CHAPTER II

SOME THEOREMS ON THE ABSOLUTE CONVERGENCE OF A FOURIER SERIES

1. It has been established by A. Zygmund[1)] that if $f \in \mathrm{Lip}\,\alpha(0,2\pi)$, $\alpha > 0$, and f is of bounded variation, then

$$(1) \qquad \sum_{n=1}^{\infty}(|a_n|^{\beta} + |b_n|^{\beta})$$

is convergent for $\beta > 2/(\alpha+2)$, where a_n, b_n are the Fourier coefficients of the function f. The conclusion (1) is not necessarily true when $\beta = 2/(\alpha+2)$.

A natural question that arises in this connection is as to whether the series (1), corresponding to the function f, can be made convergent for $\beta = 2/(\alpha+2)$ by imposing some stronger conditions on the function f. We prove a theorem giving a sufficient condition for the convergence of the series (1) when $\beta = 2/(\alpha+2)$. Our theorem is:

<u>Theorem 1.</u> <u>If f is of bounded variation and satisfies the condition:</u>

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1) Zygmund [46] Marcinkiewicz [42]

$$|f(x+h)-f(x)| \leq \frac{c\, h^{\alpha}}{\left[l_1(h)l_2(h)\ldots l_k^{1+\varepsilon}(h)\right]^{\alpha+2}}$$

<u>where</u> $\varepsilon > 0$, $\alpha \geq 0$ <u>and</u>

$$l_1(h) = \log(e+h^{-1}) ,$$

$$l_2(h) = \log\log(e^{e}+h^{-1}) , \text{ etc.},$$

<u>then the series</u> (1) <u>is convergent for</u> $\beta = 2/(\alpha+2)$.

Analogous extensions of the classical theorem I.1 of S. Bernstein[1] and of its generalization viz. Theorem I.6 due to O. Szàsz[2] were given respectively by L. Neder[3] and A. C. Zaanen[4].

If we take $\alpha = 0$ and $k = 1$ in Theorem 1 it reduces to the more general form of Zygmund's theorem I.8[5].

We have also studied the convergence of the series(1) under different conditions. We prove the following theorem:

<u>Theorem 2.</u> <u>Let</u> $f(x)$ <u>be continuous and periodic in</u> $(0,2\pi)$.
<u>If</u>

(i) $f(x) \in \mathrm{Lip}\,\alpha\ (0\ ,\ 2\pi),\ 0 < \alpha \leq 1$,

(ii) a_n <u>and</u> b_n <u>are positive,</u>

<u>then the series</u> (1) <u>converges for</u> $\beta > 1/(1+\alpha)$.

As a corollary of Theorem 2 we have the following:

<u>Theorem 3.</u> <u>Let</u> $g(x)$ <u>be a periodic and continuous</u>

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1) Bernstein [1] 2) Szàsz [34] 3) Neder [32]

4) Zaanen [44] 5) Zygmund [45]

function in $(0\,,\,2\pi)$ *and*

$$g(x) \sim (a_0'/2) + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx)\,,$$

and denote by $\omega_1(\delta)$ *the modulus of continuity of* g.
If (i) $\omega_1(\delta) \leq \omega(\delta)$, $a_n' \geq - a_n$, $b_n' \geq - b_n$,

(ii) $f(x)$ *satisfies the conditions of Theorem 2,*

then $\sum_{n=1}^{\infty}(|a_n'|^{\beta} + |b_n'|^{\beta})$

converges for $\beta > 1/(1+\alpha)$.

Proof of theorem 1. We shall prove the theorem for $k = 2$.
We have

$$f(x+h) - f(x-h) \sim 2 \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \sin nh.$$

Therefore

$$(2) \qquad (1/\pi) \int_0^{2\pi} \{f(x+h) - f(x-h)\}^2 dx = 4 \sum_{n=1}^{\infty} \rho_n^2 \sin^2 nh\,,$$

where $\rho_n^2 = a_n^2 + b_n^2$.

Now

$$(1/\pi) \int_0^{2\pi} \{f(x+h) - f(x-h)\}^2 dx \leq \frac{1}{\pi}\, \omega(h) \int_0^{2\pi} |f(x+h) - f(x-h)|\, dx$$

where ω denotes the modulus of continuity of f.
Putting $h = \pi/(2N)$, where N is a positive integer, we get

$$(3) \qquad (1/\pi) \int_0^{2\pi} f(x+\tfrac{\pi}{2N}) - f(x-\tfrac{\pi}{2N})\ {}^2\, dx$$

$$\leq \frac{1}{\pi}\, \omega(\pi/N) \int_0^{2\pi} |f(x+\tfrac{\pi}{2N}) - f(x-\tfrac{\pi}{2N})|\, dx\,,$$

and

$$\int_0^{2\pi} \left|f(x+\frac{\pi}{2N}) - f(x-\frac{\pi}{2N})\right| dx$$

$$= \int_0^{2\pi} \left|f(x+\frac{\pi}{N}) - f(x)\right| dx$$

$$\leq \sum_{n=0}^{2N-1} \int_{n\pi/N}^{(n+1)\pi/N} \left|f(x+\frac{\pi}{N}) - f(x)\right| dx$$

$$\leq \int_0^{\pi/N} \sum_{n=0}^{2N-1} \left|f(x+\frac{n+1}{N}\pi) - f(x+\frac{n}{N}\pi)\right| dx .$$

