CHAPTER I

INTRODUCTION

1.1 The expansion of an arbitrary function into a series of Bessel functions of order zero was first given by J. B. Joseph Fourier[1], in 1822, when he was studying the radial flow of heat in a solid circular cylinder. He was led to consider the expansion

$$(1.1.1) \qquad \sum_{n=1}^{\infty} a_n J_0(x j_n)$$

corresponding to an arbitrary function f of a real variable x, where $j_1 < j_2 < j_3 < \dots$ denote the positive zeros of the Bessel function $J_0(x)$ of first kind of order zero, arranged in the ascending order of magnitude.

---

1) Fourier [35], §§ 314-320.

In fact, if the initial temperature of an infinite solid circular cylinder is given by $v = f(r)$, and if the surface $r=a$ be kept at a constant temperature, say zero, then $v(r,t)$, the temperature at time t at a point distant r from the axis, satisfies the equation:[2)]

$$(1.1.2) \qquad \frac{\partial v}{\partial t} = K\left\{\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right\}, \qquad 0 < r < a,$$

with $v(a,t) = 0$ and $v(r,0) = f(r)$, where K is a constant depending upon the density, thermalconductivity and specific heat of the material of the cylinder.

In particular, if

$$v = e^{-K\alpha^2 t}u,$$

where u is a function of r only, then we get

$$(1.1.3) \qquad \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \alpha^2 u = 0.$$

This is Bessel's equation of order zero. An equation of order different from zero is obtained when the initial temperature at $r=a$ is taken to be different from zero.

Fourier-Bessel expansions for arbitrary real functions were also used in Daniell Bernoulli's problem[3)]

---

2) Fourier [33], §§ 118-120; Carslaw and Jaeger [20], Ch.7; Watson [103], § 1.5.

3) Bernoulli [15]; Watson [103], pp. 3,4.

of oscillations of a chain hanging under gravity and in Euler's problem[1] of the vibrations of a stretched circular membrane.

In Euler's problem, r is considered to be the radius of a circular membrane with circumference fixed. Assuming that the motion of the membrane is symmetrical about the center, the transverse displacement $z(r,\theta,t)$, at time t, at the point (r,θ), satisfies the equation:

$$(1.1.4) \qquad \frac{\partial^2 z}{\partial t^2} = c^2 \left\{ \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} \right\},$$

where c is a constant depending upon density and tension of the membrane.

In particular, when

$$z = u \sin(\alpha t + A) \sin(\beta\theta + B),$$

where u is a function of r and α, β, A and B are all constants, the Bessel's equation

$$(1.1.5) \qquad \frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\frac{\alpha^2}{c^2} - \frac{\beta^2}{r^2} \right) u = 0.$$

is obtained.

---

1) Euler [30]; Bowman [18], Ch. II; Watson [103], pp. 5,6.

In both cases above, the series (1.1.1) is used, which is referred to as Fourier-Bessel series of first type in this thesis.

When heat is flowing inside an infinite hollow cylinder perpendicular to its axis, in which surfaces $r=a$ and $r=b$ are kept at zero temperature, the temperature $v(r,t)$ at time t at a radius r, $a<r<b$, is given by[1]

$$\frac{\partial v}{\partial t} = K\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right), \quad 0<a<r<b.$$

In particular, when

$$v = e^{-K\alpha^2 t}\, u,$$

where u is a function of r only, we get (1.1.3), i.e., Bessel's equation of zero order.

Since, in this case, r does not extend to origin, the Bessel function of second kind, $Y_0(t)$, also gets involved and we can express the Fourier-Bessel expansion

$$(1.1.6) \qquad f(x) \sim \sum_{m=1}^{\infty} d_m\, c_0(x\gamma_m, b\gamma_m), \qquad a \le x \le b,$$

where

$$c_0(\alpha,\beta) = J_0(\alpha)\, Y_0(\beta) - J_0(\beta)\, Y_0(\alpha),$$

---

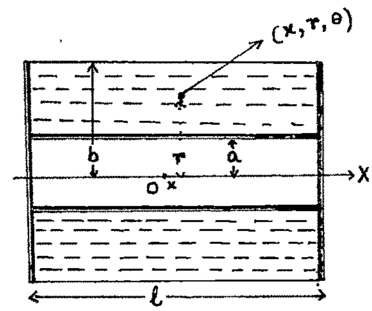
1) Carslaw and Jaeger [20], Ch. VII.

and $\gamma_1 < \gamma_2 < \gamma_3 < \dots$ are the positive zeros of $c_0(at, bt)$.

The series (1.1.6) is called Fourier-Bessel series of third type in the present thesis.

In a similar way, T. Ito[1] has considered the vibration of a cylindrical shell immersed in water, with finite closed boundary walls as shown in the figure:

═══	Rigid wall
-----	Water region
───	Wall of the cylindrical shell making vibrations.



The radial displacement w at any instant t of the shell is expressed by

$$(1.1.7) \qquad w = U \cos\frac{\pi x}{\ell} \sin\nu\theta \cos\omega t,$$

where ν is a whole number giving the number of nodal lines, and ω is the angular frequency of vibration.

He obtained the velocity potential in the form of series

$$(1.1.8) \qquad \sum_{n=1}^{\infty} B_n C_\nu(xk_n, ak_n), \qquad a \le x \le b,$$

where $C_\nu(\alpha, \beta) = J_\nu(\alpha) Y'_\nu(\beta) - J'_\nu(\beta) Y_\nu(\alpha)$, and

1) Ito [56].

$k_1 < k_2 < k_3 < \dots$ are the positive zeros of

$$E(t) = J'_\nu(at)\, Y'_\nu(bt) - J'_\nu(bt)\, Y'_\nu(at).$$

This series of special kind is called Fourier-Bessel series of fourth type.

1.2 BESSEL FUNCTIONS. It is well known that the Bessel function, $J_\nu(x)$, of the first kind of order ν, is a solution of the Bessel's equation[1)]

$$(1.2.1) \qquad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)\, y = 0.$$

If ν is an integer, $J_\nu(x)$, as a series solution of (1.2.1), is given by

$$(1.2.2) \qquad \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{\nu+2m}}{(\nu+m)!\; m!}, \qquad \nu > 0,$$

and $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$, for all integers ν.

If ν is not an integer,

$$(1.2.3) \qquad J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{\nu+2m}}{\Gamma(\nu+m+1)\, \Gamma(m+1)},$$

and, in this case, $J_\nu(x)$ and $J_{-\nu}(x)$ are two linearly independent solutions of (1.2.1), so that

---

1) Tolstov [99], Ch. 8; Whittaker and Watson [105], Ch.17.

$$y = A\,J_\nu(x) + B\,J_{-\nu}(x), \tag{1.2.4}$$

where A and B are arbitrary constants, is a general solution of (1.2.1).

