CHAPTER - V

A PARTICULAR CASE OF $f_n^c(x)$

5.1 INTRODUCTION

An interesting particular case of the class of polynomials $\{f_n^c(x)\}$ introduced in Chapter II corresponds to the choice $\gamma_n = \frac{1}{n!}$. This special class of polynomials; which we term as generalized Laguerre polynomials and denote by $\Gamma_n^c(x,y,r,m)$ or briefly by $\Gamma_n^c(x)$ when there is no ambiguity regarding other parameters, may be expressed explicitly in the form

$$(1) \qquad \Gamma_n^c(x,y,r,m) = \sum_{k=0}^{[n/m]} (-y)^k \frac{(c+rn-rmk)_k}{(n-mk)!\ k!} x^{n-mk} ,$$

where $m \geq 1$ is an integer and other parameters are unrestricted in general.

In view of the inverse series relation 3.2(7), it is easy to see that the inverse of (1) is given by the formula

$$(2) \qquad \frac{x^n}{n!} = \sum_{k=0}^{[n/m]} (-y)^k \frac{(1-c-rn)_{k-1}(rmk-c-rn)}{k!} \Gamma_{n-mk}^c(x) ,$$

As indicated by the name, the polynomials $\Gamma_n^c(x)$ provide a generalization of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$

$$(3) \qquad L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{n} \frac{(-n)_r}{(1+\alpha)_r} \frac{x^r}{r!},$$

and its extension studied by Chandel [1] .

Our interest in the class $\{\Gamma_n^c(x,y,r,m)\}$ stems from our attempt to give an extension of the expansion formula (Erdelyi [1])

$$(4) \quad L_{n_1}^{(\alpha_1)}(a_1x) \dots L_{n_p}^{(\alpha_p)}(a_px) = \sum_{s=0}^{n_1+\dots+n_p} B_s^{(n_1,\dots,n_p)} L_s^{(\beta)}(x)$$

with

$$(5) \quad B_s^{(n_1,\dots,n_p)} = \frac{(\alpha_1+1)_{n_1}}{n_1!} \frac{(\alpha_p+1)_{n_p}}{n_p!}$$

$$\cdot F_A(\beta+1,-n_1,\dots,-n_p,-s;\ \alpha_1+1,\dots,\alpha_p+1,\beta+1;a_1,\dots,a_p,1),$$

and the corresponding generating function for the coefficients $B_s^{(n_1,\dots,n_p)}$ given by Carlitz [4]

$$(6)\quad \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} B_s^{(n_1,\ldots,n_p)} u_1^{n_1} \cdots u_p^{n_p}$$

$$= \frac{(1-u_1)^{-\alpha_1-1} \cdots (1-u_p)^{-\alpha_p-1}}{\left(1+\frac{a_1u_1}{1-u_1}+\cdots+\frac{a_pu_p}{1-u_p}\right)^{\beta+s+1}} \left(\frac{a_1u_1}{1-u_1}+\cdots+\frac{a_pu_p}{1-u_p}\right)^{s},$$

where in (5) F_A stands for Lauricella's multiple hypergeometric function of $p+1$ variables.

During our investigations we found that an analogue of (4) and (6) in terms of the polynomials $f_n^c(x)$ can be obtained if we particularize γ_n to $1/n!$ and thus we are led to the polynomials $\{\Gamma_n^c(x)\}$. In section 5.2 of this chapter we obtain an expansion formula for the product

$$\Gamma_{n_1}^{c_1}(a_1x,\ y_1,\ r_1, m) \ \cdots\ \Gamma_{n_p}^{c_p}(a_px,\ y_p,\ r_p,\ m)$$

in terms of $\Gamma_s^c(x,y,r,m)$ and derive the generating relation for the connecting coefficients $C_s^{(n_1,\ldots,n_p)}$ which occur in the above mentioned expansion.

Section 5.3 deals with certain recurrence relations and other miscellaneous results associated with $\Gamma_n^c(x)$.

