CHAPTER III

ABSOLUTE CONVERGENCE OF FOURIER SERIES OF RESTRICTED Lip (α,p) FUNCTIONS

In this chapter we have studied the absolute convergence of Fourier series of functions belonging to the class Lip (α,p). It is easy to see that the conclusions of Bernstein's theorem I.1 and Szàsz's theorem I.6 remain valid if in their hypotheses the condition that $f \in \text{Lip}\,\alpha$ is replaced by the weaker condition that $f \in \text{Lip}\,(\alpha,2)$.

In the year 1942, it was proved by Min-Teh Cheng[1] that if $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$ and

$$\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left(h(\log h^{-1})^{-1-\alpha p}\right),$$

then the series

$$(1) \qquad \sum_{n=2}^{\infty} (|a_n| + |b_n|) \log^T n$$

converges for $T < \alpha + p^{-1} - 1$. Moreover the series (1) may not converge for $T = \alpha + p^{-1} - 1$.

In this chapter we have investigated some stronger

1) Cheng [6]

conditions to be imposed on the function f so that the series (1) may converge for $T = \alpha + p^{-1} - 1$. Moreover the proof given by us is simpler than that of Cheng.
Our theorems are:

Given a periodic function $f \in L^p$, $1 < p \leq 2$ we define $\omega_p^{(m)}$, the mean modulus of continuity of the function f, by

$$\omega_p^{(m)}(f,\delta) = \sup_{0 \leq t \leq \delta}\left((1/2\pi)\int_0^{2\pi} |\Delta_m f(x,t)|^p\right)^{1/p}$$

__Theorem 11.__ *If* $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, > 0 *and*

$$\omega_p^{(m)}(f,h) = O\left(h^{1/p}(\log h^{-1})^{-(\alpha+\frac{1}{p})}(\log\log h^{-1})^{-(1+\epsilon)}\right),$$

then the series (1) *converges for* $T = \alpha + (1/p) - 1$.

__Theorem 12.__ *If* $0 < \alpha \leq 1$, $1 < p \leq 2$, $h > 0$, > 0 *and*

$$\omega_p^{(m)}(f,h) = O\left(h^{\delta/p}(\log h^{-1})^{-(\alpha+\frac{1}{p})}(\log\log h^{-1})^{-(1+\epsilon)/\beta}\right),$$

then the series

$$(2) \qquad \sum_{n=2}^{\infty}(|a_n|^{\beta} + |b_n|^{\beta}) \log^T n$$

converges for $\beta = \dfrac{p(T+1)}{(1+\alpha p)}$, where

$$\delta = 1 + \frac{p(1-\beta)}{\beta}.$$

__Proof of Theorem 11.__ Let $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, where the complex Fourier coefficients c_n of f are given by the relation

$$c_n = (1/2\pi) \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \text{ , for } n = 0, \pm 1, \pm 2, \cdots$$

we obtain

$$\Delta_m(x,h) \sim \sum_{-\infty}^{\infty} c_n (e^{inh} - e^{-inh})^m e^{inx} .$$

By Hausdorff-Young inequality, we have

$$\left(\sum_{-\infty}^{\infty} |c_n (e^{inh} - e^{-inh})^m|^{p'} \right)^{\frac{1}{p'}} \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | \Delta_m(x,h)|^p \right)^{\frac{1}{p}}$$

where $(1/p) + (1/p') = 1$.

