CHAPTER - II

ON A UNIFICATION OF GENERALIZED HUMBERT AND LAGUERRE POLYNOMIALS

2.1 INTRODUCTION

Recently, having been motivated by the earlier works of Rainville [H , p.137 , Theorem 48], Chandel ([1],[2]) and Jain[1], Rekha Panda [1] introduced an elegant generalization of several known polynomial systems belonging to (or providing extensions of) the families of the classical Jacobi, Hermite and Laguerre polynomials by means of the generating relation

$$(1) \qquad (1-t)^{-c}\, G\left[\frac{xt^{s}}{(1-t)^{r}}\right] = \sum_{n=0}^{\infty} g_n^{c}(x,r,s)t^{n},$$

where

$$(2) \qquad G[z] = \sum_{n=0}^{\infty} \gamma_n z^{n}, \quad \gamma_0 \neq 0 ,$$

c is an arbitrary complex number, r is any integer, positive or negative, and s is a positive integer.

A comparision of (1) with the generating function 1.2(4) for the generalized Humbert polynomials of Gould [9] suggests that it would be interesting and worthwhile to study, a new class of polynomials $\left\{f_n^{c}(x,y,r,m)\,|\,n=0,1,2,\ldots\right\}$

defined by the generating relation

$$(3)\quad (1+yt^m)^{-c}\, G\left[\frac{xt}{(1+yt^m)^r}\right] = \sum_{n=0}^{\infty} f_n^c(x,y,r,m)t^n ,$$

where

$$(4)\quad G[z] = \sum_{n=0}^{\infty} \gamma_n z^n , \quad \gamma_0 \neq 0 ,$$

$m \geq 1$ is an integer and other parameters are unrestricted in general.

From (3) and (4) it is easy to deduce that $f_n^c(x,y,r,m)$ is a polynomial of degree n in x with its explicit representation as

$$(5)\quad f_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k \gamma_{n-mk}\, x^{n-mk} .$$

On choosing

$$\gamma_n = \frac{\prod_{j=1}^{p}(a_j)_{\alpha_j n}}{\prod_{j=1}^{q}(b_j)_{\beta_j n}\, n!} ,$$

the explicit representation (5) may also be put in the form

$$(6) \qquad f^{c}_{n,p,q}\left[((a_p,\alpha_p))\;;\;((b_q,\beta_q));\;x\right]$$

$$= x^n\left\{\prod_{j=1}^{p}(a_j)_{\alpha_j n}\right\}\Big/\left\{\prod_{j=1}^{q}(b_j)_{\beta_j n}\cdot n!\right\}$$

$$\cdot\;{}_{q+2}\Psi_{p+1}\left[\begin{matrix}(-n,m),(-c-rn+1,rm),\;((1-b_q-n\beta_q,m\beta_q))\;;\\ (-c-rn+1,rm-1),((1-a_p-n\alpha_p,m\alpha_p));\end{matrix}\;(-1)^{m(1+\sum_1^q\beta_j-\sum_1^p\alpha_j)}\frac{y}{x^m}\right],$$

where ${}_p\Psi_q$ is Wright's generalized hypergeometric function and $((a_p\,,\,\alpha_p))$ stands for the set of p parameter pairs

$$(a_1,\alpha_1)\,,\,(a_2,\alpha_2),\,\ldots\,,\,(a_p,\alpha_p)$$

with similar interpretations for $((b_q\,,\,\beta_q))$; $\alpha_j(j = 1,\,\ldots\,,\,p)$, $\beta_j(j = 1,\,\ldots\,,\,q)$ being positive and r restricted to be positive integer.

On putting $\alpha_j = 1(j = 1,\,\ldots,p)$ and $\beta_j=1(j=1,\ldots,q)$, (6) simplifies to

$$(7)\quad f^{c}_{n,p,q}\left[(a_p)\ ;\ (b_q);x\right] = x^n \left\{\prod_{j=1}^{p}(a_j)_n\right\} \Big/ \left\{\prod_{j=1}^{q}(b_j)_n\ .n!\right\}$$

$$_{m(r+q+1)}F_{m(p+r)-1}\left[\begin{matrix}\Delta(m,-n),\ \Delta(rm,1-c-rn)\quad \Delta((m,1-b_q-n))\ ; \\ \Delta(rm-1,1-c-rn),\ \Delta((m,1-a_p-n));\end{matrix}\right.$$

$$\left.\frac{(rm)^{rm}}{(rm-1)^{rm-1}}\ (-m)^{m(q-p+1)}\quad \frac{y}{x^m}\right],$$

where

$\Delta((m,\lambda_p))$ stands for the set $\Delta(m,\lambda_1),\ldots,\ \Delta(m,\lambda_p)$ and $\Delta(m,\lambda)$ as before, stands for the m parameters $\frac{\lambda}{m}$, $\frac{\lambda+1}{m}$,..., $\frac{\lambda+m-1}{m}$. Similarly (a_p) denotes the set of p parameters $a_1, \ldots ,a_p$.

