CHAPTER V

ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES

Let $f(x)$ be L-integrable over $(-\pi, \pi)$ and periodic with period 2π. Let its Fourier coefficients a_n, b_n vanish except for a strictly increasing sequence $n = n_k$ of positive integers.

Noble[1] proved the following two theorems relating to the absolute convergence of a Fourier series satisfying a certain gap condition:

Theorem F. If (i) $\lim \dfrac{N_k}{\log n_k} = \infty$, where

$$N_k = \min\{(n_{k+1} - n_k), (n_k - n_{k-1})\},$$

(ii) $f(x)$ is of bounded variation in some interval $|x - x_0| \leq \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ in $|x - x_0| \leq \delta$,

then the series

(1) $$\sum_1^\infty (|a_n| + |b_n|) < \infty.$$

1) Noble [31]

<u>Theorem G.</u> <u>If</u> (i) $\lim \dfrac{N_k}{\log n_k} = \infty$,

(ii) $f(x)$ <u>is of bounded variation in some interval</u> $|x - x_0| \le \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ <u>in</u> $|x-x_0| \le \delta$,

<u>then the series</u>

$$\sum_1^{\infty} \left(|a_n|^{\beta} + |b_n|^{\beta}\right) < \infty, \text{ for } \beta > 2/(\alpha+2). \tag{2}$$

The conclusion (2) may not hold for $\beta = 2/(\alpha+2)$.

P. B. Kennedy[1] used more powerful methods due to Paley and Wiener[2] to give a simple proof of Noble's theorem under less restricted gap hypothesis. He established the following theorem:

<u>Theorem H.</u> <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $f(x)$ <u>is of bounded variation in some interval</u> $|x - x_0| \le \delta$,

(iii) $f(x) \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$ <u>in</u> $|x-x_0| \le \delta$,

<u>then the series</u> (2) <u>is convergent.</u>

It is a well known fact that every function, which is of bounded variation, is also of bounded p^{th} variation; but the converse is not true.

We shall prove the following theorems:

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1) P. B. Kennedy [23] 2) Paley and Wiener [33]

Let $f(x) \sim \sum_{1}^{\infty}(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$, where a_{n_k}, b_{n_k} are non-vanishing Fourier coefficients of L-integrable and periodic function f of period 2π and $\{n_k\}$ is a strictly increasing sequence. Let x_0 be fixed and $\delta > 0$. Denote by ω, the modulus of continuity of the function f.

<u>Theorem 20.</u> <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $\omega(h) \leq \dfrac{c}{[l_1(h) l_2(h) \dots l_k^{1+\varepsilon}(h)]^2}$, $\varepsilon > 0$,

<u>in some interval</u> $|x - x_0| \leq \delta$,

(iii) $f(x)$ <u>is of bounded</u> 2^{nd} <u>variation</u> <u>in</u> $|x - x_0| \leq \delta$,

<u>then the series (1) is convergent.</u>

<u>Theorem 21.</u> <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $\omega(h) \leq \dfrac{c\ h^{\alpha}}{[l_1(h) l_2(h) \dots l_k^{1+\varepsilon}(h)]^{\alpha+2}}$

<u>where</u> $\varepsilon > 0$, $\alpha \geq 0$, <u>in some interval</u> $|x - x_0| \leq \delta$,

(iii) $f(x)$ <u>is of bounded</u> 2^{nd} <u>variation</u> <u>in</u> $|x - x_0| \leq \delta$,

<u>then the series (2) converges for</u> $\beta = 2/(\alpha+2)$.

In proving these theorems we shall use the following lemma:

<u>Lemma</u> (Kennedy). <u>Let</u> $\{\lambda_k\}_{-\infty}^{\infty}$ <u>be a sequence of real numbers satisfying</u> $\lambda_{k+1} > \lambda_k$; $\lambda_{k+1} - \lambda_k > 8\pi/\delta$ <u>and let</u> $\{A_k\}_{-\infty}^{\infty}$ <u>be a sequence of complex numbers such that</u> $\sum_{-\infty}^{\infty} |A_k| r^{|\lambda_k|} < \infty$, <u>for</u> $0 < r < 1$.

