$$

$$= \sum_{k=o}^{n} U_k \delta_k \quad \text{where} \quad \delta_k = \frac{1}{2^n} \sum_{v=k}^{n} \binom{n}{v} m_k$$

By Abel's transform

$$= \sum_{k=0}^{n-1} (\delta_k - \delta_{k+1}) s_k + \delta_n s_n$$

$$= \sum_{k=0}^{n-1} \delta_k' s_k + \delta_n s_n \quad \text{where } \delta_k' = \delta_k - \delta_{k+1} .$$

$$= \sum_{k=0}^{n} \delta_k' s_k .$$

Again by Abel's transform we have

$$= \sum_{k=0}^{n-1} (\delta_k' - \delta_{k+1}') (k + 1)\sigma_k + (n + 1)\sigma_n \delta_n'$$

$$= \sum_{k=0}^{n-1} \delta_k'' (k + 1)\sigma_k + (n + 1)\sigma_n \delta_n' .$$

The second difference δ_k'' may be represented as follows.

$$\delta_k'' = \delta_k' - \delta_{k+1}'$$

$$= \delta_k - 2 \delta_{k+1} + \delta_{k+2} .$$

$$= \frac{1}{2^n} \sum_{v=k}^{n} \binom{n}{v} m_k - \frac{2}{2^n} \sum_{v=k+1}^{n} \binom{n}{v} m_{k+1}$$

$$+ \frac{1}{2^n} \sum_{v=k+2}^{n} \binom{n}{v} m_{k+2} .$$

$$= \frac{1}{2^n}\left[\sum_{v=k}^{n}\binom{n}{v} m_k\right] - \frac{2}{2^n}\sum_{v=k+1}^{n}\binom{n}{v} m_{k+1} + \frac{1}{2^n}\sum_{v=k+2}^{n}\binom{n}{v} m_{k+2}$$

$$= \frac{1}{2^n}\left[\sum_{v=k}^{n}\binom{n}{v}\left\{m_k - 2m_{k+1} + m_{k+2}\right\} + 2\binom{n}{k} m_{k+1} - \binom{n+1}{k} m_{k+2}\right]$$

$$= \frac{1}{2^n}\left[\left(2^n - \sum_{v=o}^{k-1}\binom{n}{v}\right)(\Delta m_k - \Delta m_{k+1}) + 2\binom{n}{k} m_{k+1} - \binom{n+1}{k} m_{k+2}\right].$$

Therefore, we have,

$$\sum_{k=o}^{n-1}(\delta'_k - \delta'_{k+1})(k+1)\sigma_k + (n+1)\sigma_n \delta'_n$$

$$= \sum_{k=o}^{n-1}\frac{1}{2^n}\left[\left(2^n - \sum_{v=o}^{k-1}\binom{n}{v}\right)(\Delta m_k - \Delta m_{k+1}) + 2\binom{n}{k} m_{k+1} - \binom{n+1}{k} m_{k+2}\right](k+1)\sigma_k + (n+1)\sigma_n \frac{m_n}{2^n}$$

Since the hypothesis $\sigma_n = O(1)$ and $\{m_n\}$ is a convex null sequence so $n\Delta m_n = o(1)$.

$$= \frac{O(1)}{2^n}\left\{\sum_{k=o}^{n-1}\binom{n}{k}(k+1)\, m_{k+1} - \sum_{k=o}^{n-1}\binom{n+1}{k}(k+1) m_{k+2} + (n+1)\, m_n\right\}$$

$$= \frac{O(1)}{2^n} \left\{ \sum_{k=o}^{n-1} \binom{n}{k} (k+1)\Delta m_{k+2} + \sum_{k=o}^{n-1} \left(\binom{n}{k} - \binom{n+1}{k} \right) \right.$$

$$\left. (k+1)\, m_{k+2} + (n+1)\, m_n \right\}$$

$$= \frac{O(1)}{2^n} \left\{ o(1)(2^n - 1) + \frac{1}{n+1} \sum_{k=o}^{n-1} (n+1-k) \binom{n}{k} \right.$$

$$\left. (k+1)\, m_{k+2} + (n+1)\, m_n \right\}$$

$$= \frac{O(1)}{2^n} \left\{ o(1)\,(2^n - 1) + n \sum_{k=o}^{n-1} \binom{n}{k} m_{k+2} + (n+1) m_n \right\}$$

$$= \frac{O(1)}{2^n} \left\{ o(1)\,(2^n - 1) + n\, 2^n\, o(1) + o(n) \right\}$$

$$= \frac{O(1)}{2^n} \left\{ o(n\, 2^n) \right\}$$

$$= o(n)$$

with this the Lemma is proved.