In case, $\nu = n$ is an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. Hence, a second solution of (1.2.1) was found by Hankel[1], which was linearly independent of $J_n(x)$. It was defined as

$$Y_n(x) = \lim_{\nu \to n} \frac{J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n}, \tag{1.2.5}$$

and was called Hankel's function of order n.

Later, the definition was so modified by him that it could be extended to any real number ν as its order, except in case ν is half of an odd integer. He defined

$$Y_\nu(x) = 2\pi \cdot e^{\nu\pi i} \frac{J_\nu(x) \cos\nu\pi - J_{-\nu}(x)}{\sin 2\nu\pi}. \tag{1.2.6}$$

This definition is valid for integral values of as a limit. (The previous definition is a limiting case of the later as $\nu \to n$).

In order to make the definition valid for all real values of ν, it was further modified by Weber[2] and

---

[1] Hankel [41], pp. 469-472.   [2] Weber [104], p. 9.

Schläfli[1] as

$$(1.2.7) \qquad Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} = \frac{\cos\nu\pi}{\pi e^{\nu\pi i}} Y_\nu(x),$$

so that

$$(1.2.8) \qquad Y_n(x) = \lim_{\nu\to n} \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}.$$

This function was called Weber's function or Bessel function of the second kind of order ν. The symbol $Y_\nu(x)$ is after the manner of Nielsen[2].

In this way, the general solution of Bessel's equation (1.2.1), for all real values of ν, may be written as

$$(1.2.9) \qquad y = C\,J_\nu(x) + D\,Y_\nu(x),$$

where C and D are arbitrary constants. This function is usually called a cylindrical function.

In particular, Nielsen[3] considered the linear combinations $J_\nu(x) \pm i\,Y_\nu(x)$ as standard solutions of the Bessel's equation, and called them as functions of the third kind. However, in the honour of Hankel, they came to be known as Hankel's functions of the first and

---

1) Schläfli [82], p. 17. 2) Nielsen [74], p. 11.

3) Nielsen [74], p. 16.

seconds kinds of order ν, and were denoted as

(1.2.10) $$H_\nu^{(1)}(x) = J_\nu(x) + i\,Y_\nu(x),$$

and

(1.2.11) $$H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x).$$

The values of Bessel functions of order $\pm\frac{1}{2}$ are given by[1]

(1.2.12) $$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad \text{and}$$

(1.2.13) $$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x.$$

1.3 PROPERTIES OF BESSEL FUNCTIONS.

A. Recurrence relations. Let $Z_\nu(x)$ denote any of the Bessel functions $J_\nu(x)$, $Y_\nu(x)$, $H_\nu^{(1)}(x)$ or $H_\nu^{(2)}(x)$. Then the following recurrence relations are true[2]:-

(1.3.1) $$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \frac{2\nu}{x} Z_\nu(x);$$

(1.3.2) $$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2\,Z_\nu'(x);$$

(1.3.3) $$x\,Z_\nu'(x) + \nu\,Z_\nu(x) = x\,Z_{\nu-1}(x);$$

(1.3.4) $$x\,Z_\nu'(x) - \nu\,Z_\nu(x) = -x\,Z_{\nu+1}(x);$$

(1.3.5) $$\frac{d}{dx}\{x^\nu Z_\nu(x)\} = x^\nu Z_{\nu-1}(x);$$

(1.3.6) $$\frac{d}{dx}\left\{x^\nu Z_{-\nu}(x)\right\} = -x^\nu Z_{-\nu+1}(x).$$

---

1) Watson [103], p. 54. 2) Watson [103], pp. 45,66,74.

D. Asymptotic expansions. If $|z|$ is sufficiently large and $-\pi < \arg z < \pi$, then the following asymptotic expansions are valid[1]:

$$(1.3.7)\quad J_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) - \frac{4\nu^2-1}{8z}\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) + \frac{\gamma_\nu(z)}{2z^2}\right],$$

$$(1.3.8)\quad Y_\nu(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left[\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) + \frac{4\nu^2-1}{8z}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) + \frac{\delta_\nu(z)}{2z^2}\right],$$

where $\gamma_\nu(z)$ and $\delta_\nu(z)$ remain bounded as $|z| \to \infty$.

Also[2] for large values of $|z|$, real fixed ν and $-\pi + 2\delta \leq \arg z \leq 2\pi - 2\delta$, $\delta > 0$,

$$(1.3.9)\quad H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\left\{1 + \frac{a}{z} + O(|z|^{-1})\right\};$$

and for large $|z|$, real fixed ν and $-2\pi + 2\delta \leq \arg z \leq \pi - 2\delta$, $\delta > 0$,

$$(1.3.10)\quad H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\left\{1 + \frac{b}{z} + O(|z|^{-1})\right\},$$

where a and b are suitable constants.

---

1) Watson [103], p. 199; Young [112]; Gray and Mathews [37], p.70; Zygmund [114].

2) Watson [103], pp. 197-198; Mercer [66].

For any z[1],

(1.3.11) $$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}(1+\theta),$$

where $|\theta| < \exp\left\{\frac{\frac{1}{4}|z|^2}{|\nu_0+1|}\right\} - 1$, and $|\nu_0+1|$ is the smallest of the numbers $|\nu+1|$, $|\nu+2|$, $|\nu+3|$, ...

Also,

(1.3.12) $$J_\nu(z) \sim \frac{z^\nu}{2^\nu\,\Gamma(1+\nu)},$$

as $z \to 0$ uniformly if ν is not a negative integer, and

(1.3.13) $$Y_\nu(z) \sim -\frac{2^\nu\,\Gamma(\nu)}{\pi z^\nu},$$

as $z \to 0$ uniformly if ν is not a negative integer or zero.[2]

C. Inequalities. Let $\nu \geq -1/2$ be a fixed real number. Then[3]

(1.3.14) $$|J_\nu(z)| \leq \frac{|(\frac{1}{2}z)^\nu|}{\Gamma(\nu+1)}\, e^{|I(z)|},$$

for any z, where $I(z)$ denotes the imaginary part of z.

When z is real,[4]

(1.3.15) $$\sup_{z\geq 0}\left\{z^{1/2}|J_\nu(z)|\right\} = \begin{cases}(2/\pi)^{1/2}, & \text{if } -\frac{1}{2}\leq\nu\leq\frac{1}{2},\\ \text{finite and} > (2/\pi)^{1/2}, & \text{if } \nu > 1/2.\end{cases}$$

---

1) Nielsen [75], p. 230; Watson [105], p. 44.

2) Lebedev [62], pp. 104, 134, 135.

3) Watson [103], § 3.31.

4) Szegö [96], p. 167.