It is obvious that when $y = -1$, and $m = 1$, (3) would correspond to the special case $s = 1$ of (1), whereas on taking

$$\nu_n = (-m)^n \binom{p}{n},\quad c = -p \quad\text{and}\quad r = 1,\ (3)\ \text{would}$$

transform into 1.2(4) with $C = 1$. Thus the class of polynomials $\left\{f^{c}_{n}(x,y,r,m)\,|\,n = 0,\ 1,\ \ldots\right\}$ defined by (3) and (4) provides an interesting unification as well as generalization of the various polynomials included in $g^{c}_{n}(x,r,1)$ and the generalized Humbert polynomial $P_n(m,x,y,p,1)$ which itself is a generalization of several known polynomials including those of Legendre, Gegenbauer, Humbert, Tchebycheff, Princherle, and many others. For the different conditions on the parameters of $f_n^c(x,y,r,m)$ under which it reduces to the polynomials mentioned above and many others, e.g., the polynomials of sister Celine, Jacobi, Rice, reference may be made to Gould [9], Jain [1] and Panda [1].

Being motivated by the observations mentioned in the above paragraphs, we undertake here a systematic study of the polynomials $f_n^c(x,y,r,m)$. The section 2.2 of this chapter incorporates a number of recurrence relations for $f_n^c(x,y,r,m)$. In section 2.3 we derive generating function for $f_n^{c+\sigma n}(x,y,r,m)$ and discuss its various particular cases and their applications in the derivation of some expansion formulae. In section 2.4 we give generating function for

$$\frac{c-1}{c-1+\beta n}\ f_n^{c+\sigma n}(x,y,1/m,m),\ \beta = \sigma + 1/m\ .$$

In what follows, for the sake of brevity, we shall abbreviate $f_n^c(x,y,r,m)$ by $f_n^c(x)$ unless there is any ambiguity regarding other parameters.

2.2 RECURRENCE RELATIONS

If we denote the left member of 2.1(3) by U(x,t), then it is readily seen that U(x,t) satisfies the differential equation

(1) $x\left[1+(1-rm)yt^{m}\right]\frac{\partial U}{\partial x} - t(1+yt^{m})\frac{\partial U}{\partial t} = cymt^{m}U$.

Combining (1) with 2.1(3) we get the following differential recurrence relations for $f_n^c(x)$:

(2) $xD_x f_n^c(x) - nf_n^c(x) = \{n+mc-m\} yf_{n-m}^c(x) - (1-rm)yxD_x f_{n-m}^c(x)$,

(3) $xD_x f_n^c(x) - nf_n^c(x)$

$$= mcy \sum_{k=0}^{[n/m]-1} (-y)^k f_{n-mk-m}^c(x) + rmxy \sum_{k=0}^{[n/m]-1} (-y)^k D_x f_{n-mk-m}^c$$

and

(4) $xD_x f_n^c(x) - nf_n^c(x)$

$$= my \sum_{k=0}^{[n/m]-1} (mry-y)^k (c+nr-mrk-mr) f_{n-mk-m}^c(x) ,$$

where

$D_x = \frac{d}{dx}$, $n \geq m$.

In view of the general nature of the polynomials $f_n^c(x)$, the recurrence relations given above can be particularized to corresponding recurrence relations for the various classes of polynomials that are included in the definition of $f_n^c(x)$.

2.3 GENERATING FUNCTION FOR $f_n^{c+\sigma n}(x)$

The generating function that we propose to derive in this section is

$$(1) \quad \sum_{n=0}^{\infty} f_n^{c+\sigma n}(x)\, t^n = \frac{(1+yw^m)^{1-c}}{[1+y(1+\sigma m)w^m]}\, G\left[\frac{xw}{(1+yw^m)^r}\right],$$

where

$$(2) \quad w = t(1+yw^m)^{-\sigma}, \quad w(0) = 0,$$

σ is an arbitrary complex number and γ_n, the coefficient of z^n in the power series for $G[z]$ is independent of c.

