CHAPTER IX

CONVERGENCE AND RIESZ-SUMMABILITY OF A FOURIER-BESSEL SERIES OF SPECIAL KIND.

9.1. Let for $\nu \geqslant -1/2$,

$$Q_\nu(\alpha,\beta) = J_\nu(\alpha)\, Y'_\nu(\beta) - J'_\nu(\beta)\, Y_\nu(\alpha).$$

Let us denote by $k_1 < k_2 < k_3 < \dots$ the successive positive zeros of $E(t)$, where

(9.1.1) $\quad E(t) = J'_\nu(bt)\, Y'_\nu(at) - J'_\nu(at)\, Y'_\nu(bt), \quad 0 < a < b.$

The Fourier-Bessel series

(9.1.2) $$f(x) \sim \sum_{m=1}^{\infty} p_m\, Q_\nu(xk_m, ak_m), \quad 0 < a \leqslant x \leqslant b,$$

where

(9.1.3) $$p_m = \frac{1}{N(k_m)} \int_a^b t\, f(t)\, Q_\nu(tk_m, ak_m)\, dt,$$

and

$$M(k_m) = \int_a^b t\, Q_\nu^2(tk_m, ak_m)\, dt$$

(9.1.4) $$= \frac{2}{\pi^2 k_m^2}\left\{\left(1-\frac{\nu^2}{b^2k_m^2}\right)\frac{J_\nu'^2(ak_m)}{J_\nu'^2(bk_m)} - \left(1-\frac{\nu^2}{a^2k_m^2}\right)\right\},$$

corresponding to any function $f \in L^1[a,b]$, was first used by P.Kito[1] while studying the vibrations of a cylindrical shell immersed in water. We call this series as the Fourier-Bessel series of fourth type (FB-IV).

The n-th partial sum of the series (9.1.2) is given by

(9.1.5) $$S_n(x) = \sum_{m=1}^{n} p_m Q_\nu(xk_m, ak_m) = \int_a^b t\, f(t)\, V_n(t,x)\, dt,$$

where

(9.1.6) $$V_n(t,x) = \sum_{m=1}^{n} \frac{Q_\nu(xk_m, ak_m)\, Q_\nu(tk_m, ak_m)}{M(k_m)}.$$

The series (9.1.2) will be called Riesz-summable or summable(R) to a sum s, if

$$\lim_{n\to\infty} \sum_{m=1}^{n} \left(1-\frac{k_m}{L_n}\right) p_m\, Q_\nu(xk_m, ak_m) = s,$$

where $k_n < L_n < k_{n+1}$. The sum on the left hand side in the above limit is denoted by $R_n^{(\nu)}(x,f)$.

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[1] Kito [56].

By (9.1.3),

(9.1.7) $$R_n^{(\nu)}(x,f) = \int_a^b t\, f(t)\, \theta_n(t,x/R)\, dt$$

where

(9.1.8) $$\theta_n(t,x/R) = \sum_{m=1}^{n} \left(1-\frac{k_m}{L_n}\right) \frac{Q_\nu(xk_m,ak_m)\, Q_\nu(tk_m,ak_m)}{N(k_m)} .$$

F. Kito[1] proved for a Lebesgue integrable function f, which is also of bounded variation in $[a,b]$, that its Fourier-Bessel series (9.1.2) converges to the sum $\frac{1}{2}\{f(x+0)+f(x-0)\}$, when ν is a positive integer.

We have proved[2], in chapter VI of the present thesis, certain properties of (9.1.2) regarding the order of its coefficients and convergence, considering functions of various classes.

In this chapter, we extend the above theorem of Kito to a general case, in which $\nu \geqslant -1/2$, $\nu \neq 0$. We also establish Riesz-summability of this series.

Our theorems are as follows:

THEOREM 9.1. If $f \in L^1[a,b]$ and is of bounded variation in a neighbourhood of x, $a<x<b$, then the series (9.1.2) converges to the sum $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

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1) Kito [57]. 2) Agrawal and Patel [4]; [8].