<u>If</u> $\phi(x) = L^2$ - $\underset{r \to 1}{\text{limit}}\ \phi(r, x)$ in $|x - x_0| \leq \delta$,

<u>where</u> $\phi(r,x) = \sum_{-\infty}^{\infty} A_k e^{i\lambda_k x} r^{|\lambda_k|}$, $0 < r < 1$,

<u>for all real</u> x ; <u>then</u>

(3) $$\sum_{-\infty}^{\infty} |A_k|^2 \leq 8\,\delta^{-1} \int_{|x-x_0| \leq \delta} |\phi(x)|^2 dx.$$

<u>Proof of Theorem 21.</u> We write

$$g(x) = \Delta_2 f(x,h) = f(x+2h) - 2f(x) + f(x-2h) .$$

Since $$f(x) \sim (a_0/2) + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) ,$$

we obtain

$$g(x) \sim -4 \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \sin^2 nh$$

$$= \sum_{1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

$$= \sum_{-\infty}^{\infty} A_k e^{i\lambda_k x},$$

where $A_k = \frac{1}{2}(\alpha_{n_k} - i\, \beta_{n_k})$, if $k > 0$

$A_k = \bar{A}_{-k}$ if $k < 0$

$A_0 = 0$ and $\lambda_k = n_k$ if $k > 0$, $\lambda_k = -\lambda_k$ if $k < 0$.

Put
$$g(r,x) = \sum_1^\infty (\alpha_n \cos nx + \beta_n \sin nx)\, r^n , \quad 0 < r < 1 .$$

We assume without loss of generality that $I = \left[x_0 - \frac{3\delta}{4}, x_0 + \frac{3\delta}{4}\right]$ is a subinterval of $(-\pi,\pi)$. Choose $h = \pi/(2n_K)$ where K is a positive integer such that $n_K > 4\pi/\delta$.

We denote $g_1(x) = g(x)$ if $x \in I$
$= 0$ otherwise

and $g_2(x) = g(x) - g_1(x)$.

Let $\alpha_n^{(1)}$, $\beta_n^{(1)}$ and $\alpha_n^{(2)}$, $\beta_n^{(2)}$ be the Fourier coefficients of $g_1(x)$ and $g_2(x)$ respectively, and let

$$g^{(1)}(r,x) = \sum_1^\infty (\alpha_n^{(1)} \cos nx + \beta_n^{(1)} \sin nx)\, r^n ,$$
$$g^{(2)}(r,x) = \sum_1^\infty (\alpha_n^{(2)} \cos nx + \beta_n^{(2)} \sin nx)\, r^n ,$$

where $0 < r < 1$.

Since $g_1(x) \in L^2(-\pi,\pi)$, we have[1)]

(4) $$g_1(x) = L^2\text{-}\lim_{r \to 1} g^{(1)}(r,x) \text{ in } (-\pi,\pi) .$$

Further by Fejér's Theorem[2)] on A- summability of a Fourier series we have

(5) $$g^{(2)}(r,x) \to g_2(x) \text{ as } r \to 1 \text{ uniformly for}$$
$$|x - x_0| \le 3\delta/4 .$$

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1) Zygmund [49, p. 87] 2) Zygmund [49, p.51]

From (5) we deduce that

$$(6)\qquad L^2\text{-}\lim_{r\to 1} g^{(2)}(r,x) = g_2(x) \text{ uniformly in } |x-x_0|\le \frac{3\delta}{4}\,.$$

Since

$$g(r,x)= g^{(1)}(r,x)+g^{(2)}(r,x)\,,$$

it follows from (4) and (6) that

$$(7)\qquad L^2\text{-}\lim_{r\to 1} g(r,x) = g(x) \text{ uniformly in } |x-x_0|\le \frac{3\delta}{4}\,.$$

We assume that $n_{k+1}-n_k > 16\pi\,\delta^{-1}$. Choose an integer j such that

$$(8)\qquad 0\le j\le \frac{\delta\, n_K}{8\pi}\,.$$

Let

$$g_j(r,x) = g(r,x+\frac{2j\pi}{n_K}) = \sum_{-\infty}^{\infty} A_k\, e^{i\lambda_k\left(x+\frac{2j\pi}{|\lambda_k|}\right)}\, r^{|\lambda_k|}\,.$$