Since

$$\sum_{n=0}^{2N-1} \left|f(x+\frac{\pi(n+1)}{N}) - f(x+\frac{n\pi}{N})\right| \leq V ,$$

where V is the total variation of f in $(0, 2\pi)$, we have

$$\int_0^{2\pi} \left|f(x+\frac{\pi}{2N}) - f(x-\frac{\pi}{2N})\right| dx \leq \frac{\pi V}{N} .$$

Therefore it follows from (2) and (3) that

$$\sum_{n=1}^{\infty} \rho_n^2 \sin^2 (n\pi/2N) \leq \omega(\pi/N) VN^{-1} .$$

Hence

$$\sum_{n=1}^{N} \rho_n^2 \sin^2 (n\pi/2N) \leq \omega(\pi/N) V N^{-1} .$$

Putting $N = 2^m$, where m is an integer greater or equal to m_0, where $m_0 \geq (\log 2)^{-2}+3$ and taking into account only the terms with indices n exceeding $N/2$, we get from the last inequality

$$(4) \qquad \sum_{n=2^{m-1}+1}^{2^m} P_n^2 \sin^2 (n\pi/2^{m+1}) \leq \omega(\pi/2^m)\, V\, 2^{-m} .$$

Since $\sin (n\pi/2^{m+1}) > 1/\sqrt{2}$ for $2^{m-1}+1 \leq n \leq 2^m$,

we obtain

$$\sum_{n=2^{m-1}+1}^{2^m} P_n^2 \leq \omega(\pi/2^m)\, V\, 2^{-m+1} .$$

Hence by the second condition of the Theorem 1,

$$\sum_{n=2^{m-1}+1}^{2^m} P_n^2 \leq \frac{A\, 2^{-m}\, 2^{-m\alpha}}{\left[l_1(\pi/2^m) l_2^{1+\varepsilon}(\pi/2^m)\right]^{\alpha+2}}$$

$$\leq \frac{A\, 2^{-m(1+\alpha)}}{\left[l_1(\pi/2^m) l_2^{1+\varepsilon}(\pi/2^m)\right]^{\alpha+2}}$$

Now

$$\begin{aligned} l_1(\pi/2^m) &= \log (e + 2^m \pi^{-1}) \\ &> \log 2^{m-2} \\ &= (m-2) \log 2, \end{aligned}$$

$$\begin{aligned} l_2(\pi/2^m) &= \log \log (e^e + 2^m \pi^{-1}) \\ &> \log \log 2^{m-2} \\ &> \tfrac{1}{2} \log (m-2) . \end{aligned}$$

Therefore

$$\sum_{n=2^{m-1}+1}^{2^m} p_n^2 \leq \frac{A\, 2^{-m(1+\alpha)}}{\left[(m-2)\log^{1+\varepsilon}(m-2)\right]^{\alpha+2}} .$$

Applying Holder's inequality, we get

$$\sum_{2^{m-1}+1}^{2^m} p_n^\beta \leq \left(\sum_{2^{m-1}+1}^{2^m} p_n^2\right)^{\beta/2}\left(\sum_{2^{m-1}+1}^{2^m} 1\right)^{1-\frac{\beta}{2}}$$

$$\leq \frac{A\, 2^{-m(1+\alpha)\beta/2}\; 2^{m(1-\beta/2)}}{\left[(m-2)\log^{1+\varepsilon}(m-2)\right]^{(\alpha+2)\beta/2}}$$

$$= \frac{A}{(m-2)\log^{1+\varepsilon}(m-2)} ,$$

for $\beta = 2/(\alpha+2)$.

Therefore

$$\sum_{n=1}^{\infty} p_n^\beta = \sum_{m=m_0}^{\infty}\sum_{n=2^{m-1}+1}^{2^m} p_n^\beta$$

$$\leq \sum_{m=m_0}^{\infty} \frac{A}{(m-2)\log^{1+\varepsilon}(m-2)}$$

$$< \infty .$$

Since $|a_n|^\beta$, as also $|b_n|^\beta$, does not exceed p_n^β , it follows that

$$\sum (|a_n|^\beta + |b_n|^\beta) < \infty .$$

This completes the proof of the theorem.

Proof of Theorem 2. Denoting by σ_n, the arithmetic mean of order n of the partial sums of the Fourier series of f, we have

$$\sigma_n(x+h)-\sigma_n(x-h) = 2\sum_{k=1}^{n} \left(1-\frac{k}{n+1}\right)(-a_k \sin kx + b_k \cos kx)\sin kh.$$

Let us write

$$T_n(x) = (1/2) + \sum_{k=1}^{n} \left(1-\frac{k}{n+1}\right)\cos kx .$$

It is well known that

(5) $T_n(x) \geq 0$ for all $x \in (0, 2\pi)$.