THEOREM 9.2. *Let* $f \in L^1[a,b]$, $\nu \geqslant -1/2$, $\nu \neq 0$ *and let for* $a < x < b$, *the limits* $f(x \pm 0)$ *exist. Then the series FB-IV of* f *is summable(R) to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$.1)

9.2. In the following lemmas we estimate $V_n(t,x)$, $\Theta_n(t,x/R)$ and certain integrals involving them. These are needed to prove the above theorems (K_1, K_2, K_3, ..., denote suitable positive constants depending at most upon ν):

LEMMA 9.1. *Let*

$$F(w) = \pi w \frac{Q_\nu(tw,aw)\ Q_\nu(xw,bw)}{S(w)}, \quad a < t < x < b.$$

Then, on the rectangle Γ, *whose vertices are* $\pm Li$, $L_n \pm Li$ *in the w-plane, where* $k_n < L_n < k_{n+1}$, *and* L *is to be made to tend to infinity,*

$$F(w) = O\left(\frac{e^{-(x-t)|v|}}{\sqrt{xt}}\right),$$

where $w = u+iv$ *and* n *is sufficiently large.*

PROOF. As in the proof of Theorem 6.1,

$$\varphi(t) = \sqrt{t}\ Q_\nu(tw,aw)$$

is a solution of the differential equation

1) This theorem is analoguous to Fejér's theorem. Refer Titchmarsh [98], p. 414.

$$\frac{d^2\varphi}{dt^2} + \left(w^2 - \frac{\nu^2 - 1/4}{t^2}\right)\varphi = 0.$$

Hence[1], since $\varphi(a) \neq 0$, we have,

$$(9.2.1) \qquad Q_\nu(tw, aw) = \frac{2\cos(t-a)w}{\pi w\sqrt{at}} + O\left(\frac{e^{(t-a)|v|}}{|w|}\right),$$

where $a \leq t \leq b$ and $|w| \to \infty$.

Also[2],

$$(9.2.2) \qquad \varphi'(t) = -\frac{2\sin(t-a)w}{\pi\sqrt{a}} + O(e^{(t-a)|v|}),$$

where $a \leq t \leq b$ and $|w| \to \infty$.

Using (9.2.1) and (9.2.2), we get,

$$(9.2.3) \qquad S(w) = -\frac{2\sin(b-a)w}{\pi w\sqrt{ab}} + O\left(\frac{e^{(b-a)|v|}}{|w|}\right),$$

when $|w| \to \infty$.

By (9.2.1) and (9.2.3) and the inequalities (2.3.15), we obtain,

$$|F(w)| \leq 2\left|\frac{\cos(t-a)w\cos(b-x)w}{\sqrt{tx}\sin(b-a)w}\right| + O\left(\frac{e^{(t-x)|v|}}{|w|}\right) =$$
$$= O\left(\frac{e^{-(x-t)|v|}}{\sqrt{xt}}\right).$$

<u>LEMMA 9.2. The following inequalities hold true for</u> $a \leq x \leq b$, $a \leq t \leq b$:

$$(9.2.4) \qquad |V_n(t,x)| \leq \frac{K_1}{|t-x|}, \quad t \neq x;$$

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[1] Titchmarsh [97], p. 10, eqn. (1.7.3).

[2] Titchmarsh [97], p. 10, eqn. (1.7.6).

$$(9.2.5) \qquad \left|\int_a^t t^{\nu+1}\, V_n(t,x)\, dt\right| \leqslant \frac{K_2}{L_n}\left\{\frac{1}{x-t}+\frac{1}{x-a}\right\},\ x>t;$$

and

$$(9.2.6) \qquad \left|\int_t^b t^{\nu+1}\, V_n(t,x)\, dt\right| \leqslant \frac{K_3}{L_n}\left\{\frac{1}{b-x}+\frac{1}{t-x}\right\},\ t>x.$$

