CHAPTER - VII

AN EXTENSION OF $f_n^c(x)$

7.1 INTRODUCTION

An elegant extension of the generating relation 2.1(3) may be given in the form

$$(1) \quad H(t)\,\Psi[x\,S(t)] = \sum_{n=0}^{\infty} \phi_n(x)t^n ,$$

where

$$(2) \quad H(t) = \sum_{k=0}^{\infty} h_k t^{mk} , \quad (h_o = 1)$$

$$(3) \quad \Psi(z) = \sum_{n=0}^{\infty} \gamma_n z^n , \quad (\gamma_n \neq 0)$$

$$(4) \quad S(t) = \sum_{k=0}^{\infty} s_k t^{mk+1} , \quad (s_o \neq 0)$$

and m is a positive integer.

The generating relation (1) - (4) includes also the following generating relation, due to Boas and Buck [1]

$$(5) \quad A(t)\,\Psi[x\,B(t)] = \sum_{n=0}^{\infty} q_n(x)t^n ,$$

where

$$(6) \quad A(t) = \sum_{k=0}^{\infty} a_k t^k , \quad a_o \neq 0$$

$$(7) \quad B(t) = \sum_{k=0}^{\infty} b_k t^{k+1} , \quad b_o \neq 0 ,$$

to which (1)-(4) would reduce when m = 1.

In the generating relation (1)-(4) if γ_n is replaced by $\frac{1}{n!}$, the corresponding polynomials which we denote by $s_n(x)$ would provide a generalization of the polynomials $\Gamma_n^c(x)$ considered in Chapter V. In this Chapter we first derive an expansion formula for the product of an arbitrary number of polynomials $s^i_{n_i}(a_i x)$, $i=1,2,\ldots,p$ (say) in terms of $s_q(x)$, each of which possesses a generating relation 7.2(4) and then we obtain the generating relation for the coefficients $D_q^{(n_1,\ldots,n_p)}$ involved in the expansion. This result, besides generalizing the generating function given in Chapter V, also provides as with an extension of a result due to Carlitz[5].

This Chapter also contains the results of our investigations in the direction of providing a unification and generalization of the work of Brown[1], Brown and Goldberg[1], and Thakare and Madhekar[1] on characterization of polynomial sets which possess Boas and Buck type generating relation.

7.2 THE EXPANSION FORMULA

In order to derive the desired expansion formula we first observe that the polynomials $s_n(x)$ generated by 7.1(1) to 7.1(4) with $\gamma_n = \frac{1}{n!}$ are representable in the form

$$(1) \qquad s_n(x) = \sum_{k=0}^{[n/m]} \frac{s_{n,k}}{k!(n-mk)!} x^{n-mk} ,$$

and if we assume that

$$(2) \qquad [H(t)]^{-1} = \sum_{k=0}^{\infty} h'_k t^{mk} , \quad h'_o = 1 ,$$

then the inverse of (1) may be expressed in the form

$$(3) \qquad x^n = \sum_{k=0}^{[n/m]} \frac{s'_{n,k}\, n!}{k!} s_{n-mk}(x) .$$

Now, if we let

$$(4) \qquad H^{(i)}(t) \exp\left[x\, S^{(i)}(t)\right] = \sum_{n_i=0}^{\infty} s^{(i)}_{n_i}(\ \ x)t^{n_i} ,$$

$$i = 1,2,\ldots,p ,$$

where $H^{(i)}(t)$ and $S^{(i)}(t)$ possess expansions similar to 7.1(2) and 7.1(4) respectively, then in terms of the notations

$$(5) \qquad \begin{cases} n_1 + n_2 + \ldots + n_p = N \\ k_1 + k_2 + \ldots + k_p = K \\ [n_i/m] = n_i^*, \ i=1,2,\ldots,p, \ [n/m] = n^* , [N/m] = N^* \end{cases}$$