Consider the identity

$$S_n(h) \equiv \int_0^{2\pi} \{\sigma_n(x+h)-\sigma_n(x-h)\} T_n(x)\, dx$$

(6) $$= 2\int_0^{2\pi}\left[\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)(-a_k \sin kx + b_k \cos kx)\sin kh\right]\left[\frac{1}{2}+\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\cos kx\right]dx$$

$$= 2\pi \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)^2 b_k \sin kh .$$

Taking into account the inequality[1)]

$$\omega(\delta, \sigma_n) \leq \omega(\delta, f),$$

which is an immediate consequence of the representation

$$\sigma_n(x) = (1/\pi)\int_0^{2\pi} T_n(x-t) f(t)\, dt ,$$

It follows from (5) and (6) that

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1) Zygmund [48, p. 149]

(7) $\quad | S_n(h)| \leq M\, \omega(h\ ,\ f)$

where $M > 0$ is independent of n and h.
Taking $m = \left[\frac{n}{2}\right]$, where $[x]$ denotes the largest integer $\leq x$, we have

(8) $$\frac{\pi}{4m} \sum_{k=1}^{m} k\, b_k \leq \frac{2\pi}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)^2 k\, b_k\ .$$

Since

$$\sin kh \geq 2\, kh/\pi\ ,\ h = \pi/2n\ ,\ 1 \leq k \leq n\ ,$$

it follows by virtue of (6) and (8) that

(9) $$|S_n(\pi/2n)| \geq \frac{2\pi}{n} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)^2 k\, b_k \geq (\pi/4m) \sum_{k=1}^{m} k\, b_k\ .$$

Applying Hölder's inequality, we get

$$\sum_{k=1}^{m} k\, b_k^{\beta} = \sum_{k=1}^{m} (k^{1-\beta})\, (k^{\beta}\, b_k^{\beta})$$

$$\leq \left(\sum_{k=1}^{m} (k^{1-\beta})^{\frac{1}{1-\beta}}\right)^{(1-\beta)} \left(\sum_{k=1}^{m} (k^{\beta}\, b_k^{\beta})^{\beta^{-1}}\right)^{\beta}$$

$$\leq \left(\sum_{k=1}^{m} k\right)^{1-\beta} \left(\sum_{k=1}^{m} k\, b_k\right)^{\beta} .$$

Therefore, it follows from (7) and (9) that

$$\sum_{k=1}^{m} k\, b_k^{\beta} \leq \left(\sum_{k=1}^{m} k\right)^{1-\beta} \left(A\, m\, \omega\left(\frac{\pi}{2n}\ ,\ f\right)\right)^{\beta}$$

$$= A'\, m^{2(1-\beta)}\, m^{\beta} \left(\omega(\pi/2n)\right)^{\beta} .$$

Hence we have

$$(10)\quad B_m = (1/m) \sum_{k=1}^{m} k\, b_k^{\beta} \le A' \, m^{1-\beta} \left(\omega(\pi/2m) \right)^{\beta}.$$

But

$$m B_m - (m-1) B_{m-1} = m\, b_m^{\beta}$$

$$b_m^{\beta} = B_m - B_{m-1} + (B_{m-1}/m) ,$$

since $b_k \ge 0$,

$$b_m^{\beta} \le B_m - B_{m-1} + \frac{B_{m-1}}{m-1} .$$

Therefore

$$\sum_{k=2}^{m} b_k^{\beta} \le \sum_{k=2}^{m} B_k - \sum_{k=2}^{m} B_{k-1} + \sum_{k=2}^{m} \frac{B_{k-1}}{k-1}$$

$$\le B_m + \sum_{k=1}^{m-1} \frac{B_k}{k} .$$

Using (10) and the fact that

$$\left(\omega(\pi/2n , f)\right)^{\beta} \le A\, n^{-\alpha\beta}$$

we have

$$\sum_{k=2}^{m} b_k^{\beta} \le \frac{c}{m^{\beta-1}\, m^{\alpha\beta}} + c' \sum_{k=1}^{m-1} \frac{k^{1-\beta}}{k^{\alpha\beta+1}} .$$

Hence we deduce

$$\lim_{m\to\infty} \sum_{k=2}^{m} b_k^{\beta} \le \lim_{m\to\infty} \frac{c}{m^{\alpha\beta+\beta-1}} + \lim_{m\to\infty} \sum_{k=1}^{m-1} \frac{1}{k^{\alpha\beta+\beta}} .$$

But

$$\lim_{m \to \infty} \frac{1}{n^{(\beta+\alpha\cdot\beta-1)}} = 0 \text{ for } (\alpha\beta+\beta-1) > 0 .$$

Therefore

$$\sum_{k=2}^{\infty} b_k^{\beta} \leq c' \sum_{k=2}^{\infty} \frac{1}{k^{\beta(1+\alpha)}} ,$$

$$< \infty, \text{ for } \beta > 1/(1+\alpha) .$$

2. G. H. Hardy[1] proved a theorem which is connected with the chain of ideas of Bernstein[2] and Zygmund[3]. His theorem is :

<u>Theorem A.</u> <u>If</u> $f \in \operatorname{Lip} \alpha$, $0 < \alpha \leq 1$, <u>then</u>

$$(11) \qquad \sum_{n=1}^{\infty} n^{(\beta-\frac{1}{2})}(|a_n| + |b_n|) < \infty,$$

<u>for every</u> $\beta < \alpha$.

<u>Theorem B.</u> <u>If</u> f <u>is of bounded variation and</u> $f \in \operatorname{Lip} \alpha$, $0 < \alpha \leq 1$, <u>then</u>

$$(12) \qquad \sum_{n=1}^{\infty} n^{\beta/2}(|a_n| + |b_n|) < \infty, \quad \text{for } \beta < \alpha .$$

For $\beta = \alpha$, the conclusions (11) and (12) may not be ture.

A. C. Zaanen[4] extended Theorem A as follows:

<u>Theorem C.</u> If for certain $\varepsilon > 0$,

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1) Hardy [16] 2). Bernstein [1] 3) Zygmund [45]
4) Zaanen [44]

$$\omega(\delta) \leq \frac{c\,\delta^{\alpha}}{l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)} ,$$

then the series (11) *converges for* $\beta = \alpha$.

