CHAPTER - I

SOME RECENT DEVELOPMENTS IN THE FIELD OF VARIOUS GENERALIZATIONS OF CLASSICAL POLYNOMIALS

1.1 INTRODUCTION

Amongst the numerous Special Functions which find wide applications in different branches of Applied Mathematics and other disciplines, e.g., Queuing theory (see Srivastava and Kashyap [K]), Coding theory (see Sloane [1]), Statistical distributions (see Mathai and Saxena [D], [E]), Approximation theory and Numerical Analysis, the classical orthogonal polynomials, viz. those which go by the names of Legendre, Hermite, Laguerre, Chebyshev, Gegenbauer and Jacobi occupy a place of prime importance. These polynomials may be looked at from varied angles. One of these ways is to consider them as particular cases of the generalized hypergeometric series.

$$(1) \qquad {}_pF_q\left[a_1, \ldots, a_p; b_1, \ldots, b_q ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} ,$$

where,

$$(a)_n = a(a+1) \ldots (a+n-1), \ (a)_0 = 1.$$

This view point of looking at the classical polynomials is encountered quite frequently in the literature on Special Functions, a fact which may be acredited to the tremendously rich theory of hypergeometric function which has developed notably in the hands of many eminent mathematicians like F. Gauss, F. J. P. Whipple, A. C. Dixon, J. Dougall, H. Bateman, E. T. Whittaker, G. N. Watson, G. H. Hardy, P. Appell, J. Kampé de Fériet, E. W. Barnes, A. Erdelyi. J. L. Burchnall, T. W. Chaundy, W. N. Bailey, L. J. Slater, R. P. Agarwal, F. H. Jackson, L. Carlitz, H. M. Srivastava, Y. L. Luke, R. G. Buschman, Lavoie and many others quoted in the literature.

Another important way of looking at the classical polynomials is to view them as special case of general orthogonal polynomials which may be defined as follows:

Let $w(x)$ be positive and continuous over a given finite or infinite interval (a,b) except possibly at a finite set of points and let

$$\int_a^b w(x)\ |x|^n\ dx < \infty \qquad n = 0,\ 1,\ 2,\ \ldots$$

A set of real polynomials $\phi_n(x)$ of precise degree n, $n = 0, 1, 2, \ldots$ is said to be orthogonal with respect to $w(x)$ over (a,b) if

(2) $$\int_a^b w(x)\ \phi_n(x)\ \phi_m(x)dx = 0\ ,\ m \neq n.$$

As per this definition the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which include as particular cases the polynomials of Legendre $P_n(x)$, Chebyshev $T_n(x)$, and Gegenbauer $C_n^{\nu}(x)$ form a set of orthogonal polynomials with respect to the weight function $(1-x)^{\alpha}\ (1+x)^{\beta}$, $\alpha,\beta > -1$ over $(-1\ ,\ 1)$. The other two classes of classical orthogonal polynomials viz. those of Hermite and Laguerre form orthogonal sets with respect to the weight function e^{-x^2} and $x^{\alpha}\ e^{-x}$ over $(-\infty, \infty)$ and $(0\ ,\ \infty)$ respectively. The polynomials of Hermite and Laguerre may also be obtained from Jacobi polynomials through a limiting process. Another noteworthy class of orthogonal polynomials which, as pointed out by Agarwal [1], may also be obtained as a limiting case of Jacobi polynomials, is the class of Bessel polynomials :

(3) $$y_n(x,a,b) = {}_2F_0\left[-n,\ a - 1 + n\ ;—;\ -\frac{x}{b}\right].$$

These polynomials were encountered by Krall and Frink [1] during their study of the solutions of wave equation in Spherical Co-ordinates.

It is well known (see Tricomi [1]) that all the classical orthogonal polynomials possess an operational representation of the type

$$(4) \qquad \phi_n^{(\lambda)}(x) = \frac{K_n}{w(x)} \left\{ D_x^n \left[X(x)\right]^n \; w(x) \right\} ,$$

$w(x)$ is the weight function and $X(x)$, independent of n, is a polynomial in x of degree ≤ 2. The relation (4) which is commonly known as the Rodrigues formula, characterizes the classical polynomials and play a vital role in the derivation of important and interesting properties of the classical polynomials.

The generating functions provide another effective tool for developing the theory of classical polynomials. For example, the well known generating relation

$$(5) \qquad \sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2} ,$$

for the Legendre polynomials $P_n(x)$ has been at the core of extensive development of its theory.