For the sake of brevity we shall use the following notations in the subsequent sections of this Chapter.

$$
(7)\quad \begin{cases}
n_1 + n_2 + \dots + n_p = N \\
k_1 + k_2 + \dots + k_p = K \\
[n_i/m] = n_i^*,\ i = 1,2,\dots,p;\ [n/m] = n^*,\ [N/m] = N^* \\
W = \left[\dfrac{a_1\ u_1}{(1+y_1 u_1^m)^{r_1}} + \dots + \dfrac{a_p\ u_p}{(1+y_p u_p^m)^{r_p}} \right]
\end{cases}
$$

5.2 THE EXPANSION FORMULA

We first observe that 5.1(1) and 5.1(2), when combined, leads us to

$$
\Gamma_{n_1}^{c_1}(a_1x, y_1, r_1, m) \dots \Gamma_{n_p}^{c_p}(a_px, y_p, r_p, m)
$$

$$
= \sum_{k_1=0}^{n_1^*} \dots \sum_{k_p=0}^{n_p^*} \left[\prod_{j=1}^{p} \frac{(c_j + r_j n_j - r_j m k_j)_{k_j}\, a_j^{n_j - mk_j} (-y_j)^{k_j}}{k_j!\,(n_j - mk_j)!} \right]
$$

$$
\cdot \sum_{s=0}^{N^*-K} \frac{(N-mK)!\,(-y)^s\,(1-c-rN+rmK)_{s-1}\,(rms-c-rN+rmK)}{s!}
$$

$$
\cdot\, \Gamma_{N-mK-ms}^{c}(x),
$$

which in view of the easily derivable relation

$$(1) \quad \sum_{k=0}^{r} \sum_{s=0}^{r-k} \frac{A(r,s,k)}{s!} = \sum_{s=0}^{r} \sum_{k=0}^{r} \frac{A(r,r-k-s,k)}{(r-k-s)!}$$

gives us

$$(2) \quad \Gamma_{n_1}^{c_1}(a_1x, y_1, r_1, m) \cdots \Gamma_{n_p}^{c_p}(a_px, y_p, r_p, m)$$

$$= \sum_{s=0}^{N^*} C_s^{(n_1,\ldots,n_p)} \Gamma_{N+ms-mN^*}^{c}(x),$$

where the coefficients $C^{(n_1,\ldots,n_p)}$ are given by

$$(3) \quad C_s^{(n_1,\ldots,n_p)} = \sum_{k_1=0}^{n_1^*} \cdots \sum_{k_p=0}^{n_p^*} \left[\prod_{j=1}^{p} \frac{(c_j+r_jn_j-r_jmk_j)_{k_j}\, a_j^{n_j-mk_j} (-y_j)^{k_j}}{k_j!(n_j-mk_j)!} \right]$$

$$\cdot (-y)^{N^*-K-s} \frac{(N-mK)!(1-c-rN+rmK)_{N^*-K-s-1}(-c-rms-rN+rmN^*)}{(N^*-K-s)!}.$$

Hence

$$(4) \quad \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} C_s^{(n_1,\ldots,n_p)} u_1^{n_1} \cdots u_p^{n_p} = (1+y_1 u_1^m)^{-c_1} \cdots$$

$$(1+y_p u_p^m)^{-c_p} \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \frac{N!\; y^{N^*-s}(c+rms+rN-rmN^*)(1-c-rN)_{N^*-s-1}}{n_1! \cdots n_p!\; (N^* - s)!}$$

$$\cdot\ (-1)^{N^*-s-1} \left(\frac{a_1\ u_1}{(1+y_1 u_1^m)^{r_1}}\right)^{n_1} \cdots \left(\frac{a_p\ u_p}{(1+y_p u_p^m)^{r_p}}\right)^{n_p}$$

$$= (1+y_1 u_1^m)^{-c_1} \cdots (1+y_p\ u_p^m)^{-c_p}$$

$$\cdot \sum_{n=ms}^{\infty} \frac{(1-c-rn)_{n^*-s-1}(c+rms+rn-rmn^*)}{(n^* - s)!}\, y^{n^*-s}(-1)^{n^*-s-1}\, w^n.$$

Now since

$$(5) \quad \sum_{n=ms}^{\infty} A(n,n^*) = \sum_{n=s}^{\infty} \sum_{j=0}^{m-1} A(mn+j,n)$$

$$= \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} A(mn + ms + j, n + s),$$

the right hand side of the above expression may be put in the form

$$(1+y_1u_1^m)^{-c_1} \ldots (1+y_pu_p^m)^{-c_p} W^{ms}$$

$$\cdot \sum_{j=0}^{m-1} W^j \sum_{n=0}^{\infty} \frac{c+rms+rj}{c+rms+rj+rmn} \binom{c+rms+rj+rmn}{n} (yW^m)^n .$$

