CHAPTER VIII

CONVERGENCE AND RIESZ SUMMABILITY OF A SERIES OF BESSEL FUNCTIONS.

8.1. Let for $f \in L^p[a,b]$, $1 \leq p < \infty$, the integral modulus of continuity of f be denoted by

$$\omega_p(\delta, f) = \sup_{0 \leq h \leq \delta} \left\{ \int_a^b \left| f(x \pm h) - f(x) \right|^p dx \right\}^{1/p},$$

where f is even and periodic with period $2(b-a)$.

Let us define, for $\nu \geq -1/2$,

$$c_\nu(\alpha, \beta) = J_\nu(\alpha)\, Y_\nu(\beta) - J_\nu(\beta)\, Y_\nu(\alpha).$$

Suppose $j_1 < j_2 < j_3 < \dots$ are the successive positive zeros of $J_\nu(t)$, $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ are those of the function $H J_\nu(t) + t J'_\nu(t)$ and $\gamma_1 < \gamma_2 < \gamma_3 \dots$ are of $c_\nu(at, bt)$, where $0 < a < b$.

Let us consider,

$$(8.1.1)\quad \begin{cases} \varphi_\nu(t) = \sqrt{\pi t/2}\, J_\nu(t), \text{ for } t > 0, \\ \varphi_\nu(0) = \lim\limits_{t \to 0+} \varphi_\nu(t); \end{cases}$$

and

$$(8.1.2)\qquad C_m^{(\nu)}(t) = \sqrt{t}\, c_\nu(t\gamma_m, b\gamma_m), \quad a \leq t \leq b.$$

The series

$$(8.1.3)\qquad \sum_{m=1}^{\infty} a_m\, \varphi_\nu(xj_m), \quad 0 \leq x \leq 1,$$

$$(8.1.4)\qquad \sum_{m=1}^{\infty} b_m\, \varphi_\nu(x\lambda_m), \quad 0 \leq x \leq 1, \text{ and}$$

$$(8.1.5)\qquad \sum_{m=1}^{\infty} d_m\, C_m^{(\nu)}(x), \quad 0 < a \leq x \leq b,$$

for arbitrary coefficients a_m, b_m, and d_m are called Bessel series of the first, second and third type, respectively.

Corresponding to any $f \in L^1[0,1]$, the series (8.1.3) and (8.1.4) are Fourier-Bessel series of first and second type (FB-I and FB-II), respectively, if the coefficients are given by

$$(8.1.6)\quad a_m = \frac{2}{\varphi_{\nu+1}{}^2(j_m)} \int_0^1 f(t)\, \varphi_\nu(tj_m)\, dt,$$

and

$$(8.1.7)\quad b_m = \frac{4\,\lambda_m \pi^{-1} \int_0^1 f(t)\, \varphi_\nu(t\lambda_m)\, dt}{(\lambda_m^2 - \nu^2)\, J_\nu^2(\lambda_m) + \lambda_m^2\, J'^2_\nu(\lambda_m)}, \quad m = 1, 2, \ldots,$$

Similarly, for $f \in L^1[a,b]$, the series (8.1.5) is called Fourier-Bessel series of third type (FB-III), if

$$(8.1.8)\quad d_m = \frac{\pi^2\, \gamma_m^2\, J_\nu^2(a\gamma_m) \int_a^b f(t)\, C_m^{(\nu)}(t)\, dt}{2\left\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\right\}},$$

$m = 1, 2, 3, \ldots$.

8.2. Let $j_n < A_n < j_{n+1}$, and let $\gamma_n < B_n < \gamma_{n+1}$. Then, we set, for real $r > 0$, and arbitrary function f,

$$(8.2.1)\quad S_n^r(x,f) = \sum_{m=1}^{n} \left(1 - \frac{j_m^2}{A_n^2}\right)^r a_m\, \varphi_\nu(xj_m)$$

and

$$(8.2.2)\quad R_n^r(x,f) = \sum_{m=1}^{n} \left(1 - \frac{\gamma_m^2}{B_n^2}\right)^r d_m\, C_m^{(\nu)}(x).$$

R. Taberski[1] has proved certain theorems concerning the approximation of Riesz-means (8.2.1) and the convergence of series FB-I. Taberski[2] has proved similar properties for the series FB-II.

In this chapter, we extend Taberski's theorems to series FB-III and establish some more interesting theorems regarding convergence of this series.

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1) Taberski [91]. 2) Taberski [93].

The following theorems have been proved (K_1, K_2, K_3, ... , denote suitable positive constants depending at most upon r and ν only)[1]:

<u>THEOREM 8.1. Let r be a positive integer given by</u> $r \geq \nu+5/2$, $\nu \geq -1/2$, $\nu \neq 0$. <u>Suppose that</u> $f \in C[a,b]$ <u>and</u> $f(a) = f(b) = 0$. <u>Then</u>

$$\max_{a \leq x \leq b} \left| P_n^{r}(x,f) - f(x) \right| \leq K_1\, \omega(1/n,f).$$

The following corollary follows immediately:

<u>COROLLARY 8.1.1. If</u> $f \in \Lambda_\alpha[a,b]$, $0 < \alpha \leq 1$, <u>then</u>,

$$\left| P_n^{r}(x,f) - f(x) \right| = O(n^{-\alpha}), \quad n \to \infty$$

<u>THEOREM 8.2. Let</u> $f \in \Lambda_\alpha[a,b]$, $\frac{1}{p} - \frac{1}{2} < \alpha < 1$, $1 \leq p \leq 2$. <u>If</u> $f(a) = f(b) = 0$, <u>then</u>,

$$\left\{ \sum_{m=n+1}^{\infty} |a_m|^p \right\}^{1/p} = O\left(\frac{1}{n^{\alpha+1/2-1/p}} \right), \text{ as } n \to \infty.$$

The following two theorems of Bernstein type[2] and Zygmund type[3] are easily established as particular cases of Theorem 8.2, by substituting p = 1 and p = 2, respectively:

<u>THEOREM 8.3. If</u> $f \in \Lambda_\alpha[a,b]$, $1/2 < \alpha < 1$, $1 \leq p \leq 2$, $f(a) = f(b) = 0$, <u>then the series FB-III converges absolutely and uniformly in</u> $[a,b]$.

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1) Agrawal & Patel [6]. 2) Bary [12], p. 154. 3) Bary [12], p. 161.

THEOREM 8.4. If $f \in \Lambda_\alpha[a,b]$, $0<\alpha<1$, $f(a)=f(b)=0$, then the coefficients (8.1.8) have the order given by

$$d_n = O(n^{-\alpha}), \quad \text{as } n\to\infty.$$

In particular, the series FB-III converges absolutely and uniformly in $[a,b]$.