PROOF. We have[1],

$$S'(k_m) = b\left\{J''_\nu(bk_m)\, Y'_\nu(ak_m) - J'_\nu(ak_m)\, Y''_\nu(bk_m)\right\} +$$

$$+ a\left\{J'_\nu(bk_m)\, Y''_\nu(ak_m) - J''_\nu(ak_m)\, Y'_\nu(bk_m)\right\}$$

$$= -b\,\eta\,\frac{2}{\pi b k_m}\left(1-\frac{\nu^2}{b^2 k_m^2}\right) + \frac{a}{\eta}\,\frac{2}{\pi a k_m}\left(1-\frac{\nu^2}{a^2 k_m^2}\right)$$

$$\left[\text{where } \eta = \frac{J'_\nu(ak_m)}{J'_\nu(bk_m)} = \frac{Y'_\nu(ak_m)}{Y'_\nu(bk_m)}\right],$$

$$= -\frac{k_m \pi}{\eta}\, M(k_m).$$

Now, the function F of Lemma 9.1 is meromorphic having poles at $w = k_m$, $m = 1, 2, 3, \ldots$; The residue at k_m is given by,

$$\frac{\pi k_m\, Q_\nu(tk_m, ak_m)\, Q_\nu(xk_m, bk_m)}{S'(k_m)}$$

$$= -\frac{Q_\nu(tk_m, ak_m)\, Q_\nu(xk_m, ak_m)}{M(k_m)},$$

Hence, by Lemma 9.1,

$$(9.2.7) \qquad V_n(t,x) = \frac{1}{2\pi i}\int_{L_n-\infty i}^{L_n+\infty i} F(w)\, dw.$$

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[1] Watson [103], p. 76, eqn. (6).

From (9.2.7), in view of Lemma 9.1, (9.2.4) follows in case $x > t$. The case $x < t$ follows merely by interchanging x and t.

The proofs of (9.2.5) and (9.2.6) are similar to the proof of Lemma 5.3.

<u>LEMMA 9.3. For any real</u> ν, ν <u>not a negative integer or zero</u>, $a < x < b$,

$$\lim_{n \to \infty} \int_a^b t^{\nu+1} V_n(t,x)\, dt = x^{\nu}.$$

<u>PROOF.</u> By (9.1.6), using recurrence relations, we get,

$$\int_a^b t^{\nu+1} V_n(t,x)\, dt =$$

$$= \sum_{m=1}^{n} \frac{2\nu \left\{ b^{\nu-1} J_\nu'(ak_m) - a^{\nu-1} J_\nu'(bk_m) \right\} Q_\nu(xk_m, ak_m)}{k_m^3 M(k_m) J_\nu'(bk_m)}.$$

Now, the function

$$G(w) = \frac{2\nu \left\{ a^{\nu-1} Q_\nu(xw, bw) - b^{\nu-1} Q_\nu(xw, aw) \right\}}{w^2 S(w)}$$

has poles at zero and at $k_1, k_2, \ldots$, etc.

The residue of $G(w)$ at k_m is given by

$$\frac{2\nu\left\{b^{\nu-1}J_\nu'(ak_m)-a^{\nu-1}J_\nu'(bk_m)\right\}\;Q_\nu(xk_m,ak_m)}{k_m^3\,H(k_m)\,J_\nu'(bk_m)}.$$

Once again, using recurrence relations, the residue of $G(w)$ at zero is given by

$$\lim_{w\to 0}\frac{2\nu\Big[a^{\nu-1}\left\{J_\nu(xw)\,Y_\nu'(bw)-J_\nu'(bw)\,Y_\nu(xw)\right\}-b^{\nu-1}\left\{J_\nu(xw)\,Y_\nu'(aw)-J_\nu'(aw)\,Y_\nu(xw)\right\}\Big]}{w\left\{J_\nu'(bw)\,Y_\nu'(aw)-J_\nu'(aw)\,Y_\nu'(bw)\right\}}$$