The inner series of the last expression can be summed up with the help of 2.4(8), as a result of which, we finally get the generating relation

$$(6) \quad \sum_{n_1=0}^{\infty} \ldots \sum_{n_p=0}^{\infty} C_s^{(n_1,\ldots,n_p)} u_1^{n_1} \ldots u_p^{n_p}$$

$$=(1+y_1u_1^m)^{-c_1} \ldots (1+y_pu_p^m)^{-c_p} W^{ms} \eta^{c+rms} \frac{1-W^m\eta^{rm}}{1-W\eta^r} ,$$

where W is as given in 5.1(7) and

$$(7) \quad \eta = 1 + y\eta^{rm} W^m .$$

When $m = 1$ the expansion (2) would reduce to

$$(8) \quad \Gamma_{n_1}^{c_1}(a_1x,y_1,r_1,1) \ldots \Gamma_{n_p}^{c_p}(a_px,y_p,r_p,1)$$

$$= \sum_{s=0}^{N} D_s^{(n_1,\ldots,n_p)} \Gamma_s^{c}(x,y,r,1) ,$$

where the coefficients $D_s^{(n_1,\ldots,n_p)}$ are given by

(9) $$D_s^{(n_1,\ldots,n_p)} = \sum_{k_1=0}^{n_1} \cdots \sum_{k_p=0}^{n_p} \left[\prod_{j=1}^{p} \frac{(c_j+r_jk_j)_{n_j-k_j}\, a_j^{k_j}\, (-y_j)^{n_j-k_j}}{k_j!(n_j-k_j)!} \right]$$

$$\cdot \frac{K!(-y)^{K-s}(-c-rs)(1-c-rK)_{K-s-1}}{(K-s)!},$$

and the generating relation (6) would simplify to

(10) $$\sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} D_s^{(n_1,\ldots,n_p)} u_1^{n_1} \cdots u_p^{n_p}$$

$$= (1+y_1u_1)^{-c_1} \cdots (1+y_pu_p)^{-c_p}\, U^s\, V^{c+rs},$$

where U and V are defined by

(11) $$\begin{cases} U = \left[\dfrac{a_1u_1}{(1+y_1u_1)^{r_1}} + \cdots + \dfrac{a_p\,u_p}{(1+y_pu_p)^{r_p}} \right], \\ V = 1 + y\,U\,V^r. \end{cases}$$

In view of the relation

(12) $$L_n^{(\alpha)}(x) = \Gamma_n^{\alpha+1}(-x,-1,1,1),$$

the expansion formula (8) and the accompanying generating relation (10) obviously provide extensions of 5.1(4) and 5.1(6) to which they would reduce when the various parameters involved therein are particularized in accordance with (12).

5.3 ADDITIONAL RESULTS FOR $\Gamma_n^c(x,y,r,m)$

Choosing $G[z] = \exp(z)$, the generating relation 2.1(3) becomes

$$(1)\quad (1+yt^m)^{-c}\exp\left[\frac{xt}{(1+yt^m)^r}\right] = \sum_{n=0}^{\infty}\Gamma_n^c(x)t^n.$$

Differentiating (1) k times with respect to x we get

$$(2)\quad \sum_{n=k}^{\infty} D_x^k\left\{\Gamma_n^c(x)\right\}t^n$$

$$= (1+yt^m)^{-c}\exp\left[\frac{xt}{(1+yt^m)^r}\right]\frac{t^k}{(1+yt^m)^{rk}}.$$

The right hand member of (2) when viewed through (1), helps us put (2) in the form

$$\sum_{n=k}^{\infty} D_x^k\left\{\Gamma_n^c(x)\right\}t^n = \sum_{n=0}^{\infty}\Gamma_n^{c+rk}(x)t^{n+k}$$

$$= \sum_{n=k}^{\infty}\Gamma_{n-k}^{c+rk}(x)t^n,$$

in which the comparison of t^n yields

$$(3) \quad D_x^k\left\{\Gamma_n^c(x)\right\} = \Gamma_{n-k}^{c+rk}(x), \; n \geq k \, .$$

On the other hand if we replace r by $r + s$ in (2) we shall get

$$(4) \quad \sum_{n=k}^{\infty} D_x^k \left\{\Gamma_n^c(x,y,r+s,m)\right\} t^n$$

$$= (1+yt^m)^{-c} \exp\left[\frac{xt}{(1+yt^m)^{r+s}}\right] \frac{t^k}{(1+yt^m)^{(r+s)k}} \, ,$$

or equivalently

$$(5) \quad = \sum_{n=0}^{\infty} \Gamma_n^{c+sk}(x,y,r+s,m) \frac{t^{n+k}}{(1+yt^m)^{rk}}$$

$$= \sum_{n=k}^{\infty} \sum_{j=0}^{[\overline{n-k}/m]} \Gamma_{n-k-mj}^{c+sk}(x,y,r+s,m) \frac{(rk)_j}{j!}(-y)^j \, t^n \, ,$$

which yields the interesting relation

$$(6) \quad D_x^k \left\{\Gamma_n^c(x,y,r+s,m)\right\}$$

$$= \sum_{j=0}^{[\overline{n-k}/m]} \frac{(rk)_j}{j!} (-y)^j \; \Gamma_{n-k-mj}^{c+sk}(x,y,r+s,m) \, .$$