THEOREM 8.5. Suppose that $f\in L^p[a,b]$, f is even and periodic of period $2(b-a)$, $1\le p<\infty$. Then,

$$(8.2.3)\qquad \left\{\int_a^b |P_n^r(x,f)-f(x)|^p\,dx\right\}^{1/p} \le K_2\ \left\{\omega_p(1/n,f) + U_p^r(1/n,f)\right\},$$

where $n=1,2,3,\ldots$, and

$$U_p^r(\delta,f) = \left\{\int_a^{a+\delta} |f(x)|^p\,dx\right\}^{1/p} + \delta^r\left\{\int_{a+\delta}^{b-\delta}\left|\frac{f(x)}{(x-a)^r}\right|^p dx\right\}^{\frac{1}{p}} + \delta^{r+1}\left\{\int_{a+\delta}^{b-\delta}\left|\frac{f(x)}{(b-x)^{r+1}}\right|^p dx\right\}^{1/p} + \delta\left\{\int_{a+\delta}^{b-\delta}\left|\frac{f(x)}{x}\right|^p dx\right\}^{1/p} + \left\{\int_{b-\delta}^{b}|f(x)|^p\,dx\right\}^{1/p}.$$

In particular, if f is of bounded variation over $[a,b]$, and if $f\in L^1[a,b]$, then

$$(8.2.4)\qquad \int_a^b |P_n^r(x,f)-f(x)|\,dx = O(1/n), \quad \text{as } n\to\infty;$$

(ii) *the assumptions*

$$(8.2.5)\quad \begin{cases} \omega_2(\delta,f) = O(\delta^{\alpha}), \text{ as } \delta \to 0+, \ 0<\alpha<1, \text{ and} \\ |f(t+a)| + |f(b-t)| = O(t^{\alpha-1/2}), \text{ as } t \to 0+, \end{cases}$$

imply

$$(8.2.6)\qquad \left\{ \int_a^b |P_n^r(x,f) - f(x)|^2 \, dx \right\}^{1/2} = O(1/n^{\alpha}),$$

as $n \to \infty$.

THEOREM 8.6. *Let* $f \in L^2[a,b]$, $r \geqslant \nu + 3/2$, $\nu \geqslant -1/2$, $\nu \neq 0$, *and let*

$$\sum_{n=1}^{\infty} n^{-1/2} \left\{ \omega_2(1/n,f) + W_2^r(1/n,f) \right\} < \infty.$$

Then the series FB-III corresponding to f converges absolutely and uniformly in $[a,b]$.

THEOREM 8.7. *Let* $f \in C[a,b]$ *be a function of bounded variation over* $[a,b]$ *such that* $f(a) = f(b) = 0$. *If* , *moreover,*

$$\sum_{n=1}^{\infty} n^{-1} \left\{ \omega(1/n,f) \right\}^{1/2} < \infty,$$

then the series FB-III corresponding to f converges absolutely and uniformly in $[a,b]$.

Taberski[1] has also studied the Riesz means,

$$(8.2.7)\quad R_n^{\alpha}(x) = \sum_{m=1}^{n}\left(1 - \frac{j_m^{\alpha}}{A_n^{\alpha}}\right) a_m\, \varphi_{\nu}(x j_m), \quad 0 \leqslant x \leqslant 1,$$

corresponding to the Bessel series (8.1.3) for $\alpha > 0$. We, further, study in this chapter the typical Riesz means,

$$(8.2.8)\quad Q_n^{\alpha}(x) = \sum_{m=1}^{n}\left(1 - \frac{\gamma_m^{\alpha}}{B_n^{\alpha}}\right) d_m\, C_m^{(\nu)}(x), \quad a \leqslant x \leqslant b,$$

corresponding to the Bessel series (8.1.5) for $\alpha > 0$.

Whenever the series (8.1.5) is the series FB-III of a function f, the Riesz means (8.2.8) is denoted by $Q_n^{\alpha}(x,f)$.

Our further theorems are as follows:

<u>THEOREM 8.2. Let</u> (8.1.5) <u>be the series FB-III of</u> $f \in C[a,b]$, $f(a) = f(b) = 0$. <u>If</u> $\nu \geqslant -1/2$, $\nu \neq 0$, <u>then</u>,

(i) <u>for</u> $0 < \alpha < 1$,

$$(8.2.9)\quad \max_{a \leqslant x \leqslant b} \left| Q_n^{\alpha}(x,f) - f(x) \right| \leqslant K_3 \cdot \omega(1/n^{\alpha}, f), \; n \geqslant 1;$$

<u>and</u> (ii) <u>for</u> $\alpha \geqslant 1$,

$$(8.2.10)\quad \max_{a \leqslant x \leqslant b} \left| Q_n^{\alpha}(x,f) - f(x) \right| \leqslant K_4 \cdot \omega\left(\frac{\log n}{n}, f\right), \; n \geqslant 1.$$

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[1] Taberski [92].

The following corollaries follow from the above theorem:

<u>COROLLARY 8.8.1.</u> <u>If</u> $f \in \Lambda_\alpha[a,b]$, $0<\alpha<1$, $f(a)=f(b)=0$, <u>then</u>

$$\max_{a \leq x \leq b} \left| Q_n^\alpha(x,f) - f(x) \right| = O(n^{-\alpha^2}), \quad \text{as } n \to \infty.$$

<u>COROLLARY 8.8.2.</u> <u>If</u> $f \in C[a,b]$ <u>and</u> $f(a)=f(b)=0$, <u>then for</u> $\nu \geqslant -1/2$, $\nu \neq 0$, $\alpha>0$, <u>the typical Riesz means</u> $Q_n^\alpha(x,f)$ <u>converges uniformly to</u> $f(x)$ <u>in</u> $[a,b]$.

<u>COROLLARY 8.8.3.</u> <u>Let</u> $f \in L^p[a,b]$, $1 \leqslant p < \infty$. <u>Then for</u> $\nu \geqslant -1/2$, $\nu \neq 0$ <u>and</u> $\alpha>0$,

$$(8.2.11) \quad \int_a^b \left| Q_n^\alpha(x,f) - f(x) \right|^p dx \longrightarrow 0, \quad \text{as } n \to \infty.$$

<u>THEOREM 8.9.</u> <u>If the series</u> (8.1.5) <u>is the Fourier-Bessel series of third type for</u> $f \in L^p$, $1 \leqslant p < \infty$, <u>and if</u> $\nu \geqslant -1/2$, $\nu \neq 0$, $0<\alpha<1$, <u>then</u>

$$(8.2.12) \quad \left\{ \int_a^b \left| Q_n^\alpha(x,f) - f(x) \right|^p dx \right\}^{1/p} \leqslant K_6 \left\{ \omega_p(1/n^\alpha, f) + W_p'^{\alpha}(1/n, f) \right\},$$

<u>where</u> $n \geqslant 1$, <u>and</u>

$$W_p'^{\alpha}(\delta,f) = \left\{ \int_a^{a+\delta} |f(x)|^p dx \right\}^{1/p} + \delta^\alpha \left\{ \int_{a+\delta}^{b-\delta} \left| \frac{f(x)}{x} \right|^p dx \right\}^{1/p} +$$

$$+ \delta^\alpha \left\{ \int_{a+\delta}^{b-\delta} \left| \frac{f(x)}{(x-a)^\alpha} \right|^p dx \right\}^{1/p} +$$

$$+\delta^{\alpha}\left\{\int_{a+\delta}^{b-\delta}\left|\frac{f(x)}{(b-x)^{\alpha}}\right|^{p}dx\right\}^{1/p}+\delta\left\{\int_{b-\delta}^{b}|f(x)|^{p}dx\right\}^{1/p}.$$

THEOREM 8.10. A necessary and sufficient condition for the series (8.1.5) to be the Fourier-Bessel series of a function $f \in C\,[a,b]$, vanishing at a and b, is that

(8.2.13) $$\lim_{n\to\infty} Q_n^{\alpha}(x) = f(x),$$

uniformly in $[a,b]$.

THEOREM 8.11. The series (8.1.5) is the series FB-III of a function $f \in L^1$, or $f \in L^p$, $1 < p < \infty$, if and only if

(8.2.14) $$\lim_{n\to\infty}\int_a^b |Q_n^{\alpha}(x) - f(x)|\,dx = 0,$$

or

(8.2.15) $$\sup_{n\geq 1}\int_a^b |Q_n^{\alpha}(x)|^p\,dx < \infty,$$

respectively.