To prove (1) we start with the function

$$(3) \quad F(A_n, y, r, m, c) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k A_{n-mk},$$

where A_n is an arbitrary sequence such that $\sum_{n=0}^{\infty} |A_n| < \infty$, $m \geq 1$ is an integer and other parameters are unrestricted in general.

For arbitrary complex values of σ it is easy to see that

$$(4) \quad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n)t^n = \sum_{n=0}^{\infty} A_n t^n \sum_{k=0}^{\infty} \binom{-c-\sigma n-rn-\sigma mk}{k} (yt^m)^k.$$

On summing the inner series on the right hand side of (4) with the help of the following consequence of

Lagrange's expansion formula (Pólya and Szegö [G], p. 146, Problem 216) :

$$(5)\quad \sum_{n=0}^{\infty} \binom{a+bn}{n} t^n = (1+v)^{a+1} \left[1-(b-1)v\right]^{-1},$$
$$v = t(1+v)^b ,$$

we get after a little simplification the general formula

$$(6)\quad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n) t^n$$
$$= \frac{(1+yw^m)^{1-c}}{1+y(1+\sigma m)w^m} \sum_{n=0}^{\infty} \left[\frac{w}{(1+yw^m)^r}\right]^n A_n ,$$

where w is given by (2).

In (6) if we take $A_n = \gamma_n x^n$ and then make use of 2.1(5) and 2.1(4), we are immediately led to the generating function (1).

Alternatively, we may start with the following consequence of the defining relation 2.1(3)

$$(7)\qquad f_n^c(x) = \sum_{k=0}^{[n/m]} \frac{(c-b)_k (-y)^k}{k!} f_{n-mk}^b(x) ,$$

and then arrive at (1) by using the method illustrated by Singhal [1]. Yet another method of providing (1) would run parallel to that of Rekha Srivastava [1] which she employed for deriving a corresponding generating function for $g_n^{c+\sigma n}(x,r,s)$ wherein by putting $s = 1$ we shall get the case $m = 1$, $y = -1$ of our result (1).

For $\sigma = 0$, (1) evidently reduces to 2.1(3), whereas the substitution $\sigma = -1/m$ transforms it to

$$(8)\quad \sum_{n=0}^{\infty} f_n^{c-n/m}(x)t^n = (1-yt^m)^{c-1}\, G\left[xt(1-yt^m)^{r-1/m}\right].$$

On the other hand by putting $\sigma = -2/m$ we shall get

$$(9)\quad \sum_{n=0}^{\infty} f_n^{c-2n/m}(x)t^n$$

$$=(1-4yt^m)^{-1/2}\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{c} G\left[xt\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{r-2/m}\right],$$

and by putting $\sigma = 1/m$ we shall get

$$(10)\quad \sum_{n=0}^{\infty} f_n^{c+n/m}(x)\, t^n$$

$$= (1+4yt^m)^{-1/2}\left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{1-c}$$

$$G\left[xt\left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{-r-1/m}\right].$$

Various other particular cases of (1) can be given by particularizing $f_n^c(x)$ and assigning different values to σ .

It is worth mentioning here that the particular case (8) of (1), when expressed in the form

$$(11) \quad \sum_{n=0}^{\infty} f_n^{b-n/m}(x)\,(1-yt^m)^{c-b}\,t^n = (1-yt^m)^{c-1}\, G\left[xt(1-yt^m)^{r-1/m}\right],$$

yields the expansion formula

$$(12) \quad f_n^{c-n/m}(x) = \sum_{k=0}^{[n/m]} \frac{(b-c)_k}{k!}\, y^k\, f_{n-mk}^{b+k-n/m}(x) ,$$

which is analogous to the similar consequence of 2.1(3) given by equation (7) above.

2.4 ANOTHER GENERATING FUNCTION FOR $f_n^{c+\sigma n}(x)$

In this section we prove the following generating relation :

$$(1) \quad \sum_{n=0}^{\infty} \frac{c-1}{c-1+(\sigma+1/m)n}\, f_n^{c+\sigma n}(x,y,1/m,m)t^n = u^{c-1}\Psi\left[xt\ u^{\sigma+1/m}\right],$$

where

$$(2) \quad u = 1 - yt^m\, u^{\sigma m+1}, \quad u(0) = 1,$$

and

$$(3) \quad \Psi(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1+(\sigma+1/m)k}\, \gamma_k z^k .$$

The above generating relation may also be expressed in the alternative form :

(4) $$\sum_{n=0}^{\infty} \frac{c - 1}{c - 1 + (\sigma+1/m)n} f_n^{c+\sigma n}(x,y,1/m,m)t^n$$

$$= (1 + y\eta^m)^{c-1} \Psi\left[(-1)^{1/m} x\eta\right]$$

where

(5) $$\eta = (-1)^{1/m} t \, (1+y\eta^m)^{\sigma+1/m}$$

and $\Psi(z)$ is given by (3).