For any positive k,[1]

(1.3.16) $\quad |J_\nu(z)| \le A(k)\,|z|^{R(\nu)}$, for $|z| \le k$;

and if $\nu \ne 0$,

(1.3.17) $\quad |\varphi(z)| \le A(k)\,|z|^{-R(\nu)}$, for $|z| \le k$,

where $\varphi(z)$ denotes $Y_\nu(z)$ or $H_\nu^{(1)}(z)$ or $H_\nu^{(2)}(z)$ and $R(\nu)$ denotes the real part of ν. $A(k)$ is a positive constant depending upon k.

Also, for $|z| \ge k$,

(1.3.18) $\quad |z^{1/2} J_\nu(z)| \le A_1(k)\, e^{|I(z)|}$,

(1.3.19) $\quad |z^{1/2} Y_\nu(z)| \le A_1(k)\, e^{|I(z)|}$,

and for $|z| \ge k$, $I(z) \ge 0$,

(1.3.20) $\quad |z^{1/2} H_\nu^{(1)}(z)| \le A_1(k)\, e^{-I(z)}$.

For an unrestricted z, the following inequalities are true:-

(1.3.21) $\quad |z^{1/2} J_\nu(z)| \le A\, e^{|I(z)|}$, for $R(\nu) \ge -1/2$,

(1.3.22) $\quad |z^{1/2} Y_\nu(z)| \le A\, e^{|I(z)|}$, $-1/2 \le R(\nu) \le 1/2$,

(1.3.23) $\quad |z^{1/2} H_\nu^{(1)}(z)| \le A$, for $I(z) \ge 0$, $-\frac{1}{2} \le R(\nu) \le \frac{1}{2}$,

the special case $\nu = 0$ is not exceptional for (1.3.22) and (1.3.23), since $|z^{1/2} \log z| \le A$, for $|z| \le 1$.

---

1) Cherry [23], pp. 25-26; Mac-Robert [65].

When $|z|$ is sufficiently large, it is also true that[1],

$$(1.3.24) \qquad \left| z^{1/2} J_\nu(z) \right| \geqslant A e^{|I(z)|}, \quad \nu+1/2 \geqslant 0.$$

D. Zeros of Bessel functions. It was for the first time stated by Danieli Bernoulli[2] and J.B.J.Fourier[3] that $J_0(z)$ has infinitely many real zeros. This was proved by F.W.Bessel[4]. Subsequently, Lommel[5] observed that $J_\nu(z)$ has an infinity of real zeros, for any given real value of ν. From the expansion (1.2.3), it is clear that corresponding to each positive zero, there is a negative zero also. It is also known that for any real $\nu > -1$, $J_\nu(z)$ does not have purely imaginary zeros[6]. Moreover, the zeros of $J_\nu(z)$ are interlaced with the zeros of $J_{\nu+1}(z)$ for any real $\nu > -1$; i.e., between any two zeros of $J_{\nu+1}(z)$, there is one and only one zero of $J_\nu(z)$ and vice versa. The same is true about the zeros of $J_\nu(z)$ and $J'_\nu(z)$.

---

1) Watson [103], p. 584; Cherry [23], p. 29.

2) Bernoulli [15]. 3) Fourier [33], § 308.

4) Bessel [16], p. 29. 5) Lommel [62].

6) Gray and Mathews [37], p. 81; Watson [103]; Bôcher [17]; Gegenbauer [34]; Porter [78].

The zeros of the functions of the form

(1.3.25) $$A\,J_\nu(z) + B\,z\,J'_\nu(z)$$

are also of interest. A. C. Dixon[1] has shown that if A and B are real and $\nu > -1$, then the expression (1.3.25) has all its zeros real except that it has two purely imaginary zeros when $\frac{A}{B} + \nu < 0$. The cylinder function (1.2.9), for real constants C and D, also has infinitely many positive zeros.[2]

Let $j_1 < j_2 < j_3 < \dots$ denote the positive zeros of $J_\nu(z)$ arranged in the ascending order of magnitude. It is known[3] that when n is large, the asymptotic expansion of the n-th zero is given by

(1.3.26) $$j_n = (n + \tfrac{\nu}{2} - \tfrac{1}{4})\pi + \mu_n ,$$

where

$$\mu_n = \frac{1 - 4\nu^2}{8(n+\nu/2 - 1/4)\pi} + O(n^{-3}).$$

If $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ are the successive positive zeros of (1.3.25), where A and B are positive constants, it is known from the works of E.W.Hobson[4], C.N.Moore[5],

---

1) Dixon [26], p. 7; Lebedev [62], p. 127; Watson [103], p. 480.
2) Watson [103], p. 481. 3) Watson [103], p. 506.
4) Hobson [47], p. 366. 5) Moore [67]; [68]; [69].

M.G. Scherberg[1], W.H. Young[2] etc. that the asymptotic expansion of λ_m, when m is large, is given by

$$\lambda_m = \left(m + \frac{\nu}{2} + \frac{1}{4}\right)\pi + s\pi + \frac{c(m)}{m}, \tag{1.3.27}$$

where s is a fixed, integer, positive or negative, and $c(m)$ is a function bounded for all m.

We are also interested in the zeros of the following functions:

$$c_\nu(az, bz) = J_\nu(az)\, Y_\nu(bz) - J_\nu(bz)\, Y_\nu(az) \tag{1.3.28}$$

and

$$f(z) = J'_\nu(bz)\, Y'_\nu(az) - J'_\nu(az)\, Y'_\nu(bz), \tag{1.3.29}$$

where $0 < a < b$.

Let $\gamma_1 < \gamma_2 < \gamma_3 < \dots$ be the positive zeros of (1.3.28). From the works of D. Naylor and D.M. Willis[3], it follows that for large values of m,

$$\gamma_m = \frac{m\pi}{b-a} + \frac{(4\nu^2-1)(b-a)}{8m\pi ab} + O(m^{-3}). \tag{1.3.30}$$

In a similar way, if $k_1 < k_2 < k_3 < \dots$ are the successive positive zeros of (1.3.29), the asymptotic expansions of k_m, for large m and real ν, are given by[4]

$$k_m = \frac{m\pi}{b-a} + \frac{(4\nu^2+3)(b-a)}{8m\pi ab} + O(m^{-3}). \tag{1.3.31}$$

---

1) Scherberg [91]. 2) Young [112], p. 166.
3) Naylor [72], p. 69; Willis [106]. 4) Naylor [72], p. 70.