$$=\frac{\lim\limits_{w\to 0}\left[2a^\nu\,\frac{J_\nu(bw)}{J_\nu(aw)}\cdot\frac{Y_\nu(bw)}{Y_\nu(aw)}\left\{\frac{J_\nu(xw)}{J_\nu(bw)}-\frac{Y_\nu(xw)}{Y_\nu(bw)}\right\}-2b^\nu\left\{\frac{J_\nu(xw)}{J_\nu(aw)}-\frac{Y_\nu(xw)}{Y_\nu(aw)}\right\}\right]}{\lim\limits_{w\to 0}\left\{\frac{J_\nu(bw)}{J_\nu(aw)}-\frac{Y_\nu(bw)}{Y_\nu(aw)}\right\}}$$

$$=\frac{2a^\nu\,\frac{b^\nu}{a^\nu}\cdot\frac{a^\nu}{b^\nu}\left\{\frac{x^\nu}{b^\nu}-\frac{b^\nu}{x^\nu}\right\}-2b^\nu\left\{\frac{x^\nu}{a^\nu}-\frac{a^\nu}{x^\nu}\right\}}{\frac{b^\nu}{a^\nu}-\frac{a^\nu}{b^\nu}}$$

$$=-2x^\nu,$$

since, $\lim_{w\to 0} \frac{J_\nu(aw)}{J_\nu(bw)} = \frac{a^\nu}{b^\nu}$ and $\lim_{w\to 0} \frac{Y_\nu(aw)}{Y_\nu(bw)} = \frac{b^\nu}{a^\nu}$,

when ν is not a negative integer or zero.

Taking Γ, as in Lemma 9.1, to be the contour of integration, we obtain,

$$\int_a^b t^{\nu+1} V_n(t,x)\, dt = \frac{1}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} G(w)\, dw + x^\nu.$$

Since,

$$\left|\int_{L_n-\infty i}^{L_n+\infty i} G(w)\, dw\right| \leqslant K_4 \int_0^\infty \frac{e^{-(x-a)v} - e^{-(b-x)v}}{L_n^2+v^2}\, dv$$

$$\leqslant \frac{K_4}{L_n^2}\left\{\frac{1}{x-a} + \frac{1}{b-x}\right\},$$

the Lemma follows.

LEMMA 9.4. (Khoti[1]): If $|\lambda| < b-a$, then

$$\lim_{L\to\infty} \int_{L_n-Li}^{L_n+Li} \frac{\sin \lambda w}{w \sin (b-a)w}\, dw = O(1/n), \quad \text{as } n\to\infty.$$

LEMMA 9.5. For any real ν, ν not a negative integer or zero and $a < x < b$,

$$(9.2.8) \qquad \lim_{n\to\infty} \int_a^x t^{\nu+1} V_n(t,x)\, dt = \tfrac{1}{2} x^\nu$$

and

$$(9.2.9) \qquad \lim_{n\to\infty} \int_x^b t^{\nu+1} V_n(t,x)\, dt = \tfrac{1}{2} x^\nu.$$

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[1] Khoti [55], Ch. VII, Lemma 4.

PROOF. We have, by (9.2.7),

$$\int_a^x t^{\nu+1} V_n(t,x)\,dt = \frac{1}{2i}\int_{L_n-\infty i}^{L_n+\infty i} \frac{Q_\nu(xw,bw)}{S(w)} \times$$

$$\times\Big[x^{\nu+1}\left\{J_{\nu+1}(xw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(xw)\right\} -$$

$$- a^{\nu+1}\left\{J_{\nu+1}(aw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(aw)\right\}\Big]\,dw$$

(9.2.10) $\quad = I_1 + I_2$, say.

By the recurrence relations and Lemma 9.1,

$$|I_2| = \frac{a^{\nu+1}}{2}\left|\int_{L_n-\infty i}^{L_n+\infty i} \frac{\nu\,Q_\nu(xw,bw)\,Q_\nu(aw,aw)}{a\,w\,S(w)}\,dw\right|$$

$$\leqslant \nu\,a^\nu K_5\int_0^\infty \frac{e^{-(x-a)v}}{L_n^2+v^2}\,dv$$

(9.2.11) $\leqslant \dfrac{\nu a^\nu K_5}{L_n^2(x-a)} \to 0$, as $n\to\infty$

for $a<x$.