To prove the relation (1) we first note that the explicit representation of $f_n^{c+\sigma n}(x)$ is given by

(6) $$f_n^{c+\sigma n}(x,y,1/m,m) = \sum_{k=0}^{[n/m]} \binom{-c-n/m-\sigma n+k}{k} \gamma_{n-mk} x^{n-mk} y^k ,$$

and therefore,

(7) $$\sum_{n=0}^{\infty} \frac{c - 1}{c-1+(\sigma+1/m)n} f_n^{c+\sigma n}(x,y,1/m,m)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{c - 1}{c-1+(\sigma+1/m)(n+mk)} \binom{c+\sigma n+\sigma mk+n/m+k-1}{k}$$

$$\gamma_n(xt)^n (-yt^m)^k ,$$

which by the following consequence of Lagrange's expansion formula (Pólya and Szegö[G], p.146, Problem 212)

(8) $$\sum_{n=0}^{\infty} \frac{a}{a + bn} \binom{a + bn}{n} t^n = z^a , \quad z = 1 + tz^b$$

yields (1).

The substitution $\sigma = -2/m$ transforms (1) to

$$(9)\quad \sum_{n=0}^{\infty} \frac{c-1}{c-1-n/m} f_n^{c-2n/m}(x,y,1/m,m)t^n = \left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{c-1} \Phi\left[xt\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{-1/m}\right]$$

where

$$(10)\quad \Phi(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1-k/m} \gamma_k z^k ,$$

whereas for $\sigma = -1/m$ it evidently reduces to 2.3(8) with $r = 1/m$, that is

$$(11)\quad \sum_{n=0}^{\infty} f_n^{c-n/m}(x)t^n = (1-yt^m)^{c-1} G[xt] .$$

On the other hand on putting $\sigma = 0$, we shall get

$$(12)\quad \sum_{n=0}^{\infty} \frac{c-1}{c-1+n/m} f_n^{c}(x,y,1/m,m)t^n = (1+yt^m)^{1-c} \theta\left(\frac{xt}{(1+yt^m)^{1/m}}\right) ,$$

where

$$(13)\quad \theta(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1+k/m} \gamma_k z^k .$$

And on putting $\sigma = 1/m$, (1) would reduce to

$$(14)\quad \sum_{n=0}^{\infty} \frac{c-1}{c-1+2n/m} f_n^{c+n/m}(x,y,1/m,m)t^n = \left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{1-c} \phi\left[xt\left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{-2/m}\right] ,$$

where

(15) $\phi(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1+2k/m} \gamma_k z^k .$

The relation (1) may also be stated in the form of following theorem :

Theorem 1:

Let $\{p_n(x)\}$ be the polynomials possessing the generating relation,

(16) $$\sum_{n=0}^{\infty} p_n(x)t^n = \frac{(1+yw^m)^{1-c}}{1+y(1+\sigma m)w^m} \quad G\left[\frac{xw}{(1+yw^m)^{1/m}}\right],$$

where w is given by 2.3(2) and $G[z]$ is given by 2.1(2), then

(17) $$\sum_{n=0}^{\infty} \frac{c-1}{c-1+(\sigma+1/m)n} p_n(x)t^n = u^{c-1} \Psi\left[xt\ u^{\sigma+1/m}\right],$$

where u and Ψ are given by (2) and (3) respectively.

When $\sigma = -2$, $c = 1-a$, $y = 1$ and $m = 1$, Theorem 1 would particularize to the known Theorem 1 given in (Mc Bride [F], p.85), whereas if we put $\sigma = 1$, $y = -1$, $m = 1$ and replace c by $1+a$, our Theorem 1 would correspond to its particular case given in (Mc Bride [F], Theorem 2,p.85).

Another worth mentioning particular case of our Theorem 1 is the known result

$$(18)\quad \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} L_n^{(\alpha+\beta n)}(x)t^n$$

$$=(1+\zeta)^{\alpha} {}_1F_1\left[\begin{matrix}\alpha/(\beta+1)\ ; \\ 1+\alpha/(\beta+1);\end{matrix} \quad -x\zeta\right],$$

where

$$(19)\qquad \zeta = t(1+\zeta)^{\beta+1}, \quad \zeta(0) = 0\ ;$$

derived earlier by Srivastava ([11], Eq. 2.4).