E. Orthogonality - Lommel's integrals. Let α and β be non-negative real numbers. Then[1]

$$(1.3.32)\qquad \int x\, J_\nu(\alpha x)\, J_\nu(\beta x)\, dx = x\left\{\alpha J'_\nu(\alpha x)\, J_\nu(\beta x) - \beta J'_\nu(\beta x)\, J_\nu(\alpha x)\right\} / (\beta^2-\alpha^2)$$

for $\alpha \neq \beta$, and

$$(1.3.33)\qquad \int x\, J_\nu^2(\alpha x)\, dx = \frac{1}{2}\left\{x^2\, J'^2_\nu(\alpha x) + \left(x^2-\frac{\nu^2}{\alpha^2}\right) J_\nu^2(\alpha x)\right\}.$$

If α and β are any two zeros of $J_\nu(z)$, then evaluating the integrals between the limits 0 and 1, we get for $\nu > -1$,

$$(1.3.34)\qquad \int_0^1 x\, J_\nu(\alpha x)\, J_\nu(\beta x)\, dx = 0, \qquad \alpha \neq \beta,$$

and

$$(1.3.35)\qquad \int_0^1 x\, J_\nu^2(\alpha x)\, dx = \frac{1}{2}\, J'^2_\nu(\alpha).$$

The relation (1.3.34) establishes the fact that the system of functions $\left\{J_\nu(xj_n):\ n = 1, 2, \dots\right\}$, where $\nu > -1$, is an orthogonal system over $[0,1]$ with weight x. In other words, the system $\left\{\sqrt{x}\, J_\nu(xj_n):\ n = 1, 2, \dots\right\}$ is an orthogonal system over $[0,1]$.

If, however, α and β are zeros of $H\, J_\nu(z) + z\, J'_\nu(z)$, (1.3.34) is still true and

$$(1.3.36)\qquad \int_0^1 x\, J_\nu^2(\alpha x)\, dx = \frac{1}{2}\left\{J'^2_\nu(\alpha) + \left(1-\frac{\nu^2}{\alpha^2}\right) J_\nu^2(\alpha)\right\}.$$

---

[1] Bowman [13], p. 101; Tolstov [99], pp. 216-217.

If γ_n is the n-th positive zero of $c_\nu(az, bz)$, then[1) for $0 < a < b$,

$$\int_a^b x\, c_\nu(x\gamma_m, b\gamma_m)\, c_\nu(x\gamma_n, b\gamma_n)\, dx = 0, \text{ if } m \neq n, \tag{1.3.37}$$

and

$$\int_a^b x\, c_\nu^2(x\gamma_n, b\gamma_n)\, dx = \frac{2\left\{J_\nu^2(a\gamma_n) - J_\nu^2(b\gamma_n)\right\}}{\pi^2\, \gamma_n^2\, J_\nu^2(a\gamma_n)}. \tag{1.3.38}$$

Also, if

$$Q_\nu(\alpha, \beta) = J_\nu(\alpha)\, Y_\nu'(\beta) - J_\nu'(\beta)\, Y_\nu(\alpha), \tag{1.3.39}$$

and k_n is the n-th positive zero of $Q(z)$, then for $0 < a < b$,

$$\int_a^b x\, Q_\nu(xk_m, ak_m)\, Q_\nu(xk_n, ak_n)\, dx = 0, \text{ if } m \neq n, \tag{1.3.40}$$

and

$$\int_a^b x\, Q_\nu^2(xk_n, ak_n)\, dx = \frac{2}{\pi^2 k_n^2} \left[\left(1 - \frac{\nu^2}{b^2 k_n^2}\right) \times \left(\frac{J_\nu'(ak_n)}{J_\nu'(bk_n)}\right)^2 - \left(1 - \frac{\nu^2}{a^2 k_n^2}\right) \right]. \tag{1.3.41}$$

In this way, each of the sequences $\left\{x^{1/2} c_\nu(x\gamma_n, b\gamma_n)\right\}$ and $\left\{x^{1/2}\, Q_\nu(xk_n, ak_n)\right\}$ is an orthogonal sequence on the interval $[a, b]$.

---

1) Titchmarsh [96], p. xlv.

1.4. FOURIER-BESSEL SERIES. The importance of Fourier-Bessel expansions in the study of several problems of Mathematical Physics has already been emphasized in § 1.1 of the present chapter. In view of the orthogonality properties of the Bessel function $J_\nu(x)$ and some connected functions [e.g. those given by (1.3.25), (1.3.28) and (1.3.39)], these expansions can be treated as Fourier expansions of a given function f with respect to the corresponding orthogonal systems[1], and are known as Fourier-Bessel series of f.

In the present work, mainly four types of Fourier-Bessel series are considered.

Any series of the type

$$(1.4.1) \qquad \sum_{m=1}^{\infty} a_m J_\nu(x j_m), \qquad 0 \leq x \leq 1,$$

in which the coefficients a_m form a sequence of constants, is called a series of Bessel functions or a Bessel series.

If, for any $f \in L^1[0,1]$,

$$(1.4.2) \quad a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t\, f(t)\, J_\nu(t j_m)\, dt, \quad m = 1, 2, \ldots,$$

where $j_1 < j_2 < j_3 < \ldots$ are positive zeros of $J_\nu(t)$, the series (1.4.1) is called the Fourier-Bessel series of the

---

1) Sansone [80], pp. 10-11; Alexits [9], p. 6.

first type (FB-I) associated with f. This result was first stated by Lommel[1]. However, the validity of this expansion was not discussed in his work, neither in general nor in any particular case.

In order to obtain the coefficients (1.4.2), we assume the uniform convergence of the expansion

$$f(x) = \sum_{n=1}^{\infty} a_n \, J_\nu(xj_n), \qquad a \le x \le b,$$

multiply its both sides by $x\,J_\nu(xj_n)$ and integrate term-by-term between 0 and 1. In view of the integrals (1.3.34) and (1.3.35), (1.4.2) is obtained. Fourier[2] had obtained the coefficients a_n in this way for $\nu = 0$. There is, however, no a priori reason for supposing that the above expansion for a given function f, is uniformly convergent. Hence, instead of starting with the series, we start with the function f, obtain the coefficients by the formula (1.4.2) and write FB-I by

$$(1.4.3) \qquad f(x) \sim \sum_{n=1}^{\infty} a_n \, J_\nu(xj_n), \qquad a \le x \le b.$$

The validity of the expansion (1.4.3), for $\nu \geq -1/2$, was attempted, in the first part, by

---

1) Lommel [63], pp. 69-73. 2) Fourier [53], pp. 316-319.

Hankel[1], Schläfli[2] and Harnack[3].

Dini[4], later gave a more general expansion

$$(1.4.4) \qquad f(x) \sim \sum_{n=1}^{\infty} b_n J_\nu(x\lambda_n), \qquad 0 \leq x \leq 1,$$

where $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ are successive positive zeros of the function

$$(1.4.5) \qquad H J_\nu(z) + z J_\nu'(z),$$

when $\nu \geq -1/2$ and H is anygiven constant, and

$$(1.4.6) \qquad b_n = \frac{2\lambda_n^2}{(\lambda_n^2 - \nu^2) J_\nu^2(\lambda_n) + \lambda_n^2 J_\nu'^2(\lambda_n)} \times \int_0^1 t\, f(t)\, J_\nu(t\lambda_n)\, dt.$$

The series (1.4.4) is called the Fourier-Bessel series of second type (FB-II) or Fourier-Dini series corresponding to f.