The unwaning utility of the classical orthogonal polynomials over the years has led to the introduction of numerous generalizations and extensions of these polynomials which help to study them in a unified way and, at times, also provide new and interesting insight into their study. Most of these generalizations that have been introduced into analysis are by way of modifying the generating functions or the hypergeometric representations of the classical polynomials or by changing the Rodrigues formula. For the purpose of summarizing the various generalizations we classify them into three groups. Section 1.2 of this chapter incorporates a brief summary of the important generalizations of the classical polynomials which have been introduced either by modifying the generating relations or by making appropriate changes in the hypergeometric representations. In section 1.3 we present the various generalizations through Rodrigues formulae and section 1.4 deals with certain other types of extensions of the classical polynomials.

1.2 GENERALIZATIONS THROUGH GENERATING RELATIONS AND HYPERGEOMETRIC REPRESENTATIONS

As mentioned above the well known generating function 1.1(5) for the Legendre polynomials $P_n(x)$ has been at the core of extensive development of its theory, which inspired the mathematicians to study its different variations and generalizations. For example, the Gegenbauer (or Ultraspherical) polynomial $C_n^{\lambda}(x)$ is a generalization of Legendre polynomial. and is generated by the relation

$$(1) \quad (1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n .$$

The generating relation (1), which is itself a particular case of a generating relation for Jacobi polynomials, reduces to that for the Chebyshev polynomials $U_n(x)$ of second kind on putting $\nu = 1$.

During his study of generalized potential problems associated with the extended Laplace equations

$$(2) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 3\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 0 ,$$

Humbert ([2], [3]) came across with the polynomials defined by the generating relation

$$(3) \quad (1-m\, xt + t^m)^{-\nu} = \sum_{n=0}^{\infty} t^n \Pi_{n,m}^{\nu}(x) ,$$

where $m \geq 1$ is an integer. Humbert's polynomials include as particular cases the polynomials studied by Pincherle (see Humbert [1]) and Kinney [1] .

In the year 1965, Gould [9] introduced an interesting generalization of several known polynomials including those of Louville, Legendre, Chebyshev, Gegenbauer, Pincherle, Humbert and Kinney, by means of the generating relation

$$(4) \quad (C - m\,xt + yt^{m})^{p} = \sum_{n=0}^{\infty} t^{n} P_n(m, x, y, p; C) ,$$

where $m \geq 1$ and other parameters are unrestricted in general. In his study of the polynomial $P_n(m, x, y, p, C)$ which he termed as generalized Humbert polynomial, a novel pair of inverse series relation was also incorporated by Gould.

The generating relation 1.1(5) when expressed in the form

$$(5) \quad (1-t)^{-1} \, {}_1F_0 \left[\begin{matrix} \frac{1}{2}; & \dfrac{2xt}{(1-t)^2} \\ -; & \end{matrix} \right] = \sum_{n=0}^{\infty} P_n(1-x)t^{n} ,$$

led a number of researchers to study its various generalizations. For example, Bateman [1] concentrated on the polynomials $Z_n(x)$ generated by

$$(6) \quad (1-t)^{-1} \, {}_1F_1 \left[\begin{matrix} \frac{1}{2}; & -\dfrac{4xt}{(1-t)^2} \\ 1; & \end{matrix} \right] = \sum_{n=0}^{\infty} Z_n(x)t^{n} ,$$

which attracted his attention during his work on inverse Laplace transforms. On the other hand S. O. Rice [1] studied the polynomials defined by

(7) $H_n(\zeta, p, v) = {}_3F_2\,(-n,\ n+1\ ,\ \zeta\ ;\ 1,\ p\ ;\ v)$,

which satisfy the generating relation

(8) $$(1-t)^{-1}\ F\left[\begin{matrix}\zeta\ ,\ \frac{1}{2};\\ p;\end{matrix}\ -\frac{4vt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} H_n(\zeta\ ,\ p,\ v)t^n\ .$$

The striking close resemblance of the generating relations for $P_n(1-x)$, $Z_n(x)$ and $H_n(\zeta, p, v)$ inspired Sister M. Celine Fasenmyer [1] to introduce a more general class of polynomials generated by

(9) $$(1-t)^{-1}\ {}_pF_q\left[\begin{matrix}a_1,\ \ldots,a_p;\\ b_1,\ \ldots,b_q;\end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$$

$$= \sum_{n=0}^{\infty} f_n\left[\begin{matrix}a_1,\ \ldots,a_p;\\ b_1,\ \ldots,b_q;\end{matrix}\ x\right]t^n\ ,$$

which yields

(10) $$f_n\left[\begin{matrix}a_1,\ \ldots,\ a_p;\\ b_1,\ \ldots,\ b_q;\end{matrix}\ x\right]$$

$$= {}_{p+2}F_{q+2}\left[\begin{matrix}-n,\ n+1,\ a_1,\ \ldots,a_p;\\ 1,\ 1/2,\ b_1,\ \ldots,b_q;\end{matrix}\ x\right].$$