the relations (1) and (3) help us to put the product

$$(6) \qquad \Omega = s_{n_1}^{(1)}(a_1x)\, s_{n_2}^{(2)}(a_2x) \dots s_{n_p}^{(p)}(a_px)$$

in the form

$$\Omega = \sum_{k_1=0}^{n_1^*} \dots \sum_{k_p=0}^{n_p^*} \left[\prod_{j=1}^{p} \frac{s_{n_j,k_j}^{(j)}\, a_j^{n_j-mk_j}}{k_j!(n_j - mk_j)!} \right]$$

$$\cdot \sum_{q=0}^{N^*-K} \frac{(N - mK)!}{q!}\, s'_{N-mK,q}\, s_{N-mK-mq}(x) \;,$$

which on making use of the identity 5.2(1) can be written as

$$(7) \qquad \Omega = \sum_{q=0}^{N^*} D_q^{(n_1,\dots,n_p)}\, s_{N-mN^*+mq}(x) \;,$$

where the coefficients $D_q^{(n_1,\dots,n_p)}$ are given by

$$(8) \qquad D_q^{(n_1,\dots,n_p)} = \sum_{k_1=0}^{n_1^*} \dots \sum_{k_p=0}^{n_p^*} \left[\prod_{j=1}^{p} \frac{s_{n_j,k_j}^{(j)}\, a_j^{n_j-mk_j}}{k_j!(n_j - mk_j)!} \right]$$

$$\cdot \frac{(N - mK)!}{(N^*-K-q)!}\, s'_{N-mK,\, N^*-K-q} \;.$$

Now, denoting the expression

$$(9) \qquad \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} D_q^{(n_1,\ldots,n_p)} u_1^{n_1} \cdots u_p^{n_p}$$

by Z , and making use of the relation

$$(10) \qquad H^{(i)}(t) \left[S^{(i)}(t)\right]^n = \sum_{k=0}^{\infty} \frac{s^{(i)}_{n+mk,k}}{k!} t^{n+mk} ,$$

which is an easy consequence of (4) and (1), we observe that

$$Z = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \left[\prod_{j=1}^{p} \frac{H^{(j)}(u_j) \left(a_j S^{(j)}(u_j)\right)^{n_j}}{n_j!} \right]$$

$$\cdot \frac{N!}{(N^* - q)!} s'_{N,N^*-q} .$$

The above expression for Z may be further simplified to yield

$$(11) \qquad Z = H^{(1)}(u_1) \cdots H^{(p)}(u_p) \sum_{n=mq}^{\infty} \frac{z^n}{(n^* - q)!} s'_{n,n^*-q} ,$$

wherein, for the sake of brevity, z stands for

$$(12) \qquad a_1 S^{(1)}(u_1) + \cdots + a_p S^{(p)}(u_p) .$$

Now, in view of 5.2(5), (11) becomes

$$(13)\quad Z = H^{(1)}(u_1)\ldots H^{(p)}(u_p) \sum_{j=0}^{m-1} z^{mq+j} \cdot \sum_{n=0}^{\infty} \frac{z^{mn}}{n!} s'_{m(n+q)+j,n} .$$

On the other hand, we have

$$\exp\left[x\, S(t)\right] = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[S(t)\right]^n ,$$

which, in view of the relation (3) may be put in the form

$$\exp\left[x\, S(t)\right] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{S^{n+mk}(t)}{k!} s_n(x)\, s'_{n+mk,k} ,$$

so that the generating relation

$$H(t) \exp\left[x\, S(t)\right] = \sum_{n=0}^{\infty} s_n(x) t^n$$

yields

$$(14)\quad \frac{t^n}{H(t)} = S^n(t) \sum_{k=0}^{\infty} \frac{s'_{n+mk,k}}{k!} S^{mk}(t) .$$