The series FB-II, in case $\nu = 0$, was used by J.B.J. Fourier[5] while solving the problems of radiations of heat from a circular cylinder. In his work, H is considered to be the ratio of the external conductivity of the cylinder to the internal conductivity. Some writers,

---

1) Hankel [42]. 2) Schläfli [82]. 3) Harnack [46].
4) Dini [27]. 5) Fourier [33], § 308.

e.g., Hobson[1], Dixon[2] etc. have also considered FB-I as a particular case of FB-II, when $H \to \infty$, but this view has been found to be incorrect[3]. Dini[4] himself had pointed out the necessity of inserting an initial term in the series FB-II, when $H+\nu = 0$, though the value given by him was incorrect. Later on, the correct value of the initial term was given by Bridgeman[5], Chree[6], and Moore[7]. The corrected series is as follows[8]:

$$(1.4.7) \qquad f(x) \sim b_0 + \sum_{m=1}^{\infty} b_m J_\nu(x\lambda_m), \qquad 0 \le x \le 1,$$

where

$$(1.4.8) \quad b_0 = \begin{cases} 0, \text{ when } H+\nu > 0; \\ 2(\nu+1)x^\nu \int_0^1 t^{1+\nu} f(t)\, dt, \text{ when } H+\nu = 0; \\ \dfrac{2\lambda_0^2 I_\nu(x\lambda_0) \int_0^1 t\, f(t)\, I_\nu(t\lambda_0)\, dt}{(\lambda_0^2+\nu^2) I_\nu^2(\lambda_0) - \lambda_0^2 I_\nu'^2(\lambda_0)}, \\ \qquad \text{when } H+\nu < 0. \end{cases}$$

In (1.4.8), when $H+\nu < 0$, $\pm i\lambda_0$ are the imaginary

---

1) Hobson [47]. 2) Dixon [28]. 3) Moore [69]; Wing [110]; *
4) Dini [27]. 5) Bridgeman [19]. * Dalzell [26].
6) Chree [24]. 7) Moore [69]. 8) Watson [103], p. 597.

zeros of $zJ'_\nu(z) + HJ_\nu(z)$ and $I_\nu(z)$ is the Bessel function of imaginary argument given by [1)]

$$(1.4.9)\quad I_\nu(z) = \begin{cases} e^{-\frac{1}{2}\nu\pi i}\ J_\nu(z\,e^{(1/2)\pi i}), & -\pi/2 < \arg z \le \pi/2; \\ e^{3/2\,\nu\pi i}\ J_\nu(z\,e^{-3/2\,\pi i}), & \pi/2 < \arg z \le \pi. \end{cases}$$

For $f \in L^1[0,1]$, Cooke[2)] defined the modified Fourier-Bessel series, corresponding to FB-I, as

$$(1.4.10)\quad f(x) \sim x^a \sum_{n=1}^{\infty} \frac{2\,J_\nu(xj_n)}{J_{\nu+1}{}^2(j_n)} \times$$

$$\times \int_0^1 t^{1-a}\, J_\nu(tj_n)\, f(t)\, dt,\ \nu \geq -\tfrac{1}{2}.$$

If $a=0$, this series reduces to the series FB-I. If $a=\nu=1/2$, this series reduces to Fourier-sine series and when $a=-\nu=1/2$, it becomes a Fourier-cosine series[3)].

Corresponding to any function $f \in L^1[a,b]$, where $0 < a < b$, the expansion given by

$$(1.4.11)\quad f(x) \sim \sum_{n=1}^{\infty} d_n\, c_\nu(x\gamma_n, b\gamma_n), \qquad a \le x \le b,$$

for any real ν, where

$$(1.4.12)\quad dn = \frac{\pi^2\gamma_n{}^2 J_\nu{}^2(a\gamma_n) \int_a^b t\, f(t)\, c_\nu(t\gamma_n, b\gamma_n)\, dt}{2\left\{J_\nu{}^2(a\gamma_n) - J_\nu{}^2(b\gamma_n)\right\}},$$

---

1) Watson [105], p.77. 2) Cooke [25]. 3) Watson [105], p.94.

had been used in the solution of physical problems concerning the flow of heat through an infinite hollow cylinder[1]. No mathematical theory of this series appeared till Titchmarsh[2], in 1924, discussed it for the first time. This series is different from the series FB-I and FB-II, and we shall call it the Fourier-Bessel series of third type (FB-III) in this thesis.

F. Nito[3], while studying vibrations of a cylindrical shell immersed in water, came across the series:

$$(1.4.13) \qquad f(x) \sim \sum_{n=1}^{\infty} D_n \, Q_\nu(xk_n, ak_n), \qquad a \leq x \leq b,$$

corresponding to a function $f \in L^1[a,b]$, $0 < a < b$, and ν a positive integer. This series of special kind has been referred to as Fourier-Bessel series of fourth type (FB-IV). The coefficients of series FB-IV are given by

$$(1.4.14) \qquad D_n = \frac{1}{N(k_n)} \int_a^b t\, f(t)\, Q_\nu(tk_n, ak_n)\, dt,$$

where

$$(1.4.15) \qquad N(k_n) = \frac{2}{\pi^2 k_n^2} \left[\left(1 - \frac{\nu^2}{b^2 k_n^2}\right) \frac{J_\nu'^2(ak_n)}{J_\nu'^2(bk_n)} - \left(1 - \frac{\nu^2}{a^2 k_n^2}\right) \right].$$

---

1) Carslaw and Jaeger [20], Ch. VII.

2) Titchmarsh [96].

3) Nito [56].

The series FB-III and FB-IV are also Sturm-Liouville expansions of the function f, since no singularity of the function f lies in the interval (a,b), when $0 < a < b$.[1]

1.5 CONVERGENCE OF FOURIER-BESSEL SERIES. The problem of convergence of FB-I, as has already been pointed out earlier, was first investigated by Hankel[2] in 1869, which being incomplete, was later completed by Schläfli[3] in 1875. The work of Hankel and Schläfli was very important, as they developed the method of contour integration to obtain the order estimates of certain kernels connected with FB-I. Their method was also used, later on, by W.H.Young[4] and L.C.Young[5] to FB-II and FT-I respectively. In 1887, Harnack[6] used the method employed for Fourier-trigonometric series to investigate the expansion FB-I. Kneser[7], in 1903, established the convergence of FB-I at a point interior to [0,1] when the generating function is of bounded variation.

The first reasonably satisfactory work in this direction was produced by Hobson[8] in 1908. He proved that if the generating function f is of bounded variation

---

1) Hobson [48]; Titchmarsh [97].   2) Hankel [42].
3) Schläfli [82].   4) Young [112].   5) Young [111].
6) Harnack [46].   7) Kneser [50].   8) Hobson [47].

in a small neighbourhood of x and if $\int_0^1 t^{1/2} f(t)\,dt < \infty$, then the series FB-II for f converges at an interior point of $[0,1]$, provided that $\nu \geq -1/2$.