THEOREM 8.12. Let $f \in L^2[a,b]$, $0<\alpha<1$, $\nu \geq -1/2$, $\nu \neq 0$ and let

$$\sum_{n=1}^{\infty} n^{-1/2}\left\{\omega_2(1/n^{\alpha},f) + W_2^{\alpha}(1/n,f)\right\} < \infty.$$

Then the series FB-III of f converges absolutely and uniformly in $[a,b]$.

THEOREM 8.13. Let $G\,[a,b]$ be a function of bounded *variation over* $[a,b]$ *such that* $f(a) = f(b) = 0$. *If, moreover,*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \omega(1/n^{\alpha}, f) \right\}^{1/2} < \infty, \quad 0 < \alpha < 1, \tag{8.2.16}$$

then the series FB-III of f *converges absolutely and uniformly in* $[a,b]$.

THEOREM 8.14.[1] *If* $f \in L^2[a,b]$ *and* $f \in \Lambda_{\alpha}[a,b]$, $\nu \geqslant -1/2$, $\nu \neq 0$, *then the series FB-III converges uniformly and absolutely when* $1/2 < \alpha < 1$.

A theorem, concerning Fourier series corresponding to a function of Lipschitz class $\Lambda_{\alpha}[a,b]$, has been discussed in the Appendix (§ 0.13).

8.3. The Riesz means given by (8.2.2) can be represented in the following form:

$$R_n^r(x,f) = \int_a^b f(t)\, \theta_n^r(t,x)\, dt, \tag{8.3.1}$$

where

$$\theta_n^r(t,x) = \sum_{n=1}^{n} \left(1 - \frac{\gamma_n^2}{R_n^2}\right)^r \frac{\pi^2 \gamma_n^2 J_{\nu}^2(a\gamma_n)\, C_n^{(\nu)}(x)\, C_n^{(\nu)}(t)}{2\left\{J_{\nu}^2(a\gamma_n) - J_{\nu}^2(b\gamma_n)\right\}}$$

$$= \sqrt{xt} \sum_{n=1}^{n} \left(1 - \frac{\gamma_n^2}{R_n^2}\right)^r \frac{\pi^2 \gamma_n^2 J_{\nu}^2(a\gamma_n)}{2\left\{J_{\nu}^2(a\gamma_n) - J_{\nu}^2(b\gamma_n)\right\}} \times \tag{8.3.2}$$

$$\times\, c_{\nu}(x\gamma_n, b\gamma_n)\, c_{\nu}(t\gamma_n, b\gamma_n);$$

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[1] Agrawal [1].

The following lemmas are needed to prove the Theorems 8.1 to 8.7:

LEMMA 8.1. *If r is a non-negative integer, then*

(8.3.3) $\left|\theta_n^{\,r}(t,x)\right| \leq n K_7,$ *for* $a \leq x \leq b,\ a \leq t \leq b,\ n \geq 1,$

(8.3.4) $\left|\theta_n^{\,r}(t,x)\right| \leq \dfrac{K_8}{n^r |t-x|^{r+1}},$ *for* $t, x \in [a,b],\ n \geq 1,$ $t \neq x.$

PROOF. Let

$$F(w) = \pi w \sqrt{xt}\left(1 - \frac{w^2}{B_n^{\,2}}\right)^r \frac{c_\nu(xw,aw)\ c_\nu(tw,bw)}{c_\nu(aw,bw)},$$

where $0 < a \leq x < t \leq b.$

As in the proof of Lemma 2.6, the residue of $F(w)$ at $w = \gamma_m$ is given by

$$\sqrt{xt}\left(1 - \frac{\gamma_m^{\,2}}{B_n^{\,2}}\right)^r \frac{\pi^2\ \gamma_m^{\,2}\ J_\nu^{\,2}(a\gamma_m)\ c_\nu(x\gamma_m, b\gamma_m)\ c_\nu(t\gamma_m, b\gamma_m)}{2\left\{J_\nu^{\,2}(a\gamma_m) - J_\nu^{\,2}(b\gamma_m)\right\}}.$$

Hence, if we take the rectangle Γ, as in Lemma 2.5, to be the contour of integration, we shall have,

(8.3.5) $$\theta_n^{\,r}(t,x) = \frac{1}{2\pi i}\int_{B_n - i\infty}^{B_n + i\infty} F(w)\, dw,$$

because, $F(w)$ being an odd function of w, its integral between $-i\infty$ and $i\infty$ vanishes, and since by Lemma 5.2,

(8.3.6) $$|F(w)| \leq K_9\, e^{-(t-x)|v|}\left|1 - \frac{(u+iv)^2}{B_n^{\,2}}\right|^r,$$

where $w = u+iv$ lies on the above rectangle, we have,

$$\lim_{B\to\infty} \int_{\pm Bi}^{B_n \pm Bi} F(w)\, dw = 0.$$

Now, by (8.3.5) and (8.3.6), it follows that

$$\left|e_n^r(t,x)\right| \leq K_9 5^{r/2} \left\{ \int_0^{B_n} \left(\frac{v}{B_n}\right)^r e^{-(t-x)v}\, dv + \right.$$

$$\left. + \int_{B_n}^{\infty} \left(\frac{v}{B_n}\right)^{2r} e^{-(t-x)v}\, dv \right\} =$$

$$= \frac{K_9\, 5^{r/2}}{B_n^{\,r}(t-x)^{r+1}} \left\{ \int_0^{(t-x)B_n} u^r e^{-u}\, du + \right.$$

$$\left. + \frac{1}{B_n^{\,r}(t-x)^r} \int_{(t-x)B_n}^{\infty} u^{2r} e^{-u}\, du \right\}$$

$$\leq \frac{K_8}{n^r(t-x)^{r+1}}, \quad a \leq x < t \leq b,$$

on account of Lemma 2.4.

Similarly, if $a \leq t < x \leq b$, by an interchange of x and t, (8.3.4) follows in this case.

To consider any $x, t \in [a,b]$, let us, now, take Γ' to be the rectangle with vertices at $\pm B_n i$, $B_n \pm B_n i$. By Lemma 3.2,

$$|F(w)| = O\left(\frac{1}{\sqrt{xt}}\right), \quad w \in \Gamma'.$$

Hence,

$$\left| \int_{\pm B_n i}^{B_n \pm B_n i} F(w)\, dw \right| \leq K_{10}\, B_n,$$

and

$$\left| \int_{B_n - D_n i}^{B_n + D_n i} F(w)\, dw \right| \leq K_{10}\, B_n .$$

(8.3.3), now, follows.