The relation (14) under the assumption of the existence of a power series, I(t) such that

$$S(I(t)) = I(S(t)) = t ,$$

gives us

$$(15)\quad \frac{(I(t))^{qm+j}}{H(I(t))} = t^{qm+j} \sum_{k=0}^{\infty} \frac{s'_{qm+j+mk,k}}{k!} t^{mk} .$$

In view of (15), the equation (13) may be written in the form

$$Z = H^{(1)}(u_1) \ldots H^{(p)}(u_p) \sum_{j=0}^{m-1} \frac{(I(z))^{qm+j}}{H(I(z))},$$

which leads us to the desired generating function

$$(16) \quad \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} D_q^{(n_1,\ldots,n_p)} u_1^{n_1} \ldots u_p^{n_p}$$

$$= \frac{H^{(1)}(u_1) \ldots H^{(p)}(u_p)}{H(I(z))} I^{qm}(z) \frac{I^m(z)-1}{I(z)-1}.$$

<u>Particular Cases</u>:

On putting m=1, we obtain the corresponding expansion formula for $s_{n_1}^{(1)}(a_1x) \ldots s_{n_p}^{(p)}(a_px)$ wherein the coefficients $D_q^{(n_1,\ldots,n_p)}$ are generated by the relation

$$(17) \quad \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} D_q^{(n_1,\ldots,n_p)} u_1^{n_1} \ldots u_p^{n_p}$$

$$= \frac{H^{(1)}(u_1) \ldots H^{(p)}(u_p)}{H(I(z))} I^q(z).$$

In (17) if we put p=2 we shall get the formula

$$(18) \quad \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} D_q^{(n_1,n_2)} u_1^{n_1} u_2^{n_2} = \frac{H^{(1)}(u_1)\, H^{(2)}(u_2)}{H(I(a_1 S^{(1)}(u_1)+a_2 S^{(2)}(u_2)))} .$$

$$. \; I^q(a_1 S^{(1)}(u_1) + a_2 S^{(2)}(u_2)) ,$$

which corresponds to the generating relation given by Carlitz [5] .

On the other hand, if

$$S(t) = \frac{t}{(1 + yt^m)^r} , \; H(t) = (1 + yt^m)^{-c} ,$$

the polynomials $s_n(x)$ would correspond to $\Gamma_n^c(x)$, discussed in Chapter V. Thus the expansion formula (7)-(8) and the generating relation (16) would simplify to 5.2(2) - 5.2(3) and 5.2(6) - 5.2(7) respectively.

7.3 CHARACTERIZATION

With reference to the set $\{q_n(x)\}$ possessing a generating relation of Boas and Buck type as given by 7.1(5) to 7.1(7), it is said that $q_n(x) \in [\Psi]$ where $[\Psi]$ denotes the subclass of all such sets whose Boas and Buck type generating relation involves a fixed Ψ .

A few years ago, in an attempt to obtain certain characterizations of the polynomial sets generated by

$$(1) \quad (1-t)^{-\alpha} A(t) \Psi[x B(t)] = \sum_{n=0}^{\infty} q_n^{(\alpha)}(x) t^n ,$$

which reduce to Boas and Buck generating relation when $\alpha = 0$, Brown and Goldberg [1] (see also Goldberg [1]) replaced the parameter α by an arbitrary function $\alpha(n)$ of the index n and proved that $\{q_n^{\alpha(n)}(x)\} \in [\Psi]$ if and only if the function $\alpha(n)$ be linear for $n \geq 1$.

Following the work of Brown and Goldberg [1], Thakare and Madhekar [1] considered the polynomials $\{q_n(\alpha_1,\ldots,\alpha_r;x)\}$ generated by

$$(2) \quad A(t)\Psi[x B(t)] \prod_{j=1}^{r} (1-w_j t)^{-\alpha_j} = \sum_{n=0}^{\infty} q_n(\alpha_1,\ldots,\alpha_r;x)t^n ,$$

and proved that

$q_n(\delta_1(n),\ldots,\delta_r(n);x) \in [\Psi]$ if and only if for each $j = 1,\ldots,r$; $\delta_j(n)$ is a linear function of n for $n \geq 1$.