Therefore

$$2^{mp'} \sum_{-\infty}^{\infty} |c_n|^{p'} |\sin n h|^{m p'} \leq \left(\omega_p^{(m)}(f,h) \right)^{p'}$$

$$2^{mp'+1} \sum_{1}^{\infty} |c_n|^{p'} |\sin nh|^{mp'} \leq \left(\omega_p^{(m)}(f,h) \right)^{p'}$$

Choosing $h = (\pi/2N)$ where N is a positive integer we get

$$2^{mp'+1} \sum_{N/2+1}^{N} |c_n|^{p'} |\sin(n\pi/2N)|^{mp'} \leq \left(\omega_p^{(m)}(f, \frac{\pi}{2N}) \right)^{p'} .$$

Since

$$\sin(n\pi/2N) \geq 1/\sqrt{2} \text{ for } (N/2) \leq n \leq N$$

we get

$$2^{mp'+1-(mp'/2)} \; |c_n|^{p'} \leq \left\{\omega_p^{(m)}\left(f, \frac{\pi}{2N}\right)\right\}^{p'} .$$

Taking $N = 2^r$, where r is an integer greater or equal to $r_0 \geq (\log 2)^{-2} + 3$, we get from the last inequality

$$(3) \qquad \sum_{2^{r-1}+1}^{2^r} |c_n|^{p'} \leq \frac{1}{2^{1+mp'/2}} \left\{\omega_p^{(m)}\left(f, \frac{\pi}{2^r}\right)\right\}^{p'} .$$

Now applying Hölder's inequality we get

$$\sum_{2^{r-1}+1}^{2^r} |c_n| \leq \left(\sum_{2^{r-1}+1}^{2^r} |c_n|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{2^{r-1}+1}^{2^r} 1\right)^{1-\frac{1}{p'}}$$

$$(4) \qquad \sum_{2^{r-1}+1}^{2^r} |c_n| \leq A \; \omega_p^{(m)}\left(f, \frac{\pi}{2^r}\right) . \; 2^{(r-1)(1-\frac{1}{p'})} .$$

Hence in virtue of the hypothesis of the theorem we get

$$\sum_{2^{r-1}+1}^{2^r} |c_n| \leq A \frac{2^{r/p} \; 2^{-r/p}}{\left(\log(2^r/\pi)\right)^{\alpha+\frac{1}{p}} \left(\log \log(2^r/\pi)\right)^{1+\varepsilon}} .$$

Therefore

$$\sum_{2^{r-1}+1}^{2^r} |c_n| \log^T n \leq \log^T(2^r) \sum_{2^{r-1}+1}^{2^r} |c_n|$$

$$= O\left(\frac{\log^T(2^r)}{\left(\log(2^r/\pi)\right)^{\alpha+\frac{1}{p}} \left(\log \log(2^r/\pi)\right)^{1+\varepsilon}}\right)$$

$$= O\left(\frac{1}{r \log^{1+\varepsilon} r}\right), \text{ for } T = \alpha + p^{-1} - 1 .$$

Hence

$$\sum_{n=2}^{\infty}|c_n| \log^T n = \sum_{r=2}^{\infty}\sum_{2^{r-1}+1}^{2^r}|c_n| \log^T n$$

$$= O\left(\sum_{r=2}^{\infty}\frac{1}{r\log^{1+\varepsilon} r}\right)$$

$$= O(1) .$$

This proves theorem 11.

<u>Proof of Theorem 12.</u> Using Holder's inequality we obtain

$$\sum_{2^{r-1}+1}^{2^r}|c_n|^{\beta} \le \left(\sum_{2^{r-1}+1}^{2^r}|c_n|^{p'}\right)^{\beta/p'}\left(\sum_{2^{r-1}+1}^{2^r}1\right)^{\left(1-\frac{\beta}{p'}\right)}.$$

Hence it follows from (3) that

$$\sum_{2^{r-1}+1}^{2^r}|c_n|^{\beta} \le A\left\{\omega_p^{(m)}\left(f, \frac{\pi}{2^r}\right)\right\}^{\beta} 2^{r\left(1-\frac{\beta}{p'}\right)}$$

$$= O\left(\frac{2^{r\left(1-\frac{\beta}{p'}\right)}\, 2^{-r\,\delta\,\beta/p}}{\left(\log(2^r/\pi)\right)^{\left(\alpha+\frac{1}{p}\right)\beta}\left(\log\log(2^r/\pi)\right)^{1+\varepsilon}}\right).$$