Similarly, by using (9.2.1), (9.2.3), the recurrence relations and the asymptotic expansions, we obtain,

$$I_1 = \frac{x^\nu}{2\pi i}\int_{L_n-\infty i}^{L_n+\infty i}\left\{\frac{2\cos(b-x)w\,\sin(x-a)w}{w\sin(b-a)w} + O(|w|^{-2})\right\}dw$$

$$= \frac{x^\nu}{2\pi i}\int_{L_n-\infty i}^{L_n+\infty i}\left\{\frac{1}{w} - \frac{\sin(b+a-2x)w}{w\sin(b-a)w}\right\}dw + O(L_n^{-1})$$

(9.2.12) $\quad = \frac{1}{2}x^\nu + O(\frac{1}{n})$, $\qquad$ by Lemma 9.4.

It is clear that (9.2.8) follows from (9.2.10) to (9.2.12). The limit (9.2.9) follows from (9.2.8) and Lemma 9.3.

LEMMA 9.6. For $a < t \leq b$, $a < x \leq b$, we have,

$$\int_a^t t^{\nu+1} V_n(t,x)\, dt = O(1/n), \text{ as } n \to \infty.$$

PROOF. From the proofs of Lemmas 9.2 and 9.5, we get,

$$(9.2.13) \quad \int_a^t t^{\nu+1} V_n(t,x)\, dt = \lim_{L\to\infty} \frac{1}{2i}\left(\int_{L_n-Li}^{L_n+Li} + \int_{-Li}^{L_n-Li} + \int_{Li}^{L_n+Li}\right) H(w)\, dw,$$

where

$$H(w) = \frac{Q_\nu(xw,aw)}{S(w)}\Big[t^{\nu+1}\left\{J_{\nu+1}(tw)Y_\nu'(aw) - J_\nu'(aw)Y_{\nu+1}(tw)\right\} - a^{\nu+1}\left\{J_{\nu+1}(aw)Y_\nu'(aw) - J_\nu'(aw)Y_{\nu+1}(aw)\right\}\Big]$$

$$= \frac{2t^{\nu+1/2}}{\pi\sqrt{x}}\left\{\frac{\sin(b+t-a-x)}{w\sin(b-a)w} - \frac{\sin(b+a-t-x)w}{w\sin(b-a)w}\right\} + O(|w|^{-2}).$$

Now, for $|\lambda| < b-a$, by using inequalities (2.3.15),

$$\lim_{L\to\infty}\left|\int_{\pm Li}^{L_n\pm Li} \frac{\sin\lambda w}{\sin(b-a)w}\, dw\right| \leq \lim_{L\to\infty}\int_0^{L_n} K_6\, e^{-L(b-a-\lambda)}du$$

$$(9.2.14) \qquad = 0;$$

and if $|\lambda| = b-a$, then

$$(9.2.15) \qquad \int_{-Li}^{L_n-Li} - \int_{Li}^{L_n+Li} \frac{\sin \lambda w}{\sin (t-a)w}\, dw = 0.$$

The lemma now follows from Lemma 9.4 and (9.2.13) to (9.2.15).

LEMMA 9.7. If $f \in L^1[a,b]$, $a \leq A < B \leq b$, and x lies outside the interval (A,B), then

$$\lim_{n \to \infty} \int_A^B t\, f(t)\, V_n(t,x)\, dt = 0.$$

The lemma follows exactly on the line of Watson[1], by using the inequalities of Lemma 9.2.