Another masterpiece work regarding the convergence of FB-II was by W.H.Young[1] in 1920. He discussed the equiconvergence of series FB-II with Fourier-trigonometric series of a function f for $\nu \geq -1/2$. He proved that if the individual terms of the series FB-II tend to zero, then at any point interior to $[0,1]$, the series FB-II behaves in respect of convergence, divergence or oscillation, uniform or otherwise, precisely like the Fourier-trigonometric series of f. In his exhaustive memoir, Young has also proved the uniqueness of series FB-I and FB-II. This uniqueness theory was later discussed by Zygmund[2] in 1952. Lumsden[3] studied in 1922 the convergence of a series of Fourier-Bessel coefficients of FB-I in case $\nu = 0$.

A detailed discussion of the works of all these authors has been given in Watson[4]. In continuation with the works of Hobson[5] and Young[6], Watson has discussed the convergence of FB-I at the end points of $[0,1]$. The

---

1) Young [112]. 2) Zygmund [114]. 3) Lumsden [64].
4) Watson [103], Ch. XVIII. 5) Hobson [47]. 6) Young [112].

uniformity of convergence in [0,1] has also been discussed by him. Moore[1] has discussed the uniform convergence of FB-II throughout the interval [0,1].

Recently, Zaborski[2] has studied the convergence and uniform convergence of FB-I in a series of papers published by him. The absolute convergence of FB-I has recently been studied by E.N.Jafarov[3].

The problem of convergence for the series FB-III seems to be first tackled by E.C.Titchmarsh[4] in 1924. He proved that for an integrable function, having a bounded variation in a neighbourhood of x, the series FB-III converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$. B.N.Khoti[5] has recently treated some problems concerning this series.

No mathematical theory concerning the series FB-IV is available, except a paper of F.Lito[6], where he has discussed the convergence of this series by taking Bessel functions of integral order only into account.

The Gibb's phenomenon of series FB-I has been dealt with by C.N.Moore[7], J.R.Wilton[8] etc. The convergence

---

1) Moore [67]. 2) Zaborski [88] to [94]. 3) Jafarov [51].
4) Titchmarsh [96]. 5) Khoti [53]; [54]; [55].
6) Lito [57]. 7) Moore [71]. 8) Wilton [108]; [109].

of differentiated Fourier-Bessel series has been discussed by W.H.Young[1]. Ford[2], in 1903, discussed the question of permissibility of term-by term differentiation of a Fourier-Bessel expansion.

Regarding the uniform convergence of orthogonal and normalised series FB-I given by

$$(1.5.1) \qquad x^{1/2} \sum_{m=1}^{\infty} a_m J_\nu(xj_m),$$

where a_m is given by (1.4.2), L.C.Young[3] has proved the following theorem:

THEOREM A. If the function $F(x) = x^{1/2} f(x)$ satisfies uniformly in x the Lipschitz condition

$$(1.5.2) \qquad |F(x+h) - F(x)| < K \left[\log|(1/h)|\right]^{-(2+c)},$$

where $c > 0$, then it is uniformly in x the sum of series (1.5.1).

Young has also conjectured that in (1.5.2) the exponent $-(2+c)$ may be replaced by -1.

Concerning the series

$$(1.5.3) \qquad f(x) \sim \sum_{m=1}^{\infty} a_m \varphi_\nu(xj_m), \qquad 0 \le x \le 1,$$

---

1) Young [113].   2) Ford [32].

3) Young [111], p. 307.

corresponding to f, where

(1.5.4) $$\varphi_\nu(t) = \begin{cases} \sqrt{t}\, J_\nu(t), & t > 0, \\ \lim_{t \to 0+} \varphi_\nu(t); & t = 0; \end{cases}$$

and

(1.5.5) $$c_m = \frac{2}{\varphi_{\nu+1}^2(j_m)} \int_0^1 f(t)\, \varphi_\nu(t j_m)\, dt, \quad m=1, 2, \ldots,$$

we have established, in chapter II of the present thesis, a theorem proving Young's conjecture to some extent. A part of the theorem reads as follows:

<u>THEOREM I.</u> <u>Let</u> $S_n(x,f)$ <u>be the n-th partial sum of series</u> (1.5.3) <u>corresponding to</u> $f \in L[0,1]$, <u>such that</u> $f(0) = f(1) = 0$. <u>Then</u>

(1.5.6) $$|f(x) - S_n(x,f)| \leq K\, \omega(1/n, f) \log n,\text{[1)]} \quad 0 \leq x \leq 1,$$

<u>where</u> K <u>is a constant independent of</u> x, n <u>and</u> ν.

Asimilar theorem concerning series FB-III has been proved in chapter III.

L.C.Young[2)] has also estimated the integrated

---

1) Let f be a continuous function defined over [a,b], and let $\delta > 0$. Then

$$\omega(\delta, f) = \sup_{|x_1 - x_2| \leq \delta} |f(x_1) - f(x_2)|, \quad x_1, x_2 \in [a,b],$$

denotes the modulus of continuity of f over [a,b]. Refer Mary [11]; p. 37.

2) Young [111].

Fourier-Bessel kernel

$$(1.5.7) \qquad G_n(t) = \int_0^t \sqrt{xt}\, T_n(t,x)\, dt,$$

where the Fourier-Bessel kernel $T_n(t,x)$ is given by

$$(1.5.8) \qquad T_n(t,x) = \sum_{m=1}^{n} \frac{2\, J_\nu(xj_m)\, J_\nu(tj_m)}{J_{\nu+1}^{2}(j_m)} ;$$

and has applied these estimations to prove a theorem on uniform and bounded convergence of series (1.5.1) [1]. We have extended these results to the series FB-III and have proved the following as one of the theorems in chapter III:[2]

THEOREM 2. *Let f be a function of bounded variation in $0 < a \leq x \leq b$, vanishing at x=a and x=b. Then*

(i) *as $n \to \infty$, we have for $a < x < b$,*

$$s_n(x,f) \longrightarrow \frac{1}{2}\{f(x+0) + f(x-0)\}$$

boundedly, where $s_n(x,f)$ is the n-th partial sum of series (1.4.11), *and,*

(ii) *if f is continuous, we have uniformly in x, for $a \leq x \leq b$,*

$$s_n(x,f) \longrightarrow f(x).$$

Further, Saborski[3] has studied the series (1.5.3)

---

1) Young [111], § 6.   2) Agrawal and Patel [7].