Therefore

$$\sum_{2^{r-1}+1}^{2^r}|c_n|^{\beta}\log^T n = O\left(\frac{2^{r\left(1-\frac{\beta}{p'}-\frac{\delta\beta}{p}\right)}\log^T 2^r}{\left(\log(2^r/\pi)\right)^{\left(\alpha+\frac{1}{p}\right)\beta}\left(\log\log(2^r/\pi)\right)^{1+\varepsilon}}\right)$$

$$= O\left(\frac{1}{r\log^{1+\varepsilon} r}\right), \text{ for } \beta = \frac{(1+T)p}{(1+\alpha p)}$$

and $\delta = 1 + \frac{p(1-\beta)}{\beta}$.

Thus we conclude that

$$\sum_{n=2}^{\infty} |c_n|^{\beta} \log^{T} n = O\left(\sum_{r=2}^{\infty}\left\{\frac{1}{r \log^{1+\varepsilon} r}\right\}\right)$$

$$< \infty .$$

This proves theorem 12.

<u>Remark.</u> Theorem 11 remains true even if

$$\omega_p^{(m)}(f,h) = O\left\{h^{1/p}(\log_1 h^{-1})^{-1}(\log_2 h^{-1})^{-1}\cdots(\log_k h^{-1})^{-(\frac{1}{p}+\alpha)}(\log_{k+1} h^{-1})^{-(1+\varepsilon)}\right\},$$

where $\log_1 h^{-1} = \log h^{-1}$ and $\log_n h^{-1} = \log \log_{n-1} h^{-1}$.

Theorem 12 remains true if

$$\omega_p^{(m)}(f,h) = O\left\{h^{\delta/p}(\log_1 h^{-1})^{-1}\cdots(\log_k h^{-1})^{-(\frac{1}{p}+\alpha)}(\log_{k+1} h^{-1})^{-(1+\varepsilon)/\beta}\right\}.$$

More general results than that of Min-Teh Cheng have been established by different authors. In particular O. Szàsz[1] proved the following generalization of Theorem I.13:

<u>Theorem D.</u> <u>If</u> $f \in \mathrm{Lip}(\alpha,p)$, $0 < \alpha < 1$, $1 \leq p \leq 2$, <u>then</u>

(5) $$\sum_{n=1}^{\infty} n^{\beta}(|a_n| + |b_n|)$$

<u>converges for every</u> $\beta < \alpha - p^{-1}$.

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1) Szàsz [35]

The series (5) may not converge for $\beta = \alpha - p^{-1}$.

We prove below a more general theorem than the theorem of O. Szász.

__Theorem 13.__ __If__ $f \in L^p$, $1 \leq p \leq 2$, __and__

$$\omega_p^{(m)}(h) = O\left(\frac{h^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]}\right) \text{ as } h \to 0 ,$$

__where__ $\varepsilon > 0$, __then the series__(5) __converges for__

$$\beta = \alpha - p^{-1} .$$

Theorem 13 can be extended as follows:

__Theorem 14.__ __Let__ $1 \leq p \leq 2$, $0 < p_1 \leq p \leq p_2$,

$$\rho_1 \geq 0, \rho_2 \geq 0, \quad \rho_1 p_1 + \rho_2 p_2 = p , \ \rho_1 + \rho_2 = 1 .$$

__If__ $$\omega_{p_1}^{(m)}(h) = O\left(\frac{h^{\alpha_1}}{\left[l_1(h) \ldots l_k^{1+\varepsilon}(h)\right]^{p/2\rho_1 p_1}}\right)$$

__and__ $$\omega_{p_2}^{(m)}(h) = O\left(\frac{h^{\alpha_2}}{\left[l_1(h) \ldots l_k^{1+\varepsilon}(h)\right]^{p/2\rho_2 p_2}}\right)$$