LEMMA 9.8. The following inequalities hold true for $a < x < b$, $a < t < b$,

$$(9.2.16) \qquad \left|\Theta_n(t,x/R)\right| < \frac{K_7}{L_n\,(t-x)^2}, \text{ if } x \neq t;$$

$$(9.2.17) \qquad \left|\Theta_n(t,x/R)\right| < K_7 L_n.$$

PROOF. As in the proof of Lemma 9.2, if $a < t < x < b$,

$$\left|\Theta_n(t,x/R)\right| = \left|\frac{1}{2\pi i}\int_{L_n-\infty i}^{L_n+\infty i}\left(1-\frac{w}{L_n}\right) F(w)\,dw + \frac{1}{\pi i}\int_0^{\infty} \frac{w F(w)}{L_n}\,dw\right|$$

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[1] Watson [103], pp. 589-591.

$$\leq \frac{K_8}{L_n} \int_0^{\infty} v\, e^{-(x-t)v}\, dv$$

$$\leq \frac{K_7}{L_n(x-t)^2}.$$

The same result is true for $a < x < t < b$, by an interchange of t and x. This proves (9.2.16).

To prove (9.2.17), the contour of integration is taken to be the rectangle with vertices $L_n \pm L_n i$, $\pm L_n i$. Then, for $a < t \leq x < b$, we have,

$$\left|\theta_n(t,x/R)\right| = \left| \frac{1}{2\pi i} \int_{L_n - L_n i}^{L_n + L_n i} (1 - w/L_n)\, F(w)\, dw + \right.$$

$$\left. + \frac{1}{\pi i} \int_0^{L_n i} \frac{w F(w)}{L_n}\, dw \right|$$

$$\leq K_7 L_n.$$

LEMMA 9.9. For any real ν, ν not a negative integer or zero and $a < x < b$,

(9.2.18) $$\lim_{n \to \infty} \int_a^b t^{\nu+1}\, \theta_n(t,x/R)\, dt = x^{\nu};$$

(9.2.19) $$\lim_{n \to \infty} \int_a^x t^{\nu+1}\, \theta_n(t,x/R)\, dt = \tfrac{1}{2} x^{\nu};$$

(9.2.20) $$\lim_{n \to \infty} \int_x^b t^{\nu+1}\, \theta_n(t,x/R)\, dt = \tfrac{1}{2} x^{\nu}.$$

This lemma follows from Lemmas 9.4 and 9.5 and the fact that convergence implies Riesz-summability.

9.3. PROOF OF THEOREM 9.1. By Lemma 9.5, we have,

$$\frac{1}{2}\left\{f(x+0)+f(x-0)\right\} = \lim_{n\to\infty} x^{-\nu} f(x-0)\times$$

$$(9.3.1) \qquad \times\int_a^x t^{\nu+1} V_n(t,x)\,dt +$$

$$+\lim_{n\to\infty} x^{-\nu} f(x+0) \int_x^b t^{\nu+1} V_n(t,x)\,dt.$$

Also, let

$$S_n'(x) = \int_a^x t^{\nu+1}\left\{t^{-\nu} f(t) - x^{-\nu} f(x-0)\right\} V_n(t,x)\,dt +$$

$$+\int_x^b t^{\nu+1}\left\{t^{-\nu} f(t) - x^{-\nu} f(x+0)\right\} V_n(t,x)\,dt$$

$$(9.3.2) \qquad = A + B, \text{ say.}$$

In view of (9.1.5) and (9.3.1), it is enough to prove that $\lim_{n\to\infty} S'(x) = 0$.

By hypothesis, $t^{-\nu}f(t) - x^{-\nu}f(x+0)$ is of bounded variation in a neighbourhood, say, $(x-\eta, x+\eta)$, of x. Hence, for $t\,\varepsilon\,(x, x+\eta)$, we may write,

$$t^{-\nu} f(t) - x^{-\nu} f(x+0) = g_1(t) - g_2(t),$$

where g_1 and g_2 are bounded non-negative increasing functions of t in $(x, x+\eta)$, such that $g_1(x+0) = g_2(x+0) = 0$.