<u>LEMMA 8.2.</u> <u>Let</u> $f_1(t) = t^{\nu+1/2}$, $t \in [a,b]$. <u>Then, given</u> $r \geq 1$, <u>we have, for</u> $x \in (a,b)$,

$$\left| P_n^{r}(x, f_1) - f_1(x) \right| \leq \frac{K_{11}}{n^{r+1}\,(b-x)^{r+1}} + \frac{K_{12}}{n^{r+1}\,(x-a)^{r+1}} .$$

<u>PROOF.</u> Since

$$\int_a^b t^{\nu+1}\, c_\nu(t\gamma_m, b\gamma_m)\, dt = \frac{b^{\nu+1}}{\gamma_m}\Big\{ J_{\nu+1}(b\gamma_m)\, Y_\nu(b\gamma_m) -$$

$$- J_\nu(b\gamma_m)\, Y_{\nu+1}(b\gamma_m)\Big\} - \frac{a^{\nu+1}}{\gamma_m}\Big\{ J_{\nu+1}(a\gamma_m)\, Y_\nu(a\gamma_m) -$$

$$- J_\nu(a\gamma_m)\, Y_{\nu+1}(a\gamma_m)\Big\}\ \frac{J_\nu(b\gamma_m)}{J_\nu(a\gamma_m)}$$

$$= \frac{2}{\pi \gamma_m^{2}\, J_\nu(a\gamma_m)} \Big\{ b^{\nu}\, J_\nu(a\gamma_m) - a^{\nu}\, J_\nu(b\gamma_m) \Big\},$$

by Watson[1], it follows, by (8.3.1) and (8.3.2), that

$$P_n^{r}(x, f_1) = \sum_{m=1}^{n} \sqrt{x} \left(1 - \frac{\gamma_m^{2}}{B_n^{2}}\right)^{r} \frac{\pi\, J_\nu(a\gamma_m)\, c_\nu(x\gamma_m, b\gamma_m)}{J_\nu^{2}(a\gamma_m) - J_\nu^{2}(b\gamma_m)} \times$$

$$\times \Big\{ b^{\nu}\, J_\nu(a\gamma_m) - a^{\nu}\, J_\nu(b\gamma_m) \Big\}.$$

Let us define,

$$G(w) = 2\sqrt{x}\left(1 - \frac{w^{2}}{B_n^{2}}\right)^{r} \frac{b^{\nu}\, c_\nu(xw, aw) - a^{\nu}\, c_\nu(xw, bw)}{w\, c_\nu(aw, bw)}$$

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[1] Watson [105], p. 77, equation (12).

Residue of $G(w)$ at $w = \gamma_m$ is

$$\sqrt{x}\left(1-\frac{\gamma_m^2}{B_n^2}\right)^r \frac{\pi J_\nu(a\gamma_m)\, c_\nu(x\gamma_m, b\gamma_m)\{b^\nu J_\nu(a\gamma_m) - a^\nu J_\nu(b\gamma_m)\}}{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)}.$$

Also, the residue of $G(w)$ at $w = 0$ is given by

$$2\sqrt{x}\lim_{w\to 0}\frac{b^\nu c_\nu(xw, aw) - a^\nu c_\nu(xw, bw)}{c_\nu(aw, bw)} =$$

$$= 2\sqrt{x}\,\frac{b^\nu \frac{a^\nu}{b^\nu}\cdot\frac{b^\nu}{a^\nu}\left\{\frac{x^\nu}{a^\nu}-\frac{a^\nu}{x^\nu}\right\} - a^\nu\left\{\frac{x^\nu}{b^\nu}-\frac{b^\nu}{x^\nu}\right\}}{\frac{a^\nu}{b^\nu}-\frac{b^\nu}{a^\nu}}$$

$$= -2x^{\nu+1/2},$$

since[1),

$$\lim_{w\to 0}\frac{J_\nu(aw)}{J_\nu(bw)} = \frac{a^\nu}{b^\nu}, \quad \text{and} \quad \lim_{w\to 0}\frac{Y_\nu(aw)}{Y_\nu(bw)} = \frac{b^\nu}{a^\nu},$$

when ν is not a negative integer or zero.

Considering the rectangle Γ, as in Lemma 2.5, to be the contour of integration, we now obtain,

$$P_n^r(x, f_1) = x^{\nu+1/2} + \frac{1}{2\pi i}\int_{B_n-\infty i}^{B_n+\infty i} G(w)\,dw.$$

Hence,

$$\left|P_n^r(x, f_1) - f_1(x)\right| = \left|\frac{1}{2\pi i}\int_{B_n-\infty i}^{B_n+\infty i} G(w)\,dw\right|$$

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1) Lebedev [62], pp. 104, 134, 135.

$$\leq \frac{K_{13}}{\pi} \int_0^\infty \left| \frac{v^2 - 2B_n v i}{B_n^2} \right|^r \frac{b^{\nu+1/2} e^{-(b-x)|v|} + a^{\nu+1/2} e^{-(x-a)|v|}}{\sqrt{B_n^2 + v^2}} \, dv$$

$$\leq \frac{K_{11}}{n^{r+1} (b-x)^{r+1}} + \frac{K_{12}}{n^{r+1} (x-a)^{r+1}} .$$

This proves the Lemma.

<u>LEMMA 8.3. If r is a posive integer,</u> $r \geq \nu + 3/2$, <u>then</u>

$$\left| \int_a^b \Theta_n^r(t,x) \, dt - 1 \right| \leq \frac{K_{14}}{n^r (x-a)^r} + \frac{K_{15}}{n^{r+1} (b-x)^{r+1}} + \frac{K_{16}}{nx} ,$$

<u>for</u> $x \in [a+1/n, b-1/n]$, $n \geq 2$.

<u>PROOF.</u> Let $a < d < x \leq b-1/n$, and let

$$f_2(t) = t^{\nu+1/2}, \text{ for } a \leq t \leq d,$$
$$= d^{\nu+1/2}, \text{ for } d < t \leq b.$$

Then, using f_1 to be the same as in Lemma 8.2,

$$P_n^r(x,f_2) - f_2(x) = \left(\int_a^d + \int_d^b \right) \left\{ \frac{f_2(t)}{t^{\nu+1/2}} - \frac{f_2(x)}{x^{\nu+1/2}} \right\} \times$$

$$\times \, t^{\nu+1/2} \Theta_n^r(t,x) \, dt + \frac{d^{\nu+1/2}}{x^{\nu+1/2}} \left\{ \int_a^b f_1(t) \, \Theta_n^r(t,x) \, dt - \right.$$

$$\left. - \, x^{\nu+1/2} \right\} .$$

Hence, by Lemma 8.1 and 8.2,

$$\left| P_n^r(x,f_2) - f_2(x) \right| \leq d^{\nu+1/2} \left\{ \frac{K_8}{r \, n^r (x-d)^r} + I \right\} +$$

$$+ \frac{d^{\nu+1/2}}{x^{\nu+1/2} n^{r+1}} \left\{ \frac{K_{11}}{(b-x)^{r+1}} + \frac{K_{12}}{(x-a)^{r+1}} \right\} ,$$

where

$$I = \int_d^b \left|1-(t/x)^{\nu+1/2}\right| \left|e_n^{\,r}(t,x)\right| dt$$

$$\leqslant \left\{\int_d^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{b}\right\} \left|1-(t/x)^{r-1}\right| \times$$

$$\times \left|e_n^{\,r}(t,x)\right| dt =$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

By Lemma 8.1,

$$I_1 \leqslant \frac{K_8}{x^{r-1} n^r} \int_d^{x-1/n} \frac{x^{r-1} - t^{r-1}}{(x-t)^{r+1}}\, dt \leqslant \frac{K_8}{nx}.$$

I_2, I_3 and I_4 also can be evaluated in the same way, and we obtain,

$$I = O\left(\frac{1}{nx}\right), \quad \text{as } n \to \infty.$$

Therefore,

$$\left|P_n^{\,r}(x,f_2) - f_2(x)\right| \leqslant d^{\nu+1/2}\left[\frac{K_8}{r\,n^r (x-d)^r} + \frac{K_{17}}{n\,x} + \right.$$

(8.3.7)

$$\left. + \frac{1}{n^{r+1} a^{\nu+1}}\left\{\frac{K_{11}}{(b-x)^{r+1}} + \frac{K_{12}}{(x-a)^{r+1}}\right\}\right]$$

Also,

$$P_n^{\,r}(x,f_2) - f_2(x) = \int_a^d t^{\nu+1/2}\, e_n^{\,r}(t,x)\, dt +$$

(8.3.8)

$$+ d^{\nu+1/2}\left\{\int_d^b e_n^{\,r}(t,x)\, dt - 1\right\}.$$

By (8.3.7), (8.3.8) and Lemma 8.1, we get,

$$\left| \int_a^b \theta_n^r(t,x)\, dt - 1 \right| \leq \frac{3\, K_8}{r\, n^r (x-d)^r} + \frac{K_{17}}{nx} + $$

$$+ \frac{1}{n^{r+1} a^{\nu+1}} \left\{ \frac{K_{11}}{(b-x)^{r+1}} + \frac{K_{12}}{(x-a)^{r+1}} \right\},$$

where $0 < a < d < x \leq b-1/n$, $a+1/n \leq x \leq b-1/n$.