__then the series__ (5) __is convergent for__

$$\beta = \frac{\rho_1 p_1 \alpha_1 + \rho_2 p_2 \alpha_2}{p} - \frac{1}{p} .$$

__Proof of Theorem 13.__ From (4) and using the hypothesis of the theorem we get

$$\sum_{2^{r-1}+1}^{2^r} |c_n| \leq B \frac{2^{r(1-\frac{1}{p'})} \, 2^{-r\alpha}}{l_1(\pi/2^r) \, l_2^{1+\varepsilon}(\pi/2^r)}$$

$$\leq C \frac{2^{r(1-\frac{1}{p'}-\alpha)}}{(r-2)\log^{1+\varepsilon}(r-2)} .$$

Hence

$$\sum_{2^{r-1}+1}^{2^r} n^\beta |c_n| \leq C \frac{2^{r\beta} \, 2^{r(\frac{1}{p}-\alpha)}}{(r-2)\log^{1+\varepsilon}(r-2)}$$

$$= \frac{C}{(r-2)\log^{1+\varepsilon}(r-2)} , \text{ for } \beta = \alpha - \frac{1}{p} .$$

Therefore

$$\sum_{n=1}^{\infty} n^\beta |c_n| = \sum_{r=r_0}^{\infty} \sum_{2^{r-1}+1}^{2^r} n^\beta |c_n|$$

$$\leq C \sum_{r=r_0}^{\infty} \frac{1}{(r-2)\log^{1+\varepsilon}(r-2)}$$

$$< \infty .$$

This proves Theorem 13 if we take into account that

$$c_n = (1/2)(a_n - ib_n) \text{ for } n > 0 .$$

Remark. For the convergence of the series (5) it is enough that

$$\int_0^1 h^{-\gamma}\omega_p^{(m)}(h)\,dh = (1), \quad \text{where } \gamma = \beta + p^{-1}+1 .$$

Proof. It is a well known fact that $\omega_p^{(m)}(h)$ is a non-decreasing function of h, and hence the above remark can easily be proved by following an analysis similar to the proof of Remark 1 in Chapter II.

Proof of Theorem 14. Using a result proved by Szàsz in [36] and following an analysis similar to the proof of Theorem 13 we get by virtue of condition of the theorem

$$\sum_{n=1}^{\infty} n^{\beta}|c_n|$$

$$\leq c \sum_{r=r_0}^{\infty}\left\{2^{r(\beta+p^{-1})}\right\}\left\{\frac{1}{2^{r\alpha_1\rho_1p_1+r\alpha_2\rho_2p_2}}\right\}^{\frac{1}{p}}\left\{\frac{1}{(r-2)\log^{1+\varepsilon}(r-2)}\right\}$$

$$\leq c \sum_{r=r_0}^{\infty}\left\{\frac{1}{2^{r(-\beta-\frac{1}{p})+r(\alpha_1\rho_1p_1+\alpha_2p_2\rho_2)/p}}\right\}\left\{\frac{1}{(r-2)\log^{1+\varepsilon}(r-2)}\right\}$$

$$< \infty, \text{ for } \beta = \frac{1}{p}\left(\rho_1p_1\alpha_1+\rho_2p_2\alpha_2\right)-\frac{1}{p} .$$

If in the hypotheses of above theorems 13 and 14, we replace the expression

$$l_1(h)\, l_2(h)\ldots l_k^{1+\varepsilon}(h)$$

by 1, we get the following weaker forms of Theorem 13 and Theorem 14 respectively .

<u>Theorem 15.</u> <u>If</u> $\omega_p^{(m)}(h) = O(h^{\alpha})$, <u>then the series</u> (5) <u>is convergent for</u> $\beta < \alpha - p^{-1}$.

<u>Theorem 16.</u> <u>In Theorem</u> 14 <u>if</u> $l_1(h)l_2(h)\ldots l_k^{1+\varepsilon}(h)$ <u>is replaced by</u> 1, <u>then the series</u> (5) <u>is convergent for</u> $\beta < \frac{1}{p}(p_1\rho_1\alpha_1 + p_2\rho_2\alpha_2) - \frac{1}{p}$.

It may be remarked that Theorem 15 and 16 are more general than Theorem D above.