An elegant generalization of the polynomials defined by (9) is due to Rainville [H, p.137] who considers the polynomials $f_n(x)$ generated by

$$(11) \quad (1-t)^{-c} \, \Psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x) t^n ,$$

where $\Psi(u)$ has a formal power series expansion

$$(12) \quad \Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n , \quad \gamma_0 \neq 0.$$

The explicit representation of the polynomials $f_n(x)$ defined by (11) and (12) is given by

$$(13) \quad f_n(x) = \frac{(c)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k \, (c+n)_k \, \gamma_k}{(\frac{c}{2})_k \, (\frac{1}{2}c + \frac{1}{2})_k} x^k .$$

Among other results Rainville also gives the inverse relation

$$(14) \quad x^n = \frac{(c)_{2n}}{2^{2n} \gamma_n} \sum_{k=0}^{n} \frac{(-1)^k \, (c+2k)}{(n-k)! \, (c)_{n+k+1}} f_k(x) .$$

The generating relation (11) and the well known relation

$$(15) \quad (1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n ,$$

indicated the possibility of unifying the study of the Laguerre polynomials and the polynomials included in the class of $\{f_n(x)\}$. Among those who followed this course of investigations, mention may be made of Chandel [1], Jain [1], and Panda [1].

Chandel considered the polynomials generated by

$$(16)\quad (1-t)^{-c} \exp\left[-xt\left(\frac{r}{1-t}\right)^{r}\right] = \sum_{n=0}^{\infty} f_n^c(x,r)\,t^n .$$

These polynomials are included in the more general class of polynomials introduced by Panda [1] and Jain [1].

As regards the generalizations that have been introduced by modifying the hypergeometric representations of the classical polynomials and their various extensions, we mention the following :

<u>Brafman polynomials</u> [1]:-

$$(17)\quad B_n^s\left[(a_p);\ (b_q);\ x\right] = {}_{p+s}F_q\left[\begin{matrix}\Delta(s;\ -n),\ (a_p); \\ (b_q);\end{matrix}\ x\right]$$

where $\Delta(r,\alpha)$ stands for the r parameters

$$\frac{\alpha}{r},\ \frac{\alpha+1}{r},\ \ldots,\ \frac{\alpha+r-1}{r}$$

and (a_p) for the set $a_1, a_2, \ldots, a_p$.

Generalized Rice Polynomials (Khandekar [1]) :-

$$(18) \quad H_n^{(\alpha,\beta)}(\xi, p, x) = \frac{(1+\alpha)_n}{n!} \; {}_3F_2\left[\begin{matrix} -n, & n+\alpha+\beta+1, & \xi \;; \\ 1+\alpha, & & p; \end{matrix} \; x\right]$$

Polynomials considered by Jain [1]:-

$$(19) \quad f_n^{(c,k)}\left[(a_p);\, (b_q);\, x\right]$$

$$= \frac{(c)_n}{n!} \; {}_{p+k}F_{q+k}\left[\begin{matrix} -n, & \Delta(k-1; c+n), (a_p); \\ & \Delta(k;c) \quad , (b_q); \end{matrix} \; (k-1)^{k-1} x\right]$$

where k is a positive integer.

Generalized Hypergeometric Polynomials (Agarwal and Manocha [1])

$$(20) \qquad f_n(x) = {}_{p+1}F_q\left[\begin{matrix} -n, \; \alpha_1, \ldots, \alpha_p \;; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; \frac{x}{a+bn}\right].$$

1.3 GENERALIZATIONS THROUGH RODRIGUES FORMULAE

The first generalization of the classical polynomials by the method of making modifications in the Rodrigues formulae seems to have been given by E. T. Bell [1] who introduced a generalization of the Hermite polynomials $H_n(x)$ by replacing the exponent 2 in the Rodrigues formula

$$(1) \quad H_n(x) = (-1)^n \exp(x^2)\, D^n \exp(-x^2) \quad , \; D = \frac{d}{dx}$$

by an arbitrary parameter r . But for a long time, until the investigations of Rajgopal [1] and Riordan [I] , Bell's paper went unnoticed. Following the work of earlier researchers, Gould and Hopper [1] defined what they called the generalized Hermite polynomial by the relation :

$$(2) \quad H_n^r(x,\alpha,p) = (-1)^n x^{-\alpha} \exp(px^r)\, D^n \left\{ x^\alpha \exp(-px^r) \right\} .$$

In the same paper (see also Gupta and Jain [1] and Lahiri [1]) they also introduced another generalization of the Hermite polynomials by means of the operational formula

$$(3) \qquad g_n^r(x,h) = e^{hD^r}\, x^n ,$$

which yields

$$(4) \quad g_n^r(x,h) = \sum_{k=0}^{[n/r]} \frac{n!}{k!\,(n-rk)!}\, h^k\, x^{n-rk}$$

$$(5) \qquad\qquad = x^n \;{}_rF_0\left[\Delta(r;\,-n);\,-\,;\, h\left(-\frac{r}{x}\right)^r \right] .$$

The polynomials $g_n^r(x,h)$ are obviously included in the Brafman polynomials as defined by 1.2 (17).