We prove a theorem which is an extension of Theorem B.

Theorem 4. *If* f *is of bounded variation and satisfies the condition*

$$(13) \qquad \omega(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta) l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^2}$$

then the series (12) *converges for* $\beta = \alpha$.

We have also considered the conditions under which the conclusion (12) holds for values of $\beta > \alpha$. Our theorems are :

Theorem 5. *Let* $f(x) \sim (a_0/2) + \sum_{n=1}^{\infty} a_n \cos nx$.

If (i) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$,

(ii) $a_n \geq \dfrac{-\omega(1/n)\log n}{n}$, $n \geq 1$,

(iii) $\omega(1/x)$ *is slowly increasing, then*

$$\sum_{n=1}^{\infty} n^{\beta/2} (\,|\,a_n\,|\,) < \infty, \text{ for } \beta < 2\alpha .$$

We have an analogous result for sine series.

Theorem 6. *Let* $f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$.

If (i) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$,

(ii) $\omega(1/x)$ *is slowly increasing,*

(iii) $b_n \geq \dfrac{-\omega(1/n)}{n}$, $n \geq 1$, *then*

$$\sum_{n=1}^{\infty} n^{\beta/2} (|b_n|) < \infty, \ \text{for} \ \beta < 2\alpha .$$

Theorem 7. *If* $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$, *and* a_n , b_n *are positive, then the series* (12) *converges for* $\beta < 2\alpha$.

We also have an extension of Theorem 7 which is analogous to the extension of Theorem 2 given by Theorem 3.

Theorem 8. *Under the hypothesis of Theorem 3,*

$$\sum_{n=1}^{\infty} n^{\beta/2}(|a_n'| + |b_n'|) < \infty \ , \ \text{for} \ \beta < 2\alpha.$$

Proof of the Theorem 4. We shall prove the theorem for $k = 2$. As f is a function of bounded variation, it follows from (4) that

$$\sum_{n=2^{m-1}+1}^{2^m} p_n^2 \leq c\, \omega(\pi/2^m)\, 2^{-m} ;$$

hence by the condition (13) we get

$$\sum_{n=2^{m-1}+1}^{2^m} p_n^2 \leq \frac{c\, 2^{-m(1+\alpha)}}{\left[l_1(\pi/2^m)\, l_2^{1+\varepsilon}(\pi/2^m) \right]^2}$$

$$\leq \frac{c\, 2^{-m(1+\alpha)}}{\left[(m-2) \log^{1+\varepsilon}(m-2)\right]^2} .$$

Applying Schwarz's inequality we obtain

$$\sum_{n=2^{m-1}+1}^{2^m} P_n \leq \Big(\sum_{n=2^{m-1}+1}^{2^m} P_n^2 \Big)^{1/2} 2^{m/2}$$

$$\leq \frac{c\, 2^{-m\alpha/2}}{(m-2) \log^{1+\varepsilon}(m-2)} .$$

Therefore

$$\sum_{n=2^{m-1}+1}^{2^m} n^{\alpha/2} P_n \leq \frac{c}{(m-2) \log^{1+\varepsilon}(m-2)} .$$

Now

$$\sum_{n=1}^{\infty} n^{\alpha/2} P_n = \sum_{m=m_0}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} n^{\alpha/2} P_n$$

$$\leq \sum_{m=m_0}^{\infty} \left\{ \frac{c}{(m-2) \log^{1+\varepsilon}(m-2)} \right\}$$

$$< \infty .$$

This proves that

$$\sum_{n=1}^{\infty} n^{\alpha/2} (|a_n| + |b_n|) < \infty .$$

<u>Proof of Theorem 5.</u> We write

$$\lambda_k = \omega(1/k) \log k , \quad k > 1 ,$$

$$\lambda_0 = A > 0 , \quad \lambda_1 = B > 0 .$$

It follows immediately from the condition (i) of Theorem 5 that

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k} < \infty,$$

which implies that the sum $g(x)$ of the convergent trigonometric series

$$(\lambda_0/2) + \sum_{k=1}^{\infty} \frac{\lambda_k}{k} \cos kx$$

is a continuous function and

$$f(x)+ g(x) \sim \frac{a_0 + \lambda_0}{2} + \sum_{k=1}^{\infty} (a_k + \frac{\lambda_k}{k}) \cos kx .$$

Hence we get

$$(14) \quad f(x+h)- f(x-h) + g(x+h)- g(x-h) \sim -2\sum_{k=1}^{\infty}\left(a_k+ \frac{\lambda_k}{k}\right) \sin kx \sin kh .$$

Let us consider the expression

$$S_n(h) = (1/2\pi) \int_0^{2\pi} \left\{f(x+h)-f(x-h)+g(x+h)-g(x-h)\right\} T_n(x)\, dx ,$$

where

$$T_n(x) = \sum_{k=1}^{n+1} (1- \frac{k}{n+1}) (- \sin kx) .$$

The polynomial $T_n(x)$ will be negative in $(0 , \pi)$ and positive in $(\pi , 2\pi)$ 1).
Taking $h = (\pi/2n)$, we get

1) L. Fejer [6]

$$(15)\quad |S_n(\pi/2n)| \le c_1\,\omega(f, \tfrac{\pi}{2n})\log n + \sum_{k=1}^{n+1}\left(1-\frac{k}{n+1}\right)\frac{\lambda_k}{k}\sin k\left(\frac{\pi}{2n}\right)$$

$$\le c_1\,\omega(f, \tfrac{\pi}{2n})\log n + (\pi/2n)\sum_{k=1}^{n+1}\left(1-\frac{k}{n+1}\right)\lambda_k$$