The extension of $\{f_n^c(x)\}$ which is given by 7.1(1) to 7.1(4) and the results for the polynomials $q_n(\alpha_1,\ldots,\alpha_r ; x)$ considered by Thakare and Madhekar [1], Thakare, Madhekar and Karande[1] motivate us to consider the modified polynomial sets $\{p_n(\alpha_1,\ldots,\alpha_r;x)\}$ defined by

$$(3) \quad H(t)\Psi[x\,S(t)] \prod_{j=1}^{r}(1-v_j t^m)^{-\alpha_j} = \sum_{n=0}^{\infty} p_n(\alpha_1,\ldots,\alpha_r;x)t^n ,$$

where

$H(t)$, $\Psi(z)$, $S(t)$ are given by 7.1(2), 7.1(3) and 7.1(4) respectively, and α_j, v_j $(j=1,2,\ldots,r)$ are arbitrary constants with h_k, γ_k, s_k $(k = 0,1,\ldots)$ independent of α_j, v_j $(j=1,2,\ldots,r)$; and prove the Characterization Theorem as below :

<u>Theorem 8:</u>

$p_n(\delta_1(n),\ldots,\delta_r(n);x) \in [\Psi]$ if and only if each $\delta_j(n)$ $(j=1,\ldots,r)$ is linear in $n (n \geq 1)$.

<u>Proof:</u>

Let $\delta_j(n)$ be linear in n, that is,

$$\delta_j(n) = \alpha_j + n\beta_j \quad (j = 1, \ldots, r) .$$

By Taylor's Theorem

$$p_n(\alpha_1,\ldots,\alpha_r;x) = \frac{1}{n!} D_t^n \left[H(t)\Psi[x\,S(t)] \prod_{j=1}^{r}(1-v_j t^m)^{-\alpha_j} \right]_{t=0},$$

so that

$$p_n(\alpha_1 + n\beta_1, \ldots, \alpha_r + n\beta_r; x)$$

$$= \frac{1}{n!} D_t^n \left[\left\{ H(t) \Psi[x S(t)] \prod_{j=1}^{r} (1 - v_j t^m)^{-\alpha_j} \right\} \cdot \left\{ \prod_{j=1}^{r} (1 - v_j t^m)^{-\beta_j} \right\}^n \right] \Bigg|_{t=0},$$

hence

$$(4) \qquad \sum_{n=0}^{\infty} p_n(\alpha_1 + n\beta_1, \ldots, \alpha_r + n\beta_r; x) t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_t^n \left[f(t) \{\phi(t)\}^n \right] \Bigg|_{t=0},$$

where, for convenience,

$$(5) \quad \begin{cases} f(t) = H(t) \Psi[x S(t)] \prod_{j=1}^{r} (1 - v_j t^m)^{-\alpha_j}, \\ \text{and} \\ \phi(t) = \prod_{j=1}^{r} (1 - v_j t^m)^{-\beta_j}. \end{cases}$$

On applying Lagrange's expansion formula (Pólya and Szegö [6], p.146, Problem 207) 5.3(24) to (4), we get

$$\sum_{n=0}^{\infty} p_n(\alpha_1 + n\beta_1, \ldots, \alpha_r + n\beta_r; x) t^n = \frac{f(\xi)}{1 - t\,\phi'(\xi)},$$

where

$$(6) \qquad \xi = t\,\phi(\xi) = t \prod_{j=1}^{r} (1 - v_j \xi^m)^{-\beta_j},$$

which on simplifying yields the following result,

$$(7)\quad \sum_{n=0}^{\infty} p_n(\alpha_1 + n\beta_1, \ldots, \alpha_r + n\beta_r ; x)t^n$$

$$= \frac{H(\xi)\ \psi[x\ S\ (\xi)] \prod_{j=1}^{r} (1 - v_j\ \xi^m)^{-\alpha_j}}{1 - m\ \xi^m \sum_{i=1}^{r} \beta_i\ v_i\ (1 - v_i\ \xi^m)^{-1}},$$

where ξ is given by (6).