Now

$$\sum_{-\infty}^{\infty}\left|A_k\, e^{i\lambda_k\frac{2j\pi}{|\lambda_k|}}\right| r^{|\lambda_k|} \le \sum_{-\infty}^{\infty}|A_k|\, r^{|\lambda_k|}$$

$$= \sum_{1}^{\infty}\left(\alpha_{n_k}^2+\beta_{n_k}^2\right)^{\frac12} r^{n_k}$$

$$= 4\sum_{1}^{\infty}\left(a_{n_k}^2+b_{n_k}^2\right)^{\frac12}\sin^2(n_k\pi/n_K)\; r^{n_k}$$

$$\le 4\sum_{1}^{\infty}\left(a_{n_k}^2+b_{n_k}^2\right)^{\frac12} r^{n_k}$$

$$(9)\qquad < \infty,\quad \text{for } 0<r<1\,.$$

Also we deduce from the conclusion (7) and the inequality (8) that

$$(10)\qquad L^2\text{-}\lim_{r\to 1} g_j(r,x) = g_j(x) \text{ in } |x-x_0|\le \delta/2\,.$$

Therefore, it follows from the conclusion (3) of the lemma in virtue of (9) and (10) by taking $\delta = \delta/2$ that

$$\sum_{-\infty}^{\infty} \left| A_k e^{i\lambda_k \frac{2j\pi}{|\lambda_k|}} \right|^2 \leq 16\,\delta^{-1} \int_{|x-x_0|\le\delta/2} |g_j(x)|^2 dx$$

i.e. $$2^{-1} \sum_{1}^{\infty} (\alpha_{n_k}^2 + \beta_{n_k}^2) \leq 16\,\delta^{-1} \int_{|x-x_0|\le\delta/2} \left| g(x + \frac{2j\pi}{n_k}) \right|^2 dx$$

$$= 16\,\delta^{-1} \int_{|x-x_0|\le\delta/2} \left| \Delta_2 f(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}) \right|^2 dx$$

$$\leq \omega(\frac{\pi}{n_k})\, 16\,\delta^{-1} \int_{|x-x_0|\le\delta/2} \left| \Delta_2 f(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}) \right| dx \,.$$

Therefore

$$8 \sum_{\frac{n_k}{2}}^{n_k} (a_n^2 + b_n^2) \sin^4(n\pi/2n_k)$$

$$\leq 16\,\delta^{-1} \omega(\frac{\pi}{n_k}) \int_{|x-x_0|\le\delta/2} \left| \Delta_2 f(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}) \right| dx \,.$$

Since $\sin x \geq 1/\sqrt{2}$ for $(\pi/4) \leq x \leq (\pi/2)$, we get

$$2 \sum_{\frac{n_k}{2}}^{n_k} (a_n^2 + b_n^2) \leq 16\,\delta^{-1} \omega(\frac{\pi}{n_k}) \int_{|x-x_0|\le\delta/2} \left| \Delta_2 f(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}) \right| dx$$

and hence

$$\sum_{0 \le j \le [\frac{\delta n_k}{8\pi}]} \sum_{\frac{n_k}{2}}^{n_k} (a_n^2 + b_n^2)$$

$$\leq 8\,\delta^{-1} \omega(\frac{\pi}{n_k}) \int_{|x-x_0|\le\delta/2} \sum_{j=0}^{[\frac{\delta n_k}{8\pi}]} \left| \Delta_2 f(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}) \right| dx.$$

But
$$\sum_{j=0}^{\left[\frac{\delta n_k}{8\pi}\right]} \left|\Delta_2 f\left(x + \frac{2j\pi}{n_k}, \frac{\pi}{2n_k}\right)\right| \le V ,$$

where V is the total 2^{nd} variation of $f(x)$ in $(x_0 - \frac{3\delta}{4}, x_0 + \delta)$.
Therefore,

$$\frac{\delta\, n_k}{8\pi} \sum_{\frac{n_k}{2}}^{n_k} (a_n^2 + b_n^2) \le 8\, \delta^{-1} \omega\left(\frac{\pi}{n_k}\right) V \frac{\delta}{2} ,$$

i.e.
$$\sum_{\frac{n_k}{2}}^{n_k} (a_n^2 + b_n^2) \le A\, \omega\left(\frac{\pi}{n_k}\right) n_k^{-1}$$

and hence

(11)
$$\sum_{2^{m-1}+1}^{2^m} (a_n^2 + b_n^2) \le A\, \omega\left(\frac{\pi}{2^m}\right) 2^{-m} .$$