For r fixed and $c_2 = c_2(r)$, taking into account the fact that λ_k is a slowly increasing sequence, we have

$$(16)\quad \begin{aligned}(1/n)\sum_{k=1}^{n}\lambda_k &\le (c_2/n)\sum_{k=n}^{n}\lambda_k \\ &\le (c_2/n)\sum_{k=n}^{n}\frac{k^{\varepsilon}\lambda_k}{k^{\varepsilon}} \\ &\le \frac{c_2\, n^{\varepsilon}\lambda_n}{n}\sum_{k=1}^{n}\frac{1}{k^{\varepsilon}} \\ &\le \frac{c_3}{n}\, n^{\varepsilon}\lambda_n\, n^{1-\varepsilon} = c_3\lambda_n .\end{aligned}$$

Since $(1/x)$ increases slowly, from (15) and (16) we get

$$(17)\quad |S_n(\pi/2n)| \le c_4\,\omega(f, \tfrac{1}{n})\log n$$

for all sufficiently large n where c_4 is an absolute constant independent of n.

On the other hand $T_n(x)$ being trigonometrical polynomial, $S_n(h)$ can be calculated by term by term integration and we have

$$S_n(h) = \sum_{k=1}^{n+1}\left(1-\frac{k}{n+1}\right)\left(a_k + \frac{\lambda_k}{k}\right)\sin k\,h .$$

But

$$\sin k h \geq (2/\pi) k h, \; h = (\pi/2n), \; 1 \leq k \leq n$$

Therefore

$$S_n(\pi/2n) \geq (2/\pi m) \sum_{k=1}^{m+1} k\left(1 - \frac{k}{m+1}\right)\left(a_k + \frac{\lambda_k}{k}\right).$$

Taking $m = \frac{n}{2}$, $m > p$, we get

$$S_n(\pi/2n) \geq (1/2\pi m) \sum_{k=1}^{m} k\left(a_k + \frac{\lambda_k}{k}\right)$$

$$\geq (1/2\pi m) \sum_{k=p}^{m} k\left(a_k + \frac{\lambda_k}{k}\right) = B_m ,$$

where p is an index sufficiently large but fixed.

From this inequality using (17) we obtain

(18) $$B_m \leq c_5 \,\omega\,(f \,,\, 1/n)\log n$$

and also we have

(19) $$a_m + \frac{\lambda_m}{m} < 2\pi\left(B_m - B_{m-1} + \frac{B_{m-1}}{m-1}\right).$$

Taking into account that λ_k $(k \geq p)$ is a slowly increasing sequence, we shall have in virtue of (19)

$$|a_k| - \frac{\lambda_k}{k} \leq \left|a_k + \frac{\lambda_k}{k}\right|$$

$$< 2\pi\left(B_k - B_{k-1} + \frac{B_{k-1}}{k-1}\right)$$

Therefore,

$$k^{\beta/2}\left(|a_k| - \frac{\lambda_k}{k}\right) < 2\pi\left(k^{\beta/2} B_k - k^{\beta/2} B_{k-1} + k^{\beta/2}\frac{B_{k-1}}{k-1}\right)$$

$$< \left[2\pi \; k^{\beta/2} B_k - (k-1)^{\beta/2} B_{k-1} + k^{\beta/2} \frac{B_{k-1}}{k-1} \right].$$

Hence

$$\sum_{p+1}^{m} k^{\beta/2} |a_k| - \sum_{p+1}^{m} k^{\beta/2} \frac{\lambda_k}{k}$$

$$\leq 2\pi \sum_{p+1}^{m} k^{\beta/2} B_k - 2\pi \sum_{p+1}^{m} (k-1)^{\beta/2} B_{k-1}$$

$$+ 2\pi \sum_{p+1}^{m} k^{\beta/2} \frac{B_{k-1}}{k-1}$$

$$\leq 2\pi \left(m^{\beta/2} B_m + \sum_{p+1}^{m} \left\{ (k+1)^{\beta/2} \frac{B_k}{k} \right\} \right).$$

Therefore

$$\sum_{p+1}^{m} k^{\beta/2} |a_k| \leq 2\pi \, m^{\beta/2} B_m + 2\pi \sum_{p+1}^{m} \left\{ (k+1)^{\beta/2} \frac{B_k}{k} \right\}$$

$$+ \sum_{p+1}^{m} k^{\beta/2} \frac{\lambda_k}{k}.$$

Now

$$\lim_{m\to\infty} \sum_{p+1}^{m} k^{\beta/2} |a_k| \leq 2\pi \lim_{m\to\infty} m^{\beta/2} B_m + 2\pi \lim_{m\to\infty} \sum_{p+1}^{m} \left\{ (k+1)^{\beta/2} \frac{B_k}{k} \right\}$$

$$+ 2\pi \lim_{m\to\infty} \sum_{p+1}^{m} k^{\beta/2} \frac{\lambda_k}{k}.$$

Hence

$$\sum_{p+1}^{\infty} k^{\beta/2} |a_k| \leq 2\pi \sum_{p+1}^{\infty} \left\{ (k+1)^{\beta/2} \frac{B_k}{k} \right\} + 2\pi \sum_{p+1}^{\infty} k^{\beta/2} \frac{\lambda_k}{k}$$

on account of the fact that

$$\lim_{n\to\infty} n^{\beta/2} B_n \le A \lim_{n\to\infty} n^{\beta/2} (1/n)\log n$$