For $k = 1$, (6) reduces to

$$(7)\quad D_x \left\{ \Gamma_n^c(x,y,r+s,m) \right\} = \sum_{j=0}^{[\overline{n-1}/m]} \frac{(r)_j \; (-y)^j}{j!} \; \Gamma_{n-1-mj}^{c+s}(x,y,r+s,m),$$

whereas, for $s=0$, (6) yields

$$(8)\quad D_x^k \left\{ \Gamma_n^c(x) \right\} = \sum_{j=0}^{[\overline{n-k}/m]} \frac{(rk)_j}{j!} (-y)^j \; \Gamma_{n-k-mj}^c(x).$$

Next, we observe that (5) can also be put in the alternative form

$$(1+yt^m)^{rk} \sum_{n=k}^{\infty} D_x^k \left\{ \Gamma_n^c (x,y,r+s,m) \right\} t^n$$

$$= \sum_{n=0}^{\infty} \Gamma_n^{c+sk}(x,y,r+s,m) t^{n+k} ,$$

or

$$\sum_{n=k}^{\infty} \sum_{j=0}^{[\overline{n-k}/m]} D_x^k \left\{ \Gamma_{n-mj}^c (x,y,r+s,m) \right\} \frac{(-rk)_j}{j!} (-y)^j \; t^n$$

$$= \sum_{n=k}^{\infty} \Gamma_{n-k}^{c+sk}(x,y,r+s,m) t^n ,$$

which leads to

$$(9)\quad \Gamma_{n-k}^{c+sk}(x,y,r+s,m)$$

$$= \sum_{j=0}^{[n-k/m]} \frac{(-rk)_j\ (-y)^j}{j!}\ D_x^k \left\{ \Gamma_{n-mj}^{c}(x,y,r+s,m) \right\},\ n \geq k .$$

For k = 1, (9) particularizes to

$$(10)\quad \Gamma_{n-1}^{c+s}(x,y,r+s,m)$$

$$= \sum_{j=0}^{[n-1/m]} \frac{(-r)_j}{j!}\ (-y)^j\ D_x \left\{ \Gamma_{n-mj}^{c}(x,y,r+s,m) \right\}.$$

Further, the generating relation (1) gives us

$$(11)\quad \sum_{n=0}^{\infty} \Gamma_n^{c_1+c_2+\dots+c_k}(x_1+\dots+x_k)t^n$$

$$=(1+yt^m)^{-(c_1+\dots+c_k)}\ \exp\left[\frac{(x_1+\dots+x_k)t}{(1+yt^m)^r}\right].$$

The right hand side of (11) may be expressed in the form

$$\sum_{n=0}^{\infty} \sum_{n_2=0}^{n} \sum_{n_3=0}^{n_2} \dots \sum_{n_k=0}^{n_{k-1}} \Gamma_{n-n_2}^{c_1}(x_1) \dots \Gamma_{n_{k-1}-n_k}^{c_{k-1}}(x_{k-1})\ \Gamma_{n_k}^{c_k}(x_k)t^n ,$$

as a result of which we get

(12) $$\Gamma_n^{c_1+\ldots+c_k}(x_1+\ldots+x_k) = \sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{c_1}(x_1)\ldots\Gamma_{i_k}^{c_k}(x_k).$$

In particular if $c_i = c$, $x_i = x$, $i = 1,\ldots,k$

(12) simplifies to

(13) $$\Gamma_n^{ck}(kx) = \sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{c}(x)\ldots\Gamma_{i_p}^{c}(x).$$

Similarly, the following particular case of the generating relation 2.3(8)