Now it follows by Hölder's inequality that

$$\sum_{2^{m-1}+1}^{2^m} \rho_n^{\beta} \le \left(\sum_{2^{m-1}+1}^{2^m} \rho_n^2\right)^{\beta/2} \left(\sum_{2^{m-1}+1}^{2^m} 1\right)^{1-\frac{\beta}{2}}$$

$$\le A_1\, \omega^{\beta/2}\left(\frac{\pi}{2^m}\right) 2^{-m\beta/2}\, 2^{m\left(1-\frac{\beta}{2}\right)}, \text{ where } \rho_n^2 = a_n^2 + b_n^2$$

$$\le B \frac{2^{-m\alpha\beta/2}\; 2^{-m\beta/2}\; 2^{m\left(1-\frac{\beta}{2}\right)}}{\left[l_1(\pi/2^m)\, l_2^{1+\varepsilon}(\pi/2^m)\right]^{(\alpha+2)\beta/2}}$$

$$= \frac{B}{l_1(\pi/2^m)\, l_2^{1+\varepsilon}(\pi/2^m)} , \text{ for } \beta = 2/(\alpha+2)$$

$$\le \frac{C}{(m-2) \log^{1+\varepsilon}(m-2)} .$$

Thus we conclude that

$$\sum_{1}^{\infty} \rho_n^{\beta} \leq C \sum_{m=4}^{\infty} \frac{1}{(m-2) \log^{1+\varepsilon} (m-2)}$$

$$< \infty$$

and hence

$$\sum_{1}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}) < \infty \text{ , for } \beta = 2/(\alpha+2) \text{ .}$$

Theorem 20 follows from Theorem 21 by choosing $\beta = 1$ and $\alpha = 0$.

We also prove the following theorem:

<u>Theorem 22.</u> <u>If</u> (i) $n_{k+1} - n_k \to \infty$ <u>as</u> $k \to \infty$,

(ii) $f(x)$ <u>is of bounded</u> 2^{nd} <u>variation in some interval</u> $|x - x_0| \leq \delta$,

(iii) $\omega(h) = O\left(\dfrac{h^{\alpha}}{\left[l_1(h)\, l_2(h) \ldots l_k^{1+\varepsilon}(h)\right]^2}\right)$,

$\varepsilon > 0$, $0 < \alpha < 1$, in $|x - x_0| \leq \delta$,

<u>then the series</u>

$$\sum_{1}^{\infty} n^{\beta/2} (|a_n| + |b_n|)$$

<u>converges for</u> $\beta = \alpha$.

<u>Proof.</u> By Cauchy-Schwarz's inequality we obtain

$$\sum_{2^{m-1}+1}^{2^m} \rho_n \leq \left(\sum_{2^{m-1}+1}^{2^m} \rho_n^2 \right)^{\frac{1}{2}} \left(\sum_{2^{m-1}+1}^{2^m} 1 \right)^{\frac{1}{2}}$$

and hence it follows from (11) that

$$\sum_{2^{m-1}+1}^{2^m} \rho_n \leq c_1 \sqrt{\omega(\pi/2^m)\, 2^{-m}} \sqrt{2^m} \,.$$

Therefore,

$$\begin{aligned}
\sum_{2^{m-1}+1}^{2^m} n^{\beta/2} \rho_n &\leq c_1\, 2^{m\,\beta/2} \sqrt{\omega(\pi/2^m)} \\
&\leq \frac{c_2 \quad 2^{m\,\beta/2} \quad 2^{-m\,\alpha/2}}{l_1(\pi/2^m)\; l_2^{1+\varepsilon}(\pi/2^m)} \\
&= \frac{c_2}{l_1(\pi/2^m)\; l_2^{1+\varepsilon}(\pi/2^m)}, \quad \text{for } \beta = \alpha \\
&\leq \frac{c_2}{(m-2)\, \log^{1+\varepsilon}(m-2)} \,.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{1}^{\infty} n^{\beta/2} \rho_n &\leq c_2 \sum_{m=4}^{\infty} \frac{1}{(m-2)\, \log^{1+\varepsilon}(m-2)} \\
&< \infty \,.
\end{aligned}$$

Thus we conclude that

$$\sum_{1}^{\infty} n^{\beta/2}(|a_n|+|b_n|) < \infty, \quad \text{for } \beta = \alpha \,.$$

This completes the proof of Theorem 22.