Taking limit as $d \to a$, the lemma gets proved.

8.4. PROOF OF THEOREM 8.1. We have,

$$P_n^r(x,f) - f(x) = \int_a^b \left\{ f(t) - f(x) \right\} \theta_n^r(t,x)\, dt +$$

$$+ f(x) \left\{ \int_a^b \theta_n^r(t,x)\, dt - 1 \right\}$$

(8.4.1) $$= U_n(x) + V_n(x), \text{ say.}$$

Also,

$$\left| f(t) - f(x) \right| \leq \omega(|t-x|, f)$$

(8.4.2) $$\leq (n|t-x|+1)\, \omega(1/n, f),$$

and by Lemma 8.1,

$$\int_a^b \left| \theta_n^r(t,x) \right| dt = \left\{ \int_a^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{b} \right\} \left| \theta_n^r(t,x) \right| dt$$

(8.4.3) $$\leq \frac{2\, K_8}{r} + 2\, K_7 = K_{18}, \text{ say.}$$

Therefore,

$$\left| U_n(x) \right| \leq \omega(1/n, f) \left\{ nI + \int_a^b \left| \theta_n^r(t,x) \right| dt \right\},$$

where

$$I = \int_a^b |t-x|\, |Q_n^r(t,x)|\, dt \leqslant \frac{2K_8}{n(r-1)} + \frac{K_7}{n};$$

i.e.,

(8.4.4) $|U_n(x)| \leqslant K_{19} \cdot \omega(1/n, f).$

Further, by (8.4.2), (8.4.3) and the condition $f(a) = f(b) = 0$, in case $a \leqslant x \leqslant a+1/n$, $b-1/n \leqslant x \leqslant b$, we have,

(8.4.5) $|V_n(x)| \leqslant K_{20}\, \omega(1/n, f).$

In case $a+1/n < x < b-1/n$, by Lemma 8.3,

$$|V_n(x)| \leqslant K_{14} \frac{|f(x)-f(a)|}{n^r (x-a)^r} + K_{15} \frac{|f(b)-f(x)|}{n^{r+1} (b-x)^{r+1}} + K_{16} \frac{|f(x)-f(a)|}{n x}$$

(8.4.6) $\leqslant K_{21}\, \omega(1/n, f).$

By (8.4.1) and (8.4.4) to (8.4.6), the theorem is, now, proved.

8.5. PROOF OF THEOREM 8.2. By the theorem of best approximation[1)] and Parseval's relation[2)],

(8.5.1) $\sum_{m=n+1}^{\infty} |d_m|^2 \leqslant \int_a^b |f(x) - P_n^r(x,f)|^2\, dx,$

so that by Corollary 8.1.1,

(8.5.2) $\sum_{m=n+1}^{\infty} |d_n|^2 = O(n^{-2\alpha}).$

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1) Alexits [9], §§ 1.2.4, 1.3.3; Sansone [80], p. 11.

2) Goldberg [35], p. 352.

Now, by Hölder's inequality and (8.5.2),

$$(8.5.3) \qquad \sum_{m=n.2^k+1}^{n.2^{k+1}} |d_m|^p \leqslant \frac{M_{44}}{n^{p(\alpha+1/2)-1}} \left\{ 2^{-p(\alpha+1/2)+1} \right\}^k ,$$

for $k = 0, 1, 2, \ldots$ etc..

Since,

$$\sum_{m=n+1}^{\infty} |d_m|^p = \sum_{k=0}^{\infty} \sum_{m=n.2^k+1}^{n.2^{k+1}} |d_m|^p ,$$

the result, now, follows immediately from (8.5.3).

8.6. <u>PROOF OF THEOREM 8.5.</u> We have,

$$\left\{ \int_a^b \left| P_n^r(x,f) - f(x) \right|^p dx \right\}^{1/p} \leqslant$$

$$\leqslant \left[\int_a^b \left\{ \int_a^b |f(t) - f(x)| \, |\theta_n^r(t,x)| \, dt \right\}^p dx \right]^{1/p} +$$

$$+ \left\{ \int_a^b |f(x)|^p \left| \int_a^b \theta_n^r(t,x) \, dt - 1 \right|^p dx \right\}^{1/p} =$$

$$(8.6.1) \qquad = I + I', \text{ say.}$$

Now,

$$I = \left(\int_a^{a+1/n} + \int_{a+1/n}^{b-1/n} + \int_{b-1/n}^{b} \right) \times$$

$$\times \left\{ \int_a^b |f(t) - f(x)| \, |\theta_n^r(t,x)| \, dt \right\}^p dx \Bigg]^{1/p}$$

$$(8.6.2) \qquad I_1 + I_2 + I_3, \text{ say.}$$

$$I_1 = \left[\int_a^{a+1/n}\left\{\left(\int_a^{x+1/n} + \int_{x+1/n}^{b}\right)|f(t)-f(x)|\,|e_n^{r}(t,x)|\times \times dt\right\}^p dx\right]^{1/p}$$

$$\leqslant I_{11} + I_{12}.$$

By using Lemma 8.1 and Zygmund[1)],

$$I_{11} \leqslant nK_7 \left[\int_a^{a+1/n}\left\{\int_{a-x}^{1/n}|f(x+u)-f(x)|\,du\right\}^p dx\right]^{1/p}$$

$$\leqslant nK_7 \left[\int_{-1/n}^{0}\left\{\int_{a-u}^{a+1/n}|f(x+u)-f(x)|^p dx\right\}^{1/p} du + \int_0^{1/n}\left\{\int_a^{a+1/n}|f(x+u)-f(x)|^p dx\right\}^{1/p} du\right]$$

$$\leqslant 2K_7\,\omega_p(1/n,f).$$

In a similar way,

$$I_{12} \leqslant \frac{K_8}{n^r}\left[\int_{1/n}^{b-a-1/n}\left\{\int_a^{a+1/n}|f(x+u)-f(x)|^p dx\right\}^{1/p}\times \frac{du}{u^{r+1}} + \int_{b-a-1/n}^{b-a}\left\{\int_a^{b-u}|f(x+u)-f(x)|^p dx\right\}^{1/p}\frac{du}{u^{r+1}}\right]$$

$$\leqslant \frac{K_8}{n^r}\,\omega_p(1/n,f)\int_{1/n}^{(b-a)}\frac{nu+1}{u^{r+1}}\,du$$

$$\leqslant K_{22}\,\omega_p(1/n,f).$$

---

1) Zygmund [115], Vol.I, p. 19, inequality (9.12).