On the other hand, Chak [1], Palas [1], Chatterjea [2], Singh and Srivastava [2] and Singhal and Joshi [1] made use of the Rodrigues formula for Laguerre polynomials to introduce a number of its generalizations. For example Singh and Srivastava [2] introduced the polynomials

$$(6) \quad L_n^{(\alpha)}(x,r,p) = \frac{x^{-\alpha}\exp(px^r)}{n!} D^n \left\{x^{\alpha+n}\exp(-px^r)\right\} .$$

Another chain of generalizations of the classical polynomials emerged from the Rodrigues formula for Jacobi polynomials viz.

$$(7) \quad P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!} D^n\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}.$$

Among those who have contributed to the study of such generalizations, the names of Fujiwara [1], Srivastava and Singhal [2], and Srivastava and Panda [3] are worth mentioning.

It may also be mentioned here that the literature also contains studies of various generalizations of classical polynomials introduced by replacing the operator D in the Rodrigues formulas by other linear operators e.g. $x^{k+1}D$ (Srivastava and Singhal [1]), $T_k = x(k + xD)$ (Mittal [1]), $T_{a,k} = x^k(a + xD)$, (Joshi and Prajapat [1]).

The list of other contributors to the study of classical polynomials and their generalizations through Rodrigues formula includes the names of Chandel and Bhargava [1] , Srivastava, Lavoie and Tremblay [1], Singhal and Soni [1], Das [2], Patil and Thakare [1], Joshi [1] and Joshi and Singhal [1].

1.4 OTHER TYPES OF EXTENSIONS OF CLASSICAL ORTHOGONAL POLYNOMIALS

Besides the generating relations, hypergeometric representations and Rodrigues formulae, the recurrence relations of the classical polynomials have also been employed to generate new extensions of the classical polynomials. For example the three term pure recurrence relation for the Legendre polynomials

$$(1) \quad n\, P_n(x) - x(2n-1)\, P_{n-1}(x) - (n-1)\, P_{n-2}(x) = 0 ,$$

when modified to

$$(2) \quad (n+\nu)y_n(\nu,x) - x(2n+2\nu-1)y_{n-1}(\nu,x) + (n+\nu-1)y_{n-2}(\nu,x) = 0. \qquad 2\nu \neq -1,-2,\ldots$$

yields a new class of polynomials called the associated Legendre polynomials. These polynomials have been studied by Palama [1] and Barucand and Dickinson [1] .

Yet another type of generalizations of the classical orthogonal polynomials originates from the extension of the notion of orthogonal polynomials to two sets of polynomials such that for a given appropriate weight function the polynomials satisfy an orthogonality condition analogous to 1.1(2). Such a concept was introduced by Kaunhauser [1] who defined a pair of polynomial sets $\{R_m(x)\}$ and $\{S_n(x)\}$ as biorthogonal over the interval (a,b) with respect to the admissible weight function p(x) and basic polynomials r(x) and s(x) provided the conditions

$$(3) \qquad \int_a^b p(x)\, R_m(x)\, S_n(x)\, dx \begin{cases} = 0, & m, n = 0,1,\ldots, m \neq n \\ \neq 0, & m = n \end{cases}$$

are satisfied.

A few years later Konhauser [2] made use of his theory of general biorthogonal polynomials to construct the pair of biorthogonal sets of polynomials relative to the Laguerre weight function $x^{\alpha} e^{-x}$. His work was followed by several other workers including Carlitz [7], Srivastava[3], Prabhakar [1] and Prabhakar and Kashyap [1] .

The seemingly most natural aspect of the extension of orthogonal polynomials to several variables was first taken up, by Jackson[1] but thereafter it remained unexplored for quite some time. Recently (1975)

Koornwinder [1] presented a paper on two variable analogues of the classical orthogonal polynomials at the Advanced Seminar on Special Functions held in Madison, Wisconsin, which generated lot of interest in the development of the theory of multiple variable analogues of orthogonal polynomials.