(in virtue of (18))

$$\le A' \lim_{n\to\infty} \frac{\log n}{n^{\alpha-(\beta/2)}}$$

$$= 0 \text{ , for } \beta < 2\alpha .$$

Again, due to (18), we have

$$\sum_{k=p+1}^{\infty} k^{\beta/2}(|a_k|) \le 2\pi A \sum_{k=p}^{\infty} \frac{(k+1)^{\beta/2}\omega(1/k)\log k}{k}$$

$$+ 2\pi A \sum_{k=p+1}^{\infty} \frac{k^{\beta/2}\omega(1/k)\log k}{k}$$

$$\le c \sum_{k=p+1}^{\infty} \frac{k^{\beta/2}\omega(1/k)\log k}{k}$$

$$\le c' \sum_{k=p+1}^{\infty} \frac{\log k}{k^{1+\alpha-(\beta/2)}} ,$$

$$< \infty \text{ , for } \beta < 2\alpha .$$

This completes the proof of theorem 5.

Similarly by considering

$$T_n'(x) = (1/2) + \sum_{k=1}^{n+1} \left(1 - \frac{k}{n+1}\right) \cos kx$$

and taking $\lambda_k = (1/k)$, we can prove Theorem 6.

Proof of Theorem 7. Writing

$$(20) \qquad B_m = (\pi/4m) \sum_{k=1}^{m} k\, b_k$$

we get

$$b_m = (4/\pi)\left(B_m - B_{m-1} + \frac{B_{m-1}}{m}\right)$$

$$< (4/\pi)\left(B_m - B_{m-1} + \frac{B_{m-1}}{m-1}\right), \text{ since } B_{m-1} \geq 0$$

Hence

$$\sum_{k=2}^{m} k^{\beta/2} b_k \leq (4/\pi) \sum_{k=2}^{m} k^{\beta/2} B_k - (4/\pi) \sum_{k=2}^{m} (k-1)^{\beta/2} B_{k-1}$$

$$+ (4/\pi) \sum_{k=2}^{m} k^{\beta/2} \frac{B_{k-1}}{k-1}$$

$$\leq (4/\pi)\, m^{\beta/2} B_m + (4/\pi) \sum_{k=2}^{m} k^{\beta/2} \frac{B_{k-1}}{k-1} \,.$$

Therefore in virtue of the conclusions (7) and (9) and using (20) we have

$$\sum_{k=2}^{m} k^{\beta/2} b_k \leq A\, m^{\beta/2}\, \omega(\pi/m) + A' \sum_{k=2}^{m} \frac{k^{\beta/2}\, \omega(1/k)}{k}$$

$$\leq \left\{A'' / m^{\alpha-(\beta/2)}\right\} + A''' \sum_{k=2}^{m} \frac{1}{k^{1+\alpha-(\beta/2)}}$$

taking limit as $m \to \infty$ we get

$$\sum_{k=2}^{\infty} k^{\beta/2} b_k \leq A''' \sum_{k=2}^{\infty} \frac{1}{k^{1+\alpha-(\beta/2)}}$$

since $\lim_{n \to \infty} \frac{1}{n^{\alpha - (\beta/2)}} = 0$, for $\beta < 2\alpha$.

Hence

$$\sum_{k=2}^{\infty} k^{\beta/2} b_k < \infty, \text{ for } \beta < 2\alpha .$$

This completes the proof of Theorem 7.

Proof of Theorem 8. Consider

$$G(x) = f(x) + g(x) ,$$

which is periodic and continuous. The Fourier coefficients of G are non-negative, since

$$a_n + a_n' \geq 0 , \quad b_n + b_n' \geq 0 ,$$

and its modulus of continuity $\leq 2\ (\delta)$. Hence the conclusion of Theorem 8 follows from Theorem 7.

3. Given a function f we write

$$\Delta_0(t) = f(x)$$

$$\Delta_1(t) = f(x+t) - f(x-t)$$

$$\Delta_2(t) = \Delta_1 \Delta_1 (t) = f(x+2t) + f(x-2t) - 2f(x) ,$$

and, in general

$$\Delta_m(t) = \Delta_1 \Delta_{m-1}(t) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\{x+(m-2k)t\} , \quad m=1,2,3,\ldots$$

Also we define $\omega_n(\delta, f)$, the generalized modulus of continuity of order r of the function f, as

$$\omega_n(\delta, f) = \sup_{|h| \leq \delta} |\Delta_n f(x,h)|.$$

A function $f(x)$ defined in an interval (a, b) is said to be of bounded r^{th} variation if for arbitrary values $x_0, x_1 \ldots\ldots x_n$ in Arithmetic progression, such that $a = x_0 < x_1 < \ldots\ldots < x_n = b$ and for every integral values of n,

$$\sum_{i=0}^{n-r} |\Delta_r f(x_i)| \leq M,$$

where

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

$$\ldots\ldots = \ldots\ldots$$

$$\Delta_{\ell} f(x_i) = \Delta_{\ell-1} f(x_{i+1}) - \Delta_{\ell-1} f(x_i) .$$

It is a well known fact that every function, which is of bounded variation, is also of bounded r^{th} variation, but the converse is not true. This is seen by considering the well known continuous non-differentiable function of Weierstrass viz.

$$f(x) = \sum_{n=1}^{\infty} b^{-n} \cos b^n x , \quad b > 1$$

which satisfies the condition

$$|f(x+h) + f(x-h) - 2f(x)| = O(h) \text{ as } h \to 0 \text{ uniformly}$$

in x, and consequently is of bounded second variation. But on account of its being a non-differentiable function it cannot be of bounded variation in any interval.