3) Saborski [90].

with coefficients a_n tending to zero as $n \to \infty$. He has generalized certain theorems on trigonometric series[1]. In chapter II we have extended these theorems to the Bessel series given by

$$\sum_{n=1}^{\infty} d_n \, C_n^{(\nu)}(x), \qquad a \le x \le b, \tag{1.5.9}$$

where

$$C_n^{(\nu)}(x) = \sqrt{x}\, C_\nu(x\gamma_n, b\gamma_n), \quad a \le x \le b.$$

If this series is the Fourier-Bessel series (FB-III) for $f \in L[a,b]$, $0 < a < b$, then

$$d_n = \frac{\gamma_m}{c_m} \int_a^b f(t)\, C_n^{(\nu)}(t)\, dt, \tag{1.5.10}$$

where

$$c_n = \frac{2\left\{J_\nu^2(a\gamma_n) - J_\nu^2(b\gamma_n)\right\}}{\pi^2\, \gamma_n\, J_\nu^2(a\gamma_n)}, \quad n = 1, 2, \ldots \tag{1.5.11}$$

Some of the theorems proved are as follows:[2]

THEOREM 5. Let $d_n c_n = f_n^{(\nu)}$. Suppose

$$f_n^{(\nu)} \to 0, \text{ as } n \to \infty \text{ and } \sum_{n=1}^{\infty} \left| f_n^{(\nu)} - f_{n+1}^{(\nu)} \right| < \infty. \tag{1.5.12}$$

Then series (1.5.9) converges to a function $f(x)$ continuous in $[a,b)$ such that $f(x) = O\left(\frac{1}{b-x}\right)$, as $x \to b-0$, and

---

1) Bary [11], pp. 87-91; Bary [12], pp. 200-214; Ul'yanov [101].   2) Agrawal and Patel [3].

$\text{mes.}\{x : |f(x)| \geq n\} = o(1/n)$, as $n \to \infty$. Moreover, if $\delta > 0$, then series (1.5.9) converges uniformly for $a \leq x \leq b-\delta$.

THEOREM 4. If

$$\sum_{n=2}^{\infty} \left| f_n^{(\nu)} - f_{n+1}^{(\nu)} \right| \log n < \infty, \quad f_n^{(\nu)} \to 0 \text{ as } n \to \infty,$$

then series (1.5.9) converges to a function $f \in L[a,b]$, whose Fourier-Bessel series (FB-III) is given by (1.5.9).

THEOREM 5. If condition (1.5.12) are true, then any function f given by

$$f(x) = \sum_{n=1}^{\infty} d_n\, G_n^{(\nu)}(x), \qquad a \leq x \leq b,$$

is the limit in the L^p-norm, $0 < p < 1$, of the sequence of partial sums of series (1.5.9).

A. Kolmogoroff[1)] has proved for trigonometric series

(1.5.13) $$\sum_{n=0}^{\infty} a_n \cos nx,$$

that if $a_n \to 0$, as $n \to \infty$ and the sequence $\{a_n\}$ is quasi-convex, then the series (1.5.13) converges, except at x=0, to an integrable function f(x), and is the Fourier series of f.

---

1) Kolmogoroff [C1].

B.P.Gupta[1] extended this result to series of ultraspherical functions, G.S.Pandey[2] proved a similar result for Jacobi series and B.P.Khoti[3] extended it to series FB-I and FB-II. A similar theorem has been proved for series FB-III in chapter II of our thesis[4].

F.Kito[5] proved the following theorem for FB-IV:

<u>THEOREM B.</u> Let $f \in L[a,b]$, $0 < a < b$, and let it be of bounded variation in $[a,b]$. Then its Fourier-Bessel series FB-IV converges to the sum $\frac{1}{2}\{f(x+0) + f(x-0)\}$, when ν is a positive integer.

In chapter III, the above theorem of Kito has been generalised for ν any non-zero real number greater than -1/2. Our proof is of Hankel-Schläfli type.

1.6 <u>MEAN CONVERGENCE.</u> The problems of convergence in the L^p-norm of the series FB-I and FB-II have been discussed during the last 25 years by various Mathematicians, viz., H.Pollard[6], C.M.Wing[7], G.Standish[8], H.Hochstadt[9], A.Benedek and P.Panzone[10] etc. Benedek

---

1) Gupta [38]. 2) Pandey [76]. 3) Khoti [59], Ch. II.

4) Agrawal and Patel [8-a]. 5) Kito [57].

6) Pollard [77]. 7) Wing [110]. 8) Standish [84].

9) Hochstadt [50]. 10) Benedek and Panzone [13].

and Fonzone[1], very recently, has also considered the mean convergence of FB-I of negative order.

M.L.Gol'dman[2] has proved that the system

$$\left\{\frac{\sqrt{2t}\ J_\nu(tj_m)}{J_{\nu+1}(j_m)}\right\}$$

forms an orthonormal basis for the space $L_\beta^p[0,1]$, $p>1$, $-1<\beta<p-1$, of functions whose p-th power is integrable in Lebesgue sense with weight x^β over $[0,1]$. He has also proved that if β does not lie in the above range, then there is a function in $L_\beta^p[0,1]$, whose Fourier-Bessel expansion corresponding to the above system diverges.

In chapter IV, we have proved similar results for the systems

$$(1.6.1) \qquad v_m(t) = \frac{\sqrt{t}\ J_\nu(t\lambda_m)}{f_m}, \qquad t>0,$$
$$= \lim_{t\to 0+} v_m(t), \qquad t=0,$$

where

$$(1.6.2) \qquad 2f_m^2 = \left(1-\frac{\nu^2}{\lambda_m^2}\right) J_\nu^2(\lambda_m) + J_\nu'^2(\lambda_m),\ m=1,2,\ldots;$$

and

$$(1.6.3) \qquad \theta_m(t) = \frac{\sqrt{t}\ c_\nu(t\gamma_m, b\gamma_m)}{e_m}, \qquad a\le x\le b,$$

---

1) Benedek and Panzone [14]. 2) Gol'dman [30].

where

$$(1.6.4) \qquad c_n^2 = \frac{2\left\{J_\nu^2(a\gamma_n) - J_\nu^2(b\gamma_n)\right\}}{\pi^2\, \gamma_n^2\, J_\nu^2(a\gamma_n)}, \quad n = 1, 2, \ldots.$$

Some of the theorems proved are as follows:

THEOREM 6. The system $\{v_n(t)\}$ forms a basis for the Banach space $L_p^\beta[0,1]$, where $p > 1$, $-1 < \beta < p-1$, $\nu \geq -1/2$.

THEOREM 7. For $p > 1$, if $\beta > p-1$ or if $\beta < -1$, there is a function in $L_p^\beta[0,1]$ whose Dini series (FB-II) diverges.

THEOREM 8. If $1 < p \leq 2$, $f \in L^p[a,b]$, $0 < a < b$ and $\nu \geq -1/2$, then

$$\lim_{n \to \infty} \int_a^b \left| f(x) - S_n(x,f) \right|^p dx = 0,$$

where $S_n(x,f)$ is the n-th partial sum of the Fourier-Bessel series of f corresponding to the system (1.6.5).