This proves that $p_n(\alpha_1+n\beta_1,\ldots,\alpha_r+n\beta_r;x) \in [\psi]$.

Conversely, let $p_n(\delta_1(n),\ldots,\delta_r(n);x) \in [\psi]$.

Clearly from the definition (3) it follows that $p_n(0,\ldots,0;x) = p_n(x)$ is of degree n in x having the form,

$$(8)\quad p_n(x) = C_{n,n}\ x^n + C_{n,n-m}\ x^{n-m} + \ldots + C_{n,o}\ .$$

Let

$$(9)\quad p_n(\delta_1(n),\ldots,\delta_r(n);x) = \bar{C}_{n,n}\ x^n + \bar{C}_{n,n-m}\ x^{n-m} + \ldots + \bar{C}_{n,o}\ .$$

As

$$\sum_{n=0}^{\infty} p_n(\alpha_1, \ldots, \alpha_r ; x)t^n$$

$$= H(t)\psi[x\ S\ (t)] \prod_{j=1}^{r} (1 - v_j t^m)^{-\alpha_j}$$

$$= \sum_{n=0}^{\infty} p_n(x) t^n .$$

$$\cdot \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(\alpha_1)_{n_1} \cdots (\alpha_r)_{n_r}}{n_1! \cdots n_r!} v_1^{n_1} \cdots v_r^{n_r} \; t^{m \sum_1^r n_j} ,$$

$$(10) \qquad = \sum_{n=0}^{\infty} \sum_{n_1=0}^{[n/m]} \cdots \sum_{n_r=0}^{[n/m]} p_{n-m\sum_1^r n_j}(x)$$

$$\cdot \frac{(\alpha_1)_{n_1}}{n_1!} v_1^{n_1} \cdots \frac{(\alpha_r)_{n_r}}{n_r!} v_r^{n_r} t^n ,$$

(10) gives us

$$p_n(\delta_1(n),\ldots,\delta_r(n);x)$$
$$= p_n(x)+(\delta_1(n)v_1+ \cdots +\delta_r(n)v_r)p_{n-m}(x)+\cdots ,$$

which on making use of (8) and (9) yields

$$\bar{C}_{n,n} x^n + \bar{C}_{n,n-m} x^{n-m} + \cdots$$

$$= C_{n,n} x^n + \left\{C_{n,n-m} + C_{n-m,n-m} (\delta_1(n)v_1+\cdots+\delta_r(n)v_r)\right\} x^{n-m} +\cdots,$$

and therefore

$$(12) \qquad \bar{C}_{n,n} = C_{n,n}$$

$$(13) \quad \bar{C}_{n,n-m} = C_{n,n-m}+ C_{n-m,n-m} (\delta_1(n)v_1+\cdots+\delta_r(n)v_r).$$

Denoting

$$r_n = \frac{\nu_n}{\nu_{n-m}} \quad \frac{C_{n,n-m}}{C_{n,n}} ,$$

and

$$\bar{r}_n = \frac{\nu_n}{\nu_{n-m}} \quad \frac{\bar{C}_{n,n-m}}{C_{n,n}} ,$$

it follows that

$$\bar{r}_n - r_n = \frac{\nu_n}{\nu_{n-m}} \quad \frac{C_{n-m,n-m}}{C_{n,n}} \quad \left[\delta_1(n)v_1 + \ldots + \delta_r(n)v_r\right] .$$