In view of the above remark we can extend our Theorems 1 and 4. More precisely we prove the following:

Theorem 9. If (i) $f(x)$ is of bounded r^{th} variation in $(0, 2\pi)$,

(ii) $$\omega_r(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^{\alpha+2}}, \varepsilon > 0,$$

then the series (1) is convergent for $\beta = 2/(\alpha+2)$.

Theorem 10. If (i) $f(x)$ is of bounded r^{th} variation in $(0, 2\pi)$,

(ii) $$\omega_r(\delta) \leq \frac{c\,\delta^{\alpha}}{\left[l_1(\delta)\, l_2(\delta) \ldots l_k^{1+\varepsilon}(\delta)\right]^{2}}, \varepsilon > 0,$$

then the series (12) converges for $\beta = \alpha$.

Proof of Theorem 9. We write

$$\Delta_r f(x,h) = f(x+rh) - \binom{r}{1} f(x+rh-2h) + \ldots + (-1)^{r/2}\binom{r}{\frac{r}{2}} f(x)$$
$$+ \ldots + \binom{r}{r-1} f(x-rh+2h) + \binom{r}{r} f(x-rh).$$

Let

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = (1/2\pi) \int_0^{2\pi} f(x)\, e^{-inx}\, dx .$$

we obtain

$$\Delta_r f(x,h) \sim \sum_{-\infty}^{\infty} c_n (e^{inh} - e^{-inh})^r e^{inx} .$$

Then using Bessel's inequality we have

$$\sum_{-\infty}^{\infty} | c_n (e^{inh} - e^{-inh})^r |^2 \leq (1/\pi) \int_0^{2\pi} | \Delta_r f(x,h)|^2\, dx$$

and hence

$$2^{2r+1}\sum_{n=1}^{\infty}|c_n|^2|\sin nh|^{2r} \leq (1/\pi) \int_0^{2\pi} |\Delta_h f(x,h)|^2 \, dx \, .$$

Putting $h = (\pi/2N)$ where N is a positive integer greater than or equal to r , we have

$$2^{2r+1}\sum_{n=1}^{\infty}|c_n|^2|\sin(n\pi/2N)|^{2r} \leq (1/\pi) \int_0^{2\pi} |\Delta_h f(x, \tfrac{\pi}{2N})|^2 \, dx$$

$$\leq (1/\pi)\, \omega_h(\pi/N) \int_0^{2\pi} |\Delta_h f(x, \tfrac{\pi}{2N})| \, dx$$

and hence

$$2^{2r+1}\sum_{\frac{N}{2}+1}^{N}|c_n|^2|\sin(n\pi/2N)|^{2r} \leq (1/\pi)\, \omega_h(\pi/N) \int_0^{2\pi} |\Delta_h f(x, \tfrac{\pi}{2N})| \, dx.$$

Since $\sin (n\pi/2N) \geqslant 1/\sqrt{2}$ for $(N/2) \leq n \leq N$, we get

$$\frac{2^{2r+1}}{2^r}\sum_{\frac{N}{2}+1}^{N}|c_n|^2 \leq 2^{2r+1}\sum_{\frac{N}{2}+1}^{N}|c_n|^2|\sin(n\pi/2N)|^{2r}$$

(21) $$\leq (1/\pi)\, \omega_h(\pi/N) \int_0^{2\pi} |\Delta_h f(x, \tfrac{\pi}{2N})| \, dx.$$

Also

$$\sum_{n=1}^{2N-h} \int_0^{2\pi} |\Delta_h f(x + \tfrac{n\pi}{N}, \tfrac{\pi}{2N})| \, dx$$

$$= \sum_{n=1}^{2N-h} \int_0^{2\pi} \left| f(x+\tfrac{n\pi}{N}+\tfrac{h\pi}{2N}) - \binom{h}{1} f(x+\tfrac{n\pi}{N}+\tfrac{(h-2)\pi}{2N}) + \dots + \binom{h}{h} f(x+\tfrac{n\pi}{N}-\tfrac{h\pi}{2N}) \right| dx \, .$$

On a ccount of periodicity , all the integrals on the right hand side will have the same value. Hence

$$\sum_{n=1}^{2N-h} \int_0^{2\pi} | \Delta_h f(x + \tfrac{n\pi}{N}, \tfrac{\pi}{2N}) | \, dx$$

$$= (2N-r) \int_0^{2\pi} | f(x + \tfrac{r\pi}{2N}) - \binom{h}{1} f(x + \tfrac{(h-2)\pi}{2N}) + \dots + f(x - \tfrac{h\pi}{2N}) | \, dx$$

(22) $$= (2N-r) \int_0^{2\pi} | \Delta_h f(x, \tfrac{\pi}{2N}) | \, dx$$

Therefore it follows from (21) and (22) that

$$(2N-h) \cdot 2^{r+1} \sum_{\frac{N}{2}+1}^{N} |c_n|^2 \leq \frac{1}{\pi} \omega_h(\pi/N) \sum_{n=1}^{2N-h} \int_0^{2\pi} |\Delta_h f(x + \tfrac{n\pi}{N}) | \, dx$$

$$= \omega_h(\pi/N) \frac{1}{\pi} \int_0^{2\pi} \sum_{n=1}^{2N-h} | \Delta_h f(x + \tfrac{n\pi}{N}) | \, dx$$

$$\leq \omega_h(\pi/N) \cdot \frac{V}{\pi} \; 2\pi ,$$

since $\sum_{n=1}^{2N-h} |\Delta_h f(x+\frac{n\pi}{N})| \leq V$, as f is of bounded r^{th} variation.