Masako Sato[1)] considered the Fourier series with

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1) Masako Sato [30]

a certain gap and satisfying some continuity condition at a point instead of in a small interval, and he proved the following two theorems:

Theorem I. Let $(1/2) < a < \alpha < 1$, $0 < p < (2-\alpha)/3$

and $p/2 < \alpha - a \leq (2-\alpha-p)/4$.

If $\frac{1}{k^{2\alpha-2a-p}} < n_k \leq e^{\frac{2k}{2+\alpha+p}}$,

$$|n_{k\pm1} - n_k| > 4\, o\, k\, n_k^p$$

and

(i) $$h^p \int_0^{h^p} |f(t)-f(t\pm h)|^2 dt = O(h^{2\alpha}) \text{ as } h \to 0,$$

(ii) $$\frac{1}{T}\int_0^T |f(t)-f(t\pm h)|^2 dt = O(1) \text{ unif. in } T > h^p,$$

then

(12) $$\sum_1^\infty (|a_{n_k}| + |b_{n_k}|) < \infty .$$

Theorem J. Let $(1/2) < a < \alpha < 1$, $0 < p < (1-\alpha)/2$,

$$Y > \frac{1}{2\alpha-2a-p}, \quad p/2 < \alpha - a < (1+p)/4$$

If $n_k = [k^Y]$, $(k = 1,2,3,\dots)$, and the conditions (i) and (ii) of Theorem I are satisfied, then (12) holds.

We give below generalizations of these theorems analogous to Theorems 21 and 22.

<u>Theorem 23.</u> <u>Under the hypothesis of Theorem I, (2) holds for $\beta > 2/(2\alpha+1)$.</u>

<u>Theorem 24.</u> <u>Under the hypothesis of Theorem I,</u>

$$\sum_{1}^{\infty} n^{\beta-\frac{1}{2}}(|a_n| + |b_n|) < \infty, \text{ for } \beta < \alpha .$$

Extensions to Theorem 23 and Theorem 24 can be given analogous to the extension of Theorem I given in the form of Theorem J.

<u>Proof of Theorem 23.</u> It has been proved by Sato[1)] that under the hypothesis of this theorem

$$\sum_{n_k}^{2n_k} \rho_n^{\,2} = O(n_k^{-2\alpha}) .$$

Hence, we have

$$(13) \qquad \sum_{2^{m-1}+1}^{2^m} \rho_n^{\,2} \leq A \; 2^{-2m\alpha} .$$

Applying Hölder's inequality, we get

$$\sum_{2^{m-1}+1}^{2^m} \rho_n^{\,\beta} \leq A' \; 2^{-m\alpha\beta} \; 2^{m(1-\frac{\beta}{2})} .$$

Therefore,

$$\sum_{1}^{\infty} \rho_n^{\,\beta} = \sum_{m=1}^{\infty} \sum_{2^{m-1}+1}^{2^m} \rho_n^{\,\beta}$$

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1) Sato [30]

$$\leq A' \sum_{m=1}^{\infty} \frac{1}{2^{m(\alpha\beta+\beta-1)}}$$

$$< \infty, \quad \text{for } \beta > 2/(2\alpha+1) .$$

Since $|a_n|^\beta$ and $|b_n|^\beta$ do not exceed ρ_n^{β}, we have the conclusion

$$\sum_1^{\infty} (|a_n|^\beta + |b_n|^\beta) < \infty, \text{ for } \beta > 2/(2\alpha+1).$$