(14) $$\sum_{n=0}^{\infty} \Gamma_n^{c-n/m}(x)t^n = (1-yt^m)^{c-1} \exp\left[xt(1-yt^m)^{r-1/m}\right]$$

leads us to

(15) $$\Gamma_n^{b_1+\ldots+b_k+1-k-n/m}(x_1+\ldots+x_k)$$

$$= \sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{b_1-i_1/m}(x_1)\ldots\Gamma_{i_k}^{b_k-i_k/m}(x_k),$$

which on putting $k=2$, $b_1=b_2$, $x_1=x_2$ and replacing n by 2n gives

(16) $$\Gamma_{2n}^{2b-1-2n/m}(2x) = \sum_{i_1+i_2=2n} \Gamma_{i_1}^{b_1-i_1/m}(x)\Gamma_{i_2}^{b_2-i_2/m}(x),$$

and now if we replace b by b+n/m in (16), we shall get

$$(17)\quad \Gamma_{2n}^{2b-1}(2x) = \sum_{s=-n}^{n} \Gamma_{n-s}^{b+s/m}(x)\quad \Gamma_{n+s}^{b-s/m}(x).$$

Other recurrence relations that follow from (1) through its differentiated form (with respect to t) are

$$(18)\quad n\,\Gamma_{n}^{c}(x) + cmy\,\Gamma_{n-m}^{c+1}(x) - x\,\Gamma_{n-1}^{c+r}(x)$$

$$+xymr\,\Gamma_{n-m-1}^{c+r+1}(x) = 0\ ,\quad n \geq m \geq 1.$$

$$(19)\quad \begin{cases} x\,\Gamma_{n}^{c}(x) - (n+1)\,\Gamma_{n+1}^{c}(x) = 0\ , & n=0,1,\ldots,m-2 \\ x\,\Gamma_{m-1}^{c}(x) - m\Gamma_{m}^{c}(x) - cmy\,\Gamma_{o}^{c}(x) = 0\ , & \\ \Gamma_{o}^{c}(x) = 1. & \end{cases}$$

$$(20)\quad (n+1)\,\Gamma_{n+1}^{c}(x) + y(n-m+cm+1)\,\Gamma_{n-m+1}^{c}(x)$$

$$= x\sum_{k=0}^{[n/m]} \frac{(r)_k}{k!}(-y)^k\,\Gamma_{n-mk}^{c}(x) + (1-rm)xy$$

$$\cdot \sum_{k=0}^{[n/m]-1} \frac{(r)_k}{k!}(-y)^k\,\Gamma_{n-mk-m}^{c}(x),\ n \geq m.$$

Lastly, we observe that the generating relation (1) when interpreted in the light of Taylor's theorem, leads us to

$$(21) \quad \Gamma_n^c(x) = \frac{1}{n!} D_t^n \left\{ (1+yt^m)^{-c} \exp\left[\frac{xt}{(1+yt^m)^r} \right] \right\} \Bigg|_{t=0}$$

and therefore,

$$(22) \quad \Gamma_n^{c+\sigma n}(x+nz) = \frac{1}{n!} D_t^n \left[\left\{ (1+yt^m)^{-c} \exp\left[\frac{xt}{(1+yt^m)^r} \right] \right\} \left\{ (1+yt^m)^{-\sigma} \exp\left[\frac{zt}{(1+yt^m)^r} \right] \right\}^n \right] \Bigg|_{t=0}$$

so that,

$$(23) \quad \sum_{n=0}^{\infty} \Gamma_n^{c+\sigma n}(x+nz) t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[D_t^n \ f(t) \left\{ \phi(t) \right\}^n \right] \Bigg|_{t=0} ,$$

where for convenience,

$$f(t) = (1+yt^m)^{-c} \exp\left[\frac{xt}{(1+yt^m)^r} \right] ,$$

$$\phi(t) = (1+yt^m)^{-\sigma} \exp\left[\frac{zt}{(1+yt^m)^r} \right] .$$

Use of Lagrange's expansion theorem (Pólya and Szegö[G], p.146, Problem 207)

(24) $$\frac{f(\xi)}{1-t\,\phi'(\xi)} = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} D_a^n \left\{ f(a) \left[\phi(a)\right]^n \right\}$$

with $\xi = a + t\,\phi(\xi)$,

readily converts (23) into the mixed generating relation

(25) $$\sum_{n=0}^{\infty} \Gamma_n^{c+\sigma n} (x + nz)t^n$$

$$= \frac{v^{-c} \exp(xu/v^r)}{1 - u\left[-\sigma myu^{m-1} v^{-1} + zv^{-r}\left\{1-mryu^m v^{-1}\right\}\right]}$$

where u and v are given by

(26) $$u = tv^{-\sigma} \exp(uz/v^r) , \quad v = (1 + yu^m).$$