Therefore

$$\sum_{\frac{N}{2}+1}^{N} |c_n|^2 \leq \frac{A\, \omega_h(\pi/N)}{2N - r}$$

$$\leq \frac{A\, \omega_h(\pi/N)}{N} , \quad \text{as } N \geq r$$

and hence

(23) $$\sum_{2^{m-1}+1}^{2^m} |c_n|^2 \leq A\, \omega_h(\pi/2^m) \cdot 2^{-m} .$$

Now it follows from Hölder's inequality that

$$\sum_{2^{m-1}+1}^{2^m} |c_n|^\beta \leq \left(\sum_{2^{m-1}+1}^{2^m} |c_n|^2\right)^{\beta/2} \left(\sum_{2^{m-1}+1}^{2^m} 1\right)^{1-\frac{\beta}{2}}$$

$$\leq A_1 \left\{ \omega_r(\pi/2^m) \right\}^{\beta/2} . \, 2^{-m\beta/2} . \, 2^{m(1-\frac{\beta}{2})} \tag{24}$$

and hence by following the same analysis as in the proof of Theorem 1 and using the second condition of the hypothesis, we conclude that

$$\sum_{n=1}^{\infty} |c_n|^\beta < \infty \quad , \text{ for } \beta = (2/\alpha+2) .$$

Since $c_n = \frac{1}{2}(a_n - i b_n)$, $n = 0, 1, 2, \cdots$

$$\sum_{n=1}^{\infty} (|a_n|^\beta + |b_n|^\beta) < \infty, \text{ for } \beta = (2/\alpha+2).$$

Proof of Theorem 10. By Cauchy-Schwarz inequality we get

$$\sum_{2^{m-1}+1}^{2^m} |c_n| \leq \left(\sum_{2^{m-1}+1}^{2^m} |c_n|^2\right)^{1/2} \left(\sum_{2^{m-1}+1}^{2^m} 1\right)^{1/2} .$$

In virtue of the conclusion (23) we have

$$\sum_{2^{m-1}+1}^{2^m} |c_n| \leq A_2 \sqrt{\omega_r(\pi/2^m)} \, 2^{-m/2} \, 2^{m/2}$$

Therefore

$$\sum_{2^{m-1}+1}^{2^m} n^{\beta/2} |c_n| \leq A_2 \, 2^{m\beta/2} \sqrt{\omega_r(\pi/2^m)}$$

$$\leq A_3 \frac{2^{m\beta/2} \, 2^{-m\alpha/2}}{l_1(\pi/2^m) \, l_2^{1+\varepsilon}(\pi/2^m)}$$

$$\leq \frac{A_4}{(m-2)\log^{1+\varepsilon}(m-2)} \quad , \text{ for } \beta = \alpha$$

Hence

$$\sum_{n=1}^{\infty} n^{\beta/2} |c_n| \leq A_4 \sum_{m=m_0}^{\infty} \frac{1}{(m-2)\log^{1+}(m-2)}$$

$$< \infty .$$

Thus we conclude that

$$\sum_{n=1}^{\infty} n^{\beta/2} (|a_n| + |b_n|) < \infty, \text{ for } \beta = \alpha.$$

Remark 1: For the convergence of the series (1) it is sufficient that f is of bounded r^{th} variation and

$$\int_0^1 h^{\beta-2} \omega_r^{\beta/2}(h)\, dh = O(1) .$$

Proof. Since $\omega_r(h)$ is a non-decreasing function of h, we have

$$\omega_r^{\beta/2}(\pi/2^m) \leq (2^m/\pi) \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, dh .$$

Therefore from (24) we obtain

$$\sum_{2^{m-1}+1}^{2^m} |c_n|^{\beta} \leq A_2\, 2^{m(1-\beta)} \frac{2^m}{\pi} \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, dh$$

$$= (A_2/\pi)\, 2^{m(2-\beta)} \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, h^{\beta-2}\, h^{2-\beta}\, dh$$

$$\leq (A_2/\pi)\, 2^{m(2-\beta)}\, 2^{-(m-1)(2-\beta)} \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, h^{\beta-2}\, dh$$

$$= A_3 \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, h^{\beta-2}\, dh$$

and hence

$$\sum_{n=1}^{\infty} |c_n|^{\beta} \leq A_3 \sum_{m=m_0}^{\infty} \int_{\pi/2^m}^{\pi/2^{m-1}} \omega_r^{\beta/2}(h)\, h^{\beta-2}\, dh$$

$$= A_3 \int_0^1 \omega_r^{\beta/2}(h)\, h^{\beta-2}\, dh$$

$$= O(1) .$$

Consequently

$$\sum_{n=1}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}) < \infty .$$

Remark 2: For the convergence of the series (12) it is sufficient that f is of bounded r^{th} variation and

$$\int_0^1 h^{-1-(\beta/2)}\, \omega_r^{\frac{1}{2}}(h)\, dh = O(1) .$$