Therefore, for any given $\varepsilon > 0$, it is possible to choose $\delta > 0$, $\delta \leqslant \eta$, such that

$$0 \leqslant g_1(t) < \varepsilon, \quad 0 \leqslant g_2(t) < \varepsilon, \quad \text{for } x \leqslant t \leqslant x+\delta.$$

Now,

$$B = \int_x^{x+\delta} t^{\nu+1}\, g_1(t)\, V_n(t,x)\, dt -$$

$$- \int_x^{x+\delta} t^{\nu+1}\, g_2(t)\, V_n(t,x)\, dt +$$

$$+ \int_{x+\delta}^{b} t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0) \right\} V_n(t,x)\, dt$$

$$= B_1 + B_2 + B_3, \text{ say.}$$

By the second mean value theorem, there is α, $x \leqslant \alpha \leqslant x+\delta$, such that

$$B_1 = g_1(x+\delta) \int_\alpha^{x+\delta} t^{\nu+1}\, V_n(t,x)\, dt$$

$$= O(1/n), \text{ as } n \to \infty, \qquad \text{by Lemma 9.6.}$$

Similarly,

$$B_2 = O(1/n), \text{ as } n \to \infty.$$

By Lemma 9.7,

$$B_3 = o(1), \text{ as } n \to \infty.$$

Hence,

(9.3.3) $\quad B = o(1)$, as $n \to \infty$.

In a similar way, it can be proved that, (9.3.4) $\quad A = o(1)$, as $n \to \infty$.

The theorem is, now, proved from (9.3.2) to (9.3.4).

9.4. PROOF OF THEOREM 9.2. In view of Lemma 9.9,

$$R_n^{(\nu)}(x,f) - \frac{1}{2}\left\{f(x+0)+f(x-0)\right\} =$$

$$= \int_a^x t^{\nu+1}\left\{t^{-\nu}f(t) - x^{-\nu}f(x-0)\right\}\theta_n(t,x/R)\,dt +$$

$$+ \int_x^b t^{\nu+1}\left\{t^{-\nu}f(t) - x^{-\nu}f(x+0)\right\}\theta_n(t,x/R)\,dt$$

(9.4.1) $\quad = I + I'$, say.

Given $\varepsilon > 0$, let us choose $\delta > 0$, such that

$$\left|t^{-\nu}f(t) - x^{-\nu}f(x-0)\right| < \varepsilon,$$

for $x-\delta < t < x$. If, now, n is chosen so large that $\delta > \frac{1}{L_n}$, then,

(9.4.2) $\quad I = \int_a^{x-\delta} + \int_{x-\delta}^{x-1/L_n} + \int_{x-1/L_n}^{x} = I_1 + I_2 + I_3$, say.

By (9.2.16),

$$|I_1| \leq \frac{K_7}{L_n}\int_a^{x-\delta}\frac{t^{\nu+1}\left|t^{-\nu}f(t) - x^{-\nu}f(x-0)\right|dt}{|t-x|^2}$$

$$\leq \frac{K_7}{L_n\delta^2}\int_a^{x-\delta}\left|t\,f(t) - t^{\nu+1}\,x^{-\nu}f(x+0)\right|dt$$

(9.4.3) $\quad = o(1)$, as $n \to \infty$,

since δ is fixed, for any given $\varepsilon > 0$.

Also,

$$|I_2| \leqslant \frac{K_7 \varepsilon}{L_n} \int_{x-\delta}^{x-1/L_n} \frac{t^{\nu+1}}{(x-t)^2}\, dt$$

(9.4.4) $\qquad \leqslant K_9 \varepsilon$;

and

$$|I_3| \leqslant K_7 L_n \varepsilon \int_{x-1/L_n}^{x} t^{\nu+1}\, dt$$

(9.4.5) $\qquad \leqslant K_{10} \varepsilon$.

Since, ε is arbitrary, by (9.4.2) to (9.4.5), it follows that,

(9.4.6) $\qquad I = o(1)$, as $n \to \infty$.

Similarly,

(9.4.7) $\qquad I' = o(1)$, as $n \to \infty$.

The proof is, now, complete by (9.4.1), (9.4.6) and (9.4.7).

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