Therefore,

$$I_1 \leqslant K_{23} \ \omega_p(1/n,f). \tag{8.6.3}$$

Similarly,

$$I_3 \leqslant K_{24} \ \omega_p(1/n,f). \tag{8.6.4}$$

I_2 can be treated by writing it in the form

$$\left[\int_{a+1/n}^{b-1/n} \left\{ \left(\int_a^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{b} \right) \times \right.\right.$$
$$\left.\left. \times |f(t)-f(x)| \, |\theta_n^{r}(t,x)| \, dt \right\}^p dx \right]^{1/p},$$

and we obtain, by using Lemma 8.1 and Zygmund[1),

$$I_2 \leqslant K_{25} \ \omega_p(1/n,f). \tag{8.6.5}$$

It, now, follows from (8.6.2) to (8.6.5) that,

$$I \leqslant K_2 \ \omega_p(1/n,f). \tag{8.6.6}$$

Further,

$$I' = \left\{ \left(\int_a^{a+1/n} + \int_{a+1/n}^{b-1/n} + \int_{b-1/n}^{b} \right) |f(x)|^p \times \right.$$
$$\left. \times \left| \int_a^b \theta_n^{r}(t,x) \, dt - 1 \right|^p dx \right\}^{1/p}$$

$$\leqslant I_1' + I_2' + I_3', \text{ say.} \tag{8.6.7}$$

By (8.4.3),

$$I_1' \leqslant (1+K_{18}) \left\{ \int_a^{a+1/n} |f(x)|^p \, dx \right\}^{1/p}, \tag{8.6.8}$$

---

1) Zygmund [115], Vol.I, p. 19, inequality (9.12).

and

$$(8.6.9)\qquad |I_3'| \leqslant (1 + K_{18}) \left\{\int_{b-1/n}^{b} |f(x)|^p\, dx\right\}^{1/p}.$$

Also, by Lemma 8.5,

$$I_2' \leqslant \frac{K_{14}}{n^r} \left\{\int_{a+1/n}^{b-1/n} \left|\frac{f(x)}{(x-a)^r}\right|^p dx\right\}^{1/p} +$$

$$(8.6.10)\qquad + \frac{K_{15}}{n^{r+1}} \left\{\int_{a+1/n}^{b-1/n} \left|\frac{f(x)}{(b-x)^{r+1}}\right|^p dx\right\}^{1/p} +$$

$$+ \frac{K_{16}}{n} \left\{\int_{a+1/n}^{b-1/n} \left|\frac{f(x)}{x}\right|^p dx\right\}^{1/p}.$$

(8.2.3), now, follows from (8.6.1) and (8.6.6) to (8.6.10).

If f is of bounded variation, it is bounded. Moreover, $\frac{f(t)}{t} \in L^1[a,b]$ and $\omega_1(1/n,f) = O(1/n)$, as $n \to \infty$.

Hence,

$$\int_a^b |P_n^r(x,f) - f(x)|\, dx = O(1/n), \text{ as } n \to \infty.$$

This proves (8.2.4).

Further, (8.2.5) implies that

$$\omega_2^r(1/n,f) \leqslant K_{26}\left\{\frac{n^{-\alpha}}{\sqrt{2\alpha}} + \frac{1}{n^r}.\frac{n^{r-\alpha}}{\sqrt{2r-2\alpha}} + \frac{1}{n^{r+1}}.\frac{n^{r-\alpha+1}}{\sqrt{2r-2\alpha+2}} + \right.$$

$$\left. + \frac{1}{n}.\frac{n^{1-\alpha}}{\sqrt{2-2\alpha}} + \frac{n^{-\alpha}}{\sqrt{2\alpha}}\right\} =$$

$$(8.6.11)\qquad = O(n^{-\alpha}).$$

From (8.6.11) and (8.2.3), (8.2.6) follows. The theorem is, now, completely proved.

8.7. PROOF OF THEOREM 8.6. In view of equation (2.3. 14),

$$c_n^{(\nu)}(t) = O(1/\gamma_m), \text{ as } m \to \infty.$$

Hence, in order to prove the theorem, it is enough to show the convergence of $\sum_{n=1}^{\infty} |d_n|$.

As in the proof of Theorem 8.2, we have, by Theorem 8.5,

$$(8.7.1) \quad \left\{ \sum_{m=n+1}^{\infty} |d_m|^2 \right\}^{1/2} \leqslant K_2 \left\{ \omega_2(1/n,f) + W_2^F(1/n,f) \right\}.$$

Setting

$$r_n = \sum_{m=n+1}^{\infty} |d_m|^2,$$

by (8.7.1), we have[1),]

$$\sum_{n=1}^{\infty} |d_n| \leqslant K_{27} \sum_{n=1}^{\infty} \sqrt{\frac{r_n}{n}}$$

$$\leqslant K_{27} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} K_2 \left\{ \omega_2(1/n,f) + W_2^F(1/n,f) \right\}$$

$$< \infty,$$

in view of the hypothesis. This proves the theorem.

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1) Bary [12], p. 157.

PROOF OF THEOREM 8.7. As in the proof of Theorem 8.6,

$$r_n \leq \int_a^b \left|f(x) - P_n^r(x,f)\right|^2 dx$$

$$\leq \|f(x) - P_n^r(x,f)\|_\infty \int_a^b \left|f(x) - P_n^r(x,f)\right| dx$$

$$\leq K_1\ \omega(1/n,f) \cdot O(1/n),$$

by Theorem 8.1 and (8.2.4) of Theorem 8.5.

The rest of the proof is similar to Theorem 8.6.

8.8. The typical Riesz means given by (8.2.8), corresponding to a function f, can be written in the following form:

$$(8.8.1)\quad C_n^\alpha(x,f) = \int_a^b f(t)\, \phi_n^\alpha(t,x)\, dt, \quad a \leq x \leq b,$$

where

$$(8.8.2)\quad \phi_n^\alpha(t,x) = \frac{\sqrt{tx}}{2} \sum_{m=1}^{n} \left(1 - \frac{\gamma_m^\alpha}{B_n^\alpha}\right) \frac{\pi^2 \gamma_m^2 J_\nu^2(a\gamma_m)}{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)} \times$$

$$\times\ c_\nu(x\gamma_m, b\gamma_m)\ c_\nu(t\gamma_m, b\gamma_m).$$

The following lemmas are required to prove the Theorems 8.8 - 8.13:

LEMMA 8.4. Let $\nu \geq -1/2$. Then

(i) for $\alpha > 0$,

$$(8.8.3)\quad \left|\phi_n^\alpha(t,x)\right| \leq n K_{28},\ t,x \in [a,b],\ n \geq 1;$$

(ii) for $0 < \alpha \leq 1$,

$$(8.8.4)\quad \left|\phi_n^\alpha(t,x)\right| \leq \frac{K_{29}}{n^\alpha |t-x|^{\alpha+1}},\ t,x \in [a,b],\ t \neq x,\ n \geq 1;$$

and (iii) for $\alpha > 1$,

(8.8.5) $$\left|\phi_n^{\alpha}(t,x)\right| \leq \frac{K_{30}}{n|t-x|^2}, \quad t,x \in [a,b],\ t \neq x,\ n \geq 1.$$

PROOF. Let

$$G(w) = \sqrt{xt}\left(1-\frac{w^{\alpha}}{B_n^{\alpha}}\right)\frac{\pi w\, c_{\nu}(xw,aw)\, c_{\nu}(tw,bw)}{c_{\nu}(aw,bw)},$$

where $a \leq x < t \leq b$.