<u>Lemma 6</u> :-[1)]

If $\{p_n\} \in M^{\alpha}$ and let $\{n_k\}$ be an arbitary sequence of indicies satisfying the following condition of lacunarity

$$1 < q \leqslant \frac{n_{k+1}}{n_k} \leqslant v \qquad \text{for } k = o, 1, 2, \ldots\ldots,$$

where q and v are constants, then the orthogonal series <u>(4.1.1) with (4.1.3) is $(\bar{N}, p_n)$ summable a.e. if the</u>

1) Sharma J. P. [11o]

sequence of partial sums $\left\{ S_{n_k}(x) \right\}$ is convergent almost everywhere.

Proof of Theorem 1 :-

Since

$$\sum_{n=0}^{\infty} C_n^2 < \infty ,$$

here there exist a monotone number sequence $\left\{ \mu_n^2 \right\}$, such that $\mu_n \longrightarrow \infty$ and

$$\sum_{n=0}^{\infty} C_n^2 \mu_n^2 < \infty .$$

It is also easy to construct (for instance geometrically) a strictly increasing concave sequence $\left\{ m_n \right\}$ with $m_n^2 \leqslant \mu_n^2$, $m_n^2 \longrightarrow \infty$ and

$$\sum_{n=0}^{\infty} C_n^2 m_n^2 < \infty .$$

Since $\left\{ m_n \right\}$ is concave and tending to infinity, $\left\{ \frac{1}{m_n} \right\}$ is convex null sequence. Let $S_n(m, x)$ and $\sigma_n(m, x)$ denote the n^{th} partial sum and the $n^{th}(C, 1)$ mean of the orthogonal series.

$$\sum_{n=0}^{\infty} C_n m_n \emptyset_n(x) \quad \text{respectively.}$$

From our assumption about the uniform boundedness of the Lebesgue function (4.1.2) it follows by Lemma 2 that the relation.

$$S_{2^p}(m, x) = O_x(1)$$

holds almost everywhere on E.

Hence the relation

$$\sigma_{2^p}(m, x) = O_x(1)$$

is valid almost everywhere on E due to Lemma 1.

Also

$$\sigma_n(m, x) - \sigma_{2^p}(m, x) \longrightarrow o^{1)}$$

almost everywhere with

$$2^p < n < 2^{p+1} .$$

Therefore, the relation.

$$\sigma_n(m, x) = O_x(1)$$

is valid almost everywhere on E .

Since the relation $\sum_{n=o}^{\infty} C_n \phi_n(x)$ arises from the relation.

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1) Alexits G.([5], p. 119)

(4.1.4) $$\sum_{n=o}^{\infty} C_n \, m_n \, \phi_n (x)$$

by multiplying the terms of the series (4.1.4) by the terms of the convex null sequence $\left\{ \frac{1}{m_n} \right\}$, it follows by Lemma 5 that the estimate

$$T_n (x) = o_x (n)$$

holds almost everywhere on E .

This proves the theorem completely.

<u>Proof of Theorem 2</u> :-

From the given condition we can conclude by theorem A that the orthogonal series (4.1.1) is (C, α > o) summable almost everywhere on E. Therefore, Lemma 1 implies the convergence of the sequence $\left\{ S_{2^n} (x) \right\}$ of the partial sums of the series (4.1.1). Hence by Lemma 6 it follows that the orthogonal series is ($\bar{N}$, p_n) summable almost everywhere on E.

This proves the theorem completely .

<u>Proof of Theorem 3</u> :-

We know that

$$\bar{Q}_n (x) = \int_a^b | \bar{N}_n (t, x) | \rho(t) \, dt.$$

$$= \int_a^b \left| \frac{1}{P_n} \sum_{v=o}^{n} p_v k_v(t,x) \right| \varrho(t) \; dt$$

$$= \left\{ \int_a^b (t) \; dt \int_a^b \frac{1}{P_n^2} \left[\sum_{v=o}^{n} p_v \; k_v(t,x) \right]^2 \varrho(t) dt \right\}^{\frac{1}{2}}$$

$$= O(1) \left\{ \int_a^b \frac{1}{P_n^2} \left[\sum_{v=o}^{n} p_v \sum_{k=o}^{v} \phi_k(t) \; \phi_k(x) \right]^2 \varrho(t) \; dt \right\}^{\frac{1}{2}}$$

$$= O(1) \left\{ \frac{1}{P_n^2} \left\{ \sum_{v=o}^{n} p_v^2 \right\} \left\{ \sum_{k=o}^{n} \phi_k^2 (x) \right\} \right\}^{\frac{1}{2}}$$

$$= O_x (\sqrt{n})$$

By Lemma 4, we have

$$= O_x (\sqrt{\lambda_n})$$

hence the proof.