1.7 ESTIMATES OF PARTIAL SUMS. In case, a Fourier-Bessel series fails to converge at a point, it is of great interest to know as to by how much this convergence is missing. In Fourier series, G.H.Hardy[1] has shown that if a_m and b_m are the Fourier coefficients of an integrable

1) Hardy [44].

function, then the series

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\log n}$$

converges for almost all x in $(0,2\pi)$.

The order of approximation of the n-th partial sums of the Fourier series of a periodic function f under the condition

$$\int_0^t \left| f(x+t) + f(x-t) - 2f(x) \right| dt = o(t), \text{ as } t \to 0+,$$

is known to be $o(\log n)$.[1]

A similar theorem has also been proved by Otto-Szász.[2] Scherberg[3] proved a theorem of the type of Hardy for FB-II. Khoti[4] has extended the theorems of Bary[1] and Otto-Szász for series FB-I.

In chapter V of the present thesis, we have generalised these results to the modified Fourier-Bessel series (1.4.10). Our theorems are:[5]

<u>Theorem 9.</u> *Let* $g(x) = x^{(1/2)-\alpha} f(x)$ *and let* $\varphi_x(u) = g(x \pm u) - g(x)$. *If* $g \in L[0,1]$, $g(0) = g(1) = 0$, *and if*

$$\Phi(t) = \int_0^t \left| \varphi_x(u) \right| du = o(t), \text{ as } t \to 0+,$$

---

1) Bary [11], pp. 141-142. 2) Otto-Szász [55].
3) Scherberg [61]. 4) Khoti [59], Ch. III. 5) Agrawal and Patel [2].

then for $0 < x < 1$,

$$S_n(\alpha, x) - f(x) = o(\log n), \quad \text{as } n \to \infty,$$

almost everywhere. [$S_n(\alpha, x)$ is the n-th partial sum of (1.4.10).]

THEOREM 10. Let α, φ_x and Ψ be as in the Theorem 9 and let

$$\Psi(t) = o\left(\frac{t}{\log(1/t)}\right), \quad \text{as } t \to 0+.$$

Then for $0 < x < 1$,

$$S_n(\alpha, x) - f(x) = o(\log\log n), \text{ as } n \to \infty.$$

Lhoti's theorems are obtained by putting $\alpha = 0$ in our theorems, and the theorems of Bary and Otto-Szász are obtained by putting $\alpha = \nu = 1/2$.

We have also proved similar results corresponding to series FB-III in the same chapter[1].

1.8 ORDER OF COEFFICIENTS. For Fourier-trigonometric series, it is known[2], that for a function of bounded variation in $(-\pi, \pi)$, the Fourier coefficients are of order $O(1/n)$. For series FB-I, under the same condition on the defining function, W.F.Sheppard[3] proved that the

---

1) Agrawal and Patel [5]. 2) Bary [11], pp. 71-72.
3) Sheppard [85], p. 247; Watson [105], p. 595.

n-th term, i.e., $a_n J_\nu(xj_n)$ has the order $O(1/j_n)$, when $0 < x \leq 1$.

For FB-II, W.H.Young[1] has shown that if for any $(\alpha,\beta) \subseteq [0,1]$, $\int_a^\beta \left| t^{1/2} f(t) \right| dt < \infty$, then

$$\int_a^b t\, f(t)\, J_\nu(t\lambda_n)\, dt = o(\lambda_n^{-1/2}).$$

L.P.Khoti[2] has also studied the order of coefficients of FB-II. He has extended the above theorems of Sheppard and Young to series FB-III[3].

In Chapter VI, we have proved the following theorems, giving the order of terms in FB-IV[4]:

THEOREM 11. If f is a function of bounded variation on $[a,b]$, $0 < a < b$, then the n-th term of series FB-IV has order $O(1/k_n)$, as $n \to \infty$.

THEOREM 12. If $f \in L^2[a,b]$, then

$$\int_a^b t\, f(t)\, C_\nu(tk_n, ak_n)\, dt = o(1/k_n),$$

as $n \to \infty$, and the general term of series FB-IV tends to zero for every $x \in [a,b]$.

For a function differentiable several times,

---

1) Young [112]; Watson [105], pp. 595-596. 2) Khoti [54].
3) Khoti [55], Ch. VII. 4) Agrawal and Patel [4].

G.P.Tolstov[1] has found the orders of terms of series FB-I and FB-II. B.P.Khoti[2] has proved similar results for FB-III.

We have extended these ideas to prove similar theorems for series FB-IV in chapter VI. Two of our theorems are as follows:[3]

THEOREM 13. *Let* $f(x)$ *be a function defined for* $0 < a \leq x \leq b$, *such that* f *is differentiable* $2s$ *times* $(s > 1)$ *and such that*

$$f(a) = f'(a) = f''(a) = \ldots\ldots = f^{(2s-1)}(a) = 0;$$

$$f(b) = f'(b) = f''(b) = \ldots\ldots = f^{(2s-1)}(b) = 0;$$

and $f^{(2s)}(x)$ *is bounded (this derivative may not exist at certain points). Then the coefficients* p_n *of series FB-IV of* f *satisfy the relation*

$$p_n = O(1/k_n^{2s-1}), \quad \text{as } n \to \infty.$$

THEOREM 14. *If* f *satisfies the conditions of the above theorem, then for* $\nu \geq -1/2$, *the series FB-IV converges absolutely and uniformly on* $[a,b]$.

1.9 SEMICONVERGENCE. S.H.Tung[4] has proved certain boundedness properties of the difference kernels of the

---

1) Tolstov [99], pp. 228-233. 2) Khoti [55].
3) Agrawal and Patel [8]. 4) Tung [100].

series FB-I and Fourier-cosine series corresponding to any function $f \in L[0,1]$ and has established the equi-convergence of these two series. D.P.Gupta and B.P.Moti[1] have extended these results to modified Fourier-Bessel series for $f \in L[0,1]$. E.C.Titchmarsh[2] has also established equiconvergence of series FB-I with FB-III for $f \in L[a,b]$.

In chapter VII, we have extended the theorems of Sung to series FB-II. Two of the theorems proved are as follows:

THEOREM 15. *The difference kernel of series FB-II and Fourier-cosine series is bounded independently of n and $t \in [0,1]$, at any fixed point $x \in (0,1)$.*

THEOREM 16. *For any $f \in L[0,1]$, its Dini series (FB-II) is equiconvergent with its Fourier-cosine series at every $x \in (0,1)$.*

1.10 SUMMABILITY OF FOURIER-BESSEL SERIES.

Definitions. Let $a_0 + a_1 + a_2 + \dots + a_n + \dots$ be