Then, as in the proof of Lemma 8.1,

$$\phi_n^{\alpha}(t,x) = \frac{1}{2\pi i}\int_{B_n-i\infty}^{B_n+i\infty} G(w)\,dw +$$

(8.8.6) $$+\frac{\sqrt{xt}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{w^{\alpha+1}}{B_n^{\alpha}}\,\frac{c_{\nu}(xw,aw)\, c_{\nu}(tw,bw)}{c_{\nu}(aw,bw)}\,dw.$$

Also, for $0 < \alpha \leq 1$,

(8.8.7) $$\left|1-\frac{(B_n+vi)^{\alpha}}{B_n^{\alpha}}\right| \leq K_{31}\frac{|v|^{\alpha}}{B_n^{\alpha}}, \quad -\infty < v < \infty,$$

and for $\alpha > 1$,

(8.8.8) $$\left|1-\frac{(B_n+vi)^{\alpha}}{B_n^{\alpha}}\right| \leq \begin{cases} K_{31}, & |v| \leq B_n, \\ K_{31}\dfrac{|v|^{\alpha}}{B_n^{\alpha}}, & |v| > B_n. \end{cases}$$

The lemma now follows from (8.8.6) to (8.8.8), in a way similar to Lemma 8.1.

The following lemma follows easily from Lemma 8.4:

LEMMA 8.5. For $\nu \geq -1/2$, $\nu \neq 0$ and any $\alpha > 0$,

$$\int_a^b \left|\phi_n^{\alpha}(t,x)\right| dt \leq K_{32}, \quad x \in [a,b], \quad n \geq 1.$$

The following lemmas can be proved by techniques similar to those used in Lemmas 8.2 and 8.3:

LEMMA 8.6. Let $f_1(t) = t^{\nu+1/2}$ for $t \in [a,b]$, $\nu \geqslant -1/2$, $\nu \neq 0$. Then (i) for $0 < \alpha \leqslant 2$, $a < x < b$, $n \geqslant 1$,

$$(8.8.9) \qquad \left| Q_n^{\alpha}(x,f_1) - f_1(x) \right| \leqslant \frac{K_{33}}{n^{\alpha}} \left\{ \frac{1}{(b-x)^{\alpha}} + \frac{1}{(x-a)^{\alpha}} \right\};$$

and (ii) for $\alpha > 2$, $a < x < b$, $n \geqslant 1$,

$$(8.8.10) \qquad \left| Q_n^{\alpha}(x,f_1) - f_1(x) \right| \leqslant \frac{K_{34}}{n^{2}} \left\{ \frac{1}{(b-x)^{2}} + \frac{1}{(x-a)^{2}} \right\}.$$

LEMMA 8.7. The following estimates are valid for $\nu \geqslant -1/2$, $\nu \neq 0$:

(i) if $0 < \alpha < 1$, $a < x < b$, $n \geqslant 1$,

$$(8.8.11) \qquad \left| \int_a^b \phi_n^{\alpha}(t,x)\, dt - 1 \right| \leqslant \frac{K_{35}}{n^{\alpha}} \left\{ \frac{1}{x} + \frac{1}{(x-a)^{\alpha}} + \frac{1}{(b-x)^{\alpha}} \right\};$$

(ii) if $\alpha = 1$, $a < x < b$, $n > 1$,

$$(8.8.12) \qquad \left| \int_a^b \phi_n^{\alpha}(t,x)\, dt - 1 \right| \leqslant \frac{K_{36}}{n} \left\{ \frac{\log n}{x} + \frac{1}{x-a} + \frac{1}{b-x} \right\};$$

(iii) if $1 < \alpha \leqslant 2$, $a < x < b$, $n > 1$,

$$(8.8.13) \qquad \left| \int_a^b \phi_n^{\alpha}(t,x)\, dt - 1 \right| \leqslant K_{37} \left\{ \frac{\log n}{nx} + \frac{1}{n(x-a)^2} + \frac{1}{n^{\alpha}(x-a)^{\alpha}} + \frac{1}{n^{\alpha}(b-x)^{\alpha}} \right\};$$

and (iv) if $\alpha > 2$, $a < x < b$, $n > 1$,

$$(8.8.14) \qquad \left| \int_a^b \phi_n^{\alpha}(t,x)\, dt - 1 \right| \leqslant K_{38} \left\{ \frac{\log n}{nx} + \frac{1}{n(x-a)^2} + \right.$$

$$+ \frac{1}{n^2 (x-a)^2} + \frac{1}{n^2 (b-x)^2} \Big\}.$$

8.9. PROOF OF THEOREM 8.8. We write,

$$Q_n^{\alpha}(x,f) - f(x) = \int_a^b \{f(t) - f(x)\} \phi_n^{\alpha}(t,x)\, dt +$$

$$+ f(x) \left\{ \int_a^b \phi_n^{\alpha}(t,x)\, dt - 1 \right\}$$

$$= U_n(x) + V_n(x), \text{ say.}$$

For $0 < \alpha < 1$,

$$|f(t) - f(x)| \leq \omega(1/n^{\alpha}, f) \{ n^{\alpha}|t-x| + 1 \},$$

so that by Lemmas 8.4 and 8.5, we obtain,

$$|U_n(x)| \leq \omega(1/n^{\alpha}, f) \Big\{ K_{32} + K_{29} \left(\int_a^{x-1/n} + \int_{x+1/n}^b \right) \times$$

$$\times \frac{dt}{|t-x|^{\alpha}} + n^{1+\alpha} K_{28} \int_{x-1/n}^{x+1/n} |t-x|\, dt \Big\}$$

$$\leq K_{39}\, \omega(1/n^{\alpha}, f), \qquad a \leq x \leq b.$$

Also, by using Lemma 8.7, for $x \in [a+1/n, b-1/n]$, as in the proof of Theorem 8.1,

$$|V_n(x)| \leq K_{40}\, \omega(1/n^{\alpha}, f),$$

and for $a \leq x < a+1/n$, $b-1/n < x \leq b$, the same is true by Lemma 8.5.

Hence (8.2.9) is proved.

If $\alpha = 1$, by taking the inequality

$$|f(t) - f(x)| \leqslant \omega\left(\frac{\log n}{n}, f\right)\left(\frac{n}{\log n}|t-x| + 1\right),$$

(8.2.10) is proved in a similar way.

The treatment, in case $\alpha > 1$, is also similar.

The proof of Theorem 8.9 is similar to the proof of Theorem 8.5.

8.10. PROOF OF THEOREM 8.10. The necessity of the condition (8.2.13) follows from Corollary 8.8.2.

For the sufficiency part, we have, by using (8.2.8) and the orthogonality of the sequence $C_n^{(\nu)}(x)$,

$$(8.10.1) \qquad \left(1 - \frac{\gamma_n^{\alpha}}{s_n^{\alpha}}\right) d_n = b_n \int_a^b Q_n^{\alpha}(x)\, C_n^{(\nu)}(x)\, dx,$$

where

$$b_n = \frac{\pi^2 \gamma_n^2 J_\nu^2(a\gamma_n)}{2\left\{J_\nu^2(a\gamma_n) - J_\nu^2(b\gamma_n)\right\}}.$$

In view of (8.2.13), by taking limit as $n \to \infty$ in (8.10.1), we obtain,

$$(8.10.2) \qquad d_n = b_n \int_a^b f(x)\, C_n^{(\nu)}(x)\, dx, \; n=1, 2, \ldots .$$

(8.10.2) exhibit the coefficients d_n in the series (8.1.5) as the FB-III coefficients of f, hence, the theorem is proved.

8.11. PROOF OF THEOREM 8.11. Let us consider the series (8.1.5) as the series FB-III of $f \in L^p[a,b]$, $1 < p < \infty$.