<u>Proof of Theorem 24.</u> Applying Schwarz's inequality and using the conclusion (19), we have

$$\sum_{2^{m-1}+1}^{2^m} \rho_n = \left(\sum_{2^{m-1}+1}^{2^m} \rho_n^2 \right)^{\frac{1}{2}} \left(\sum_{2^{m-1}+1}^{2^m} 1 \right)^{\frac{1}{2}}$$

$$\leq A \;\; 2^{-m\alpha} \;\; 2^{m/2} .$$

Therefore,

$$\sum_{2^{m-1}+1}^{2^m} n^{\beta-\frac{1}{2}} \rho_n \leq A' \; 2^{m(\beta-\frac{1}{2})} \; 2^{-m(\alpha-\frac{1}{2})}$$

$$\leq A' \;\; 2^{m(\alpha-\beta)}$$

and hence

$$\sum_1^{\infty} n^{\beta-\frac{1}{2}} \rho_n = \sum_{m=1}^{\infty} \sum_{2^{m-1}+1}^{2^m} n^{\beta-\frac{1}{2}} \rho_n$$

$$\leq A' \sum_{m=1}^{\infty} \frac{1}{2^{m(\alpha-\beta)}}$$

$$< \infty, \text{ for } \beta < \alpha .$$

Thus we conclude that

$$\sum_1^{\infty} n^{\beta-\frac{1}{2}} (|a_n| + |b_n|) < \infty, \text{ for } \beta < \alpha .$$

CHAPTER VI

ON BEURLING'S TEST FOR ABSOLUTE CONVERGENCE OF FOURIER SERIES

Let f and g be continuous, even functions of period 2π, with Fourier cosine coefficients a_n and c_n resp. and let f be a contraction of g.

A. Beurling[1] established a test for the absolute convergence of a Fourier series, which was subsequently generalized by R. P. Boas Jr.[2]. Boas's theorem is:

<u>Theorem K.</u> <u>If</u> (i) $|c_k| \leq r_k$, <u>and</u>

$$\text{(ii)} \quad \sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{k=1}^{n} k^2 r_k^2\right)^{1/2} + \sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{k=n+1}^{\infty} r_k^2\right)^{1/2} < \infty,$$

<u>then</u> $\sum_{n=1}^{\infty} |a_n| < \infty$,

<u>where</u> a_n <u>and</u> c_n <u>are the Fourier cosine coefficients of</u> f <u>and</u> g <u>respectively.</u>

We have generalised the above mentioned theorem of Boas. Our theorem is as follows:

<u>Theorem 25.</u> <u>Let f and g be continuous even functions of period 2π, with Fourier coefficients</u>

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1) Beurling [5] 2) Boas Jr. [4]

a_n and c_n respectively.

If $0 < \beta \leq 1$, $0 \leq \gamma < \beta/2$ and

$$\sum_{n=1}^{\infty} n^{\gamma-\frac{3}{2}\beta}\left(\sum_{k=1}^{n} k^2 c_k^2\right)^{\beta/2} + \sum_{n=1}^{\infty} n^{\gamma-(\beta/2)}\left(\sum_{k=n+1}^{\infty} c_k^2\right)^{\beta/2} < \infty,$$

Then

$$\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta} < \infty .$$

It is easy to see that our theorem reduces to Boas theorem, on taking $\gamma = 0$ and $\beta = 1$.

Proof. Taking into account the fact that f is the contraction of g, we have

$$\int_0^{\pi} |f(x+\delta) - f(x)|^2 dx \leq \int_0^{\pi} |g(x+\delta) - g(x)|^2 dx.$$

Writing $\delta = \pi/n$, where n is a positive integer and using Parseval's theorem we get

$$(1) \qquad \sum_{k=1}^{\infty} a_k^2 \sin^2(k\pi/2n) \leq \sum_{k=1}^{\infty} c_k^2 \sin^2(k\pi/2n).$$

Since $\sin x \geq (2x/\pi)$, for $0 \leq x \leq \pi/2$, we deduce from (1),

$$(2) \qquad \begin{aligned} \sum_{k=1}^{n} k^2 a_k^2 &\leq n^2 \sum_{k=1}^{n} a_k^2 \sin^2(k\pi/2n) \\ &\leq n^2 \sum_{k=1}^{\infty} a_k^2 \sin^2(k\pi/2n) \\ &\leq n^2 \sum_{k=1}^{\infty} c_k^2 \sin^2(k\pi/2n). \end{aligned}$$

Let

$$\phi_n = (1/n)\sum_{k=1}^{n} k|a_k|^{\beta} .$$

Then using Hölder's inequality we have

$$\leq \frac{1}{n}\left(\sum_{k=1}^{n}(k^{1-\beta})^{\frac{1}{1-\beta}}\right)^{1-\beta}\left(\sum_{k=1}^{n}\left(k^{\beta}|a_k|^{\beta}\right)^{\frac{1}{\beta}}\right)^{\beta}$$

and hence

$$(3) \qquad \phi_n \leq n^{2(1-\beta)-1}\left(\sum_{k=1}^{n} k|a_k|\right)^{\beta} .$$