Next, use of the identity,

$$(14) \quad \bar{r}_n - r_n = (\bar{r}_m - r_m) + \left(\frac{n-m}{m}\right) \left\{ (\bar{r}_{2m} - r_{2m}) - (\bar{r}_m - r_m) \right\}$$

and the relation $C_{n,n} = \nu_n \; a_o \; h_o^n$ gives,

$$\delta_1(n)v_1 + \ldots + \delta_r(n)v_r = (\delta_1(m)v_1 + \ldots + \delta_r(m)v_r)$$

$$+ \left(\frac{n-m}{m}\right) \left\{ \delta_1(2m)v_1 + \ldots + \delta_r(2m)v_r - \delta_1(m)v_1 - \ldots - \delta_r(m)v_r \right\}$$

and hence we are led to

$$\delta_j(n)v_j = \delta_j(m)v_j + \left(\frac{n-m}{m}\right) \left\{ \delta_j(2m)v_j - \delta_j(m)v_j \right\}$$

for each $j = 1,2, \ldots, r$, that is,

$$(15) \quad \delta_j(n) = 2\delta_j(m) + \frac{n(\delta_j(2m) - \delta_j(m))}{m} - \delta_j(2m).$$

Thus by choosing $\delta_j(n)$ as given by (15), $\delta_j(n);(j=1,\ldots,r)$ are linear in n and the proof of the theorem is completed.

7.4 APPLICATIONS

(i) Consider generalized Laguerre polynomials $\Gamma_n^c(x)$ possessing the generating relation 5.3(1). Obviously, $\Gamma_n^c(x) \in [\exp]$ and thus from above Theorem 8 we may conclude

COROLLARY 1:

$$\Gamma_n^{\delta(n)} \in [\exp] \text{ if and only if } \delta(n) \text{ is linear in } n.$$

(ii) The polynomials $\Gamma_n^c(x)$ may be generalized further by the defining relation

$$(1) \quad \sum_{n=0}^{\infty} \Gamma_n^{(c_1,\ldots,c_p)}(x)t^n = (1+y_1t^m)^{-c_1} \ldots (1+y_pt^m)^{-c_p} \exp\left[\frac{xt}{(1+yt^m)^r}\right].$$

Clearly $\Gamma_n^{(c_1,\ldots,c_p)}(x) \in [\exp]$ and hence

COROLLARY 2:

$$\Gamma_n^{\delta_1(n),\ldots,\delta_p(n)}(x) \in [\exp] \text{ if and only if each}$$

$\delta_j(n)$, $(j=1,\ldots,p)$ is linear in n.

For $p=2$, $m=1$, $y_1=-1$, $y_2=1$, $c_1 = \alpha+1$, $c_2=\beta$, $y=-1$, $r=1$ and on replacing x by -x we obtain Corollary (5) of Thakare and Madhekar [1].

(iii) It is evident that $f_n^c(x)$ given by 2.1(3) $\in [G]$ which leads to the result :

COROLLARY 3:

$$f_n^{\delta(n)} \in [G] \text{ if and only if } \delta(n) \text{ is linear in } n.$$

(iv) As in (ii), $f_n^c(x)$ may be generalized as

$$(2) \quad \sum_{n=0}^{\infty} f_n^{c_1,\ldots,c_p}(x)t^n$$

$$=(1+y_1t^m)^{-c_1}\ldots(1+y_pt^m)^{-c_p}\, G\left[\frac{xt}{(1+yt^m)^r}\right].$$

On application of Theorem 8, this yields the result

COROLLARY 4:

$$f_n^{\delta_1(n),\ldots,\delta_p(n)}(x) \in [G] \text{ if and only if each}$$

$\delta_j(n)$;$(j = 1, \ldots,p)$ is linear in n.

(v) The polynomials $g_n^c(x,r,1)$ corresponding to the case $s = 1$ of $g_n^c(x,r,s)$ studied by R. Panda [1], defined by 2.1(1) $\in$ [G]. Therefore,

<u>COROLLARY 5:</u>

$g_n^{\delta(n)}(x,r,1) \in$ [G] if and only if $\delta(n)$ is linear in n.