Then

$$\left\{\int_a^b |Q_n^{\alpha}(x,f)|^p \, dx\right\}^{1/p} \leqslant \left[\int_a^b \left\{\left(\int_a^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{b}\right) \times \right.\right.$$

$$\left.\left. \times |f(t)| \, |\phi_n^{\alpha}(t,x)| \, dt\right\}^p dx\right]^{1/p}$$

(8.11.1) $$\leqslant I_1 + I_2 + I_3, \text{say.}$$

If $0 < \alpha \leqslant 1$, then by Lemma 8.4,

$$I_1 \leqslant \frac{K_{29}}{n^{\alpha}} \left[\int_a^b \left\{\int_a^{x-1/n} \frac{|f(t)|}{(x-t)^{\alpha+1}} \, dt\right\}^p dx\right]^{1/p}$$

$$\leqslant \frac{K_{29}}{n^{\alpha}} \int_{1/n}^{b-a} \left\{\int_{u+a}^{b} |f(x-u)|^p \, dx\right\}^{1/p} \frac{du}{u^{\alpha+1}}$$

$$K_{41} \|f\|_p.$$

If $\alpha > 1$, then α has to be replaced by 1, so that for all $\alpha > 0$, we have,

(8.11.2) $\quad I_1 \leqslant K_{41} \|f\|_p$, and

(8.11.3) $\quad I_3 \leqslant K_{41} \|f\|_p$.

In a similar way,

(8.11.4) $\quad I_2 \leqslant K_{42} \|f\|_p$.

The necessity of the condition (8.2.13) follows from (8.11.1) to (8.11.4).

To prove the sufficiency, let us consider,

$$F_n(x) = \int_a^x Q_n^{\alpha}(t)\,dt, \quad a \le x \le b.$$

Then $\{F_n\}$ is a sequence of uniformly absolutely continuous functions, for if $\varepsilon > 0$ be given and

$$\sup_{n \ge 1} \| Q_n^{\alpha} \|_p \le K_{43},$$

then $\delta = (\varepsilon/K_{43})^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and for any system S of intervals (a_i, b_i) in $[a,b]$ whose total length does not exceed δ,

$$\sum_i |F_n(b_i) - F_n(a_i)| \le \int_S |Q_n^{\alpha}(t)|\,dt$$

$$\le \left\{ \int_S |Q_n^{\alpha}(t)|^p\,dt \right\}^{1/p} \cdot \delta^{1/q}$$

$$< \varepsilon.$$

In a similar way, it can be shown that the functions F_n are of bounded variation and their complete variations are bounded. Hence, by Helley's theorem[1], there is a subsequence $\{F_{n_k}\}$, which converges pointwise to a function F of bounded variation. From the uniform absolute continuity of $\{F_n\}$, it follows that F is absolutely continuous on $[a,b]$. Hence, we may set,

$$f(x) = F'(x), \quad a \le x \le b.$$

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[1] Bary [11], p. 30.

Now,

$$\int_a^b Q_n^{\alpha}(x)\, C_m^{(\nu)}(x)\, dx = \Big[F_n(x)\, C_m^{(\nu)}(x)\Big]_a^b - \int_a^b F_n(x)\, C_m^{(\nu)\prime}(x)\, dx$$

$$= - \int_a^b F_n(x)\, C_m^{(\nu)\prime}(x)\, dx;$$

so that as in the proof of Theorem 8.10,

$$d_m = \lim_{n \to \infty} b_m \int_a^b Q_n^{\alpha}(x)\, C_m^{(\nu)}(x)\, dx$$

$$= - b_m \int_a^b F(x)\, C_m^{(\nu)\prime}(x)\, dx,$$

where the limit on the right is taken through the subsequence $\{ n_k \}$ only.

Therefore, by integration by parts,

$$d_m = b_m \int_a^b F'(x)\, C_m^{(\nu)}(x)\, dx$$

$$= b_m \int_a^b f(x)\, C_m^{(\nu)}(x)\, dx.$$

Finally, since $\left\{ Q_{n_k}^{\alpha}(x) \right\}$ converges almost everywhere to $f(x)$, we obtain, by Fatou's Lemma,

$$\int_a^b |f(x)|^p\, dx \le \varliminf_{k \to \infty} \int_a^b \left| Q_{n_k}^{\alpha}(x)\right|^p dx \le K_{43}^{\ p},$$

so that $f \in L^p[a,b]$, and the proof of sufficiency is completed.

In case $f \in L^1[a,b]$, the necessity of the condition (8.2.14) follows from Corollary 8.8.3. The sufficiency part may be proved as in case of L^p.

Theorems 8.12 and 8.13 can be proved in a way similar to Theorems 8.6 and 8.7.

8.12. PROOF OF THEOREM 8.14. B.P.Khoti[1] has proved that if $\frac{f(t)}{t^{\nu+1/2}} \in \Lambda_\alpha[a,b]$, $\nu>0$, then

$$(8.12.1) \quad \left|\sum_{m=1}^{n}\left(1-\frac{\gamma_m}{\beta_n}\right) d_m\, c_m^{(\nu)}(x) - f(x)\right| = \begin{cases} O(n^{-\alpha}), & \text{when } 0<\alpha<1; \\ O(\frac{\log n}{n}), & \text{when } \alpha = 1. \end{cases}$$

In fact, (8.12.1) is also true for $\nu \geq -1/2$, $\nu \neq 0$.

The theorem, now, follows from (8.12.1), using the same technique as in the proof of Theorem 8.2.

8.13. APPENDIX. We consider the special case $\nu = 1/2$. By the identity[2],

$$(8.13.1) \quad c_{1/2}(\alpha,\beta) = \frac{2}{\pi\sqrt{\alpha\beta}} \sin(\beta-\alpha),$$

we can write

$$(8.13.2) \quad f(x) \sim \sum_{n=1}^{\infty} d_n\, c_n^{(1/2)}(x) .$$

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1) Khoti [59], Ch. VIII. 2) Watson [103], § 3.4.

Also, if $\nu = \pm 1/2$, we have, from (2.3.13),

(8.13.3) $$\gamma_m \approx \frac{m\pi}{b-a}, \quad \text{for large } n.$$

Using (8.13.1) to (8.13.3), we obtain,

(8.13.4) $$f(x) \sim \sum_{m=1}^{\infty} \left\{ a_m \cos \frac{m\pi x}{b-a} + b_m \sin \frac{m\pi x}{b-a} \right\}, \quad a \leq x \leq b,$$

where

(8.13.5) $$a_m = \frac{2(b-a)d_m \sin \frac{bm\pi}{b-a}}{m\,\pi^2 \sqrt{b}} \quad \text{and}$$

$$b_m = - \frac{2(b-a)\, d_m \cos \frac{bm\pi}{b-a}}{m\,\pi^2 \sqrt{b}}.$$

Hence, (8.13.4) represents a Fourier series corresponding to f in [a,b], which is absolutely and uniformly convergent under the conditions of Theorem 8.4. The following Theorem is, therefore, easily established:

THEOREM 8.15. *Let* $f \in \Lambda_\alpha[a,b]$, $0 < \alpha < 1$, $f(a) = f(b) = 0$, *and let its Fourier series be represented by* (8.13.4). *Then its Fourier coefficients* (8.13.5) *have the order given by*

$$a_n = O(1/n^{\alpha+1}), \quad b_n = O(1/n^{\alpha+1}).$$

In particular, the Fourier series (8.13.4) *of* f *converges absolutely and uniformly in* [a,b].

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