Now,

$$\begin{aligned} n\,\phi_n &= \sum_{k=1}^{n} k|a_k|^{\beta} \\ &= \sum_{k=1}^{n-1} k|a_k|^{\beta} + n|a_n|^{\beta} \\ &= (n-1)\,\phi_{n-1} + n|a_n|^{\beta} . \end{aligned}$$

Therefore,

$$\begin{aligned} |a_n|^{\beta} &= \phi_n - \left(\frac{n-1}{n}\right)\phi_{n-1} \\ &= \phi_n - \phi_{n-1} + \frac{\phi_{n-1}}{n} \\ &\leq \phi_n - \phi_{n-1} + \frac{\phi_{n-1}}{n-1}, \text{ since } \phi_{n-1} \geq 0 . \end{aligned}$$

Hence

$$\begin{aligned} n^{\gamma}|a_n|^{\beta} &\leq n^{\gamma}\phi_n - n^{\gamma}\phi_{n-1} + n^{\gamma}\frac{\phi_{n-1}}{n-1} \\ &\leq n^{\gamma}\phi_n - (n-1)^{\gamma}\phi_{n-1} + n^{\gamma}\frac{\phi_{n-1}}{n-1} . \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{n=1}^{N} n^{\gamma}|a_n|^{\beta} &\leq \sum_{n=1}^{N} n^{\gamma}\phi_n - \sum_{n=1}^{N}(n-1)^{\gamma}\phi_{n-1} \\ &\quad + \sum_{n=1}^{N} n^{\gamma}\frac{\phi_{n-1}}{n-1} . \end{aligned}$$

$$\leq N^{\gamma}\phi_N + \sum_{n=1}^{N} n^{\gamma}\,\frac{\phi_{n-1}}{n-1}.$$

Hence it follows from (3) that

$$\sum_{n=1}^{N} n^{\gamma}|a_n|^{\beta} \leq N^{\gamma}\,N^{2(1-\beta)-1}\Big(\sum_{k=1}^{N} k|a_k|\Big)^{\beta}$$

$$+\sum_{n=1}^{N} n^{\gamma}\,n^{2(1-\beta)-2}\Big(\sum_{k=1}^{n} k|a_k|\Big)^{\beta}$$

$$= N^{1+\gamma-2\beta}\Big(\sum_{k=1}^{N} k|a_k|\Big)^{\beta} + \sum_{n=1}^{N} n^{\gamma-2\beta}\Big(\sum_{k=1}^{n} k|a_k|\Big)^{\beta}$$

$$= I + J.$$

By Schwarz's inequality, we have

$$\sum_{k=1}^{n} k|a_k| \leq n^{1/2}\Big(\sum_{k=1}^{n} k^2\,a_k^2\Big)^{1/2}$$

and hence, using the conclusion (2), we deduce

$$(4) \qquad \sum_{k=1}^{n} k|a_k| \leq n^{3/2}\Big(\sum_{k=1}^{\infty} c_k^2\,\sin^2(k\pi/2n)\Big)^{1/2},$$

We shall show that I and J are bounded above as $N \to \infty$.

In virtue of (4) we have

$$J = \sum_{n=1}^{N} n^{\gamma-2\beta}\Big(\sum_{k=1}^{n} k|c_k|\Big)^{\beta}$$

$$\leq \sum_{n=1}^{N} n^{\gamma-2\beta}\,n^{3\beta/2}\Big(\sum_{k=1}^{\infty} c_k^2\,\sin^2(k\pi/2n)\Big)^{\beta/2}$$

$$\leq \sum_{n=1}^{N} n^{\gamma-(\beta/2)}\Big[\sum_{k=1}^{n} c_k^2\sin^2\Big(\frac{k\pi}{2n}\Big) + \sum_{k=n+1}^{\infty} c_k^2\,\sin^2\Big(\frac{k\pi}{2n}\Big)\Big]^{\beta/2}$$

$$\leq \sum_{n=1}^{N} n^{\gamma-(\beta/2)}\Big(\sum_{k=1}^{n} c_k^2\sin^2(k\pi/2n)\Big)^{\beta/2}$$

$$+\sum_{n=1}^{N} n^{\gamma-(\beta/2)}\Big(\sum_{k=n+1}^{\infty} c_k^2\sin^2(k\pi/2n)\Big)^{\beta/2}$$

$$\leq \sum_{n=1}^{N} n^{\gamma-(3\beta/2)} \left(\sum_{k=1}^{n} k^2 c_k^2 \right)^{\beta/2}$$

$$+ \sum_{n=1}^{N} n^{\gamma-(\beta/2)} \left(\sum_{k=n+1}^{\infty} c_k^2 \right)^{\beta/2}$$

and hence it follows from the hypothesis of the theorem that

(5) $$\lim_{N \to \infty} J < \infty .$$

Again using (4), we get

$$I = N^{1+\gamma-2\beta} \left(\sum_{k=1}^{N} k|c_k| \right)^{\beta/2}$$

$$\leq N^{1+\gamma-2\beta} N^{(3\beta/2)} \left(\sum_{k=1}^{\infty} c_k^2 \sin^2(k\pi/2N) \right)^{\beta/2}$$

Therefore

$$I^{2/\beta} \leq N^{(1+\gamma-2\beta)2/\beta} N^3 \sum_{k=1}^{\infty} c_k^2 \sin^2(k\pi/2N)$$

$$= N^{(1+\gamma)2/\beta} N^{-1} \sum_{k=1}^{N} c_k^2 \sin^2(k\pi/2N)$$

$$+ N^{(1+\gamma)2/\beta} N^{-1} \sum_{k=N+1}^{\infty} c_k^2 \sin^2(k\pi/2N)$$

$$\leq N^{(1+\gamma)\frac{2}{\beta}-3} \sum_{k=1}^{N} k^2 c_k^2 + N^{(1+\gamma)\frac{2}{\beta}-1} \sum_{k=N+1}^{\infty} c_k^2$$

$$= T_1 + T_2 .$$

We see that A_n increases as n increases, where

$$A_n = \left(\sum_{k=1}^{n} k^2 c_k^2 \right)^{\beta/2} .$$

Hence

$$\sum_{n=N}^{2N} n^{\gamma-(3\beta/2)} A_n \geq A_N \sum_{N}^{2N} n^{\gamma-(3\beta/2)}$$

$$\geq A_N \; N^{\gamma-(3\beta/2)+1} .$$

Therefore

$$T_1 = N^{(1+\gamma)\frac{2}{\beta}-3} \sum_{k=1}^{N} k^2 c_k^2$$

$$= N^{(1+\gamma)\frac{2}{\beta}-3} A_N^{2/\beta}$$

$$\leq N^{(1+\gamma)\frac{2}{\beta}-3} N^{(\frac{3}{2}\beta-\gamma-1)\frac{2}{\beta}} \left(\sum_{N}^{2N} n^{\gamma-\frac{3}{2}\beta} A_n\right)^{2/\beta}$$

$$= \left(\sum_{n=N}^{2N} n^{\gamma-(3\beta/2)} A_n\right)^{2/\beta} .$$

Hence it follows from the hypothesis that

(6) $$\lim_{N \to \infty} T_1 = 0 .$$

The second series in the hypothesis has decreasing terms which must therefore be $o(1/n)$ and hence we deduce that

$$c_k^2 = o\left(\frac{1}{n^{(1+\gamma-\beta/2)2/\beta}}\right) .$$

Hence

(7) $$T_2 = N^{(1+\gamma)\frac{2}{\beta}-1} \sum_{k=N+1}^{\infty} c_k^2$$

$$= o\left(\frac{N^{(1+\gamma)\frac{2}{\beta}-1}}{N^{(1+\gamma)\frac{2}{\beta}-1}}\right)$$

$$= o(1) .$$

Thus it follows from (6) and (7) that

$$I = o(1) \quad \text{as} \quad N \to \infty$$

which proves, in view of (5), that

$$\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta} = \lim_{N \to \infty} \sum_{n=1}^{N} n^{\gamma}|a_n|^{\beta}$$

$$< \infty.$$

This completes the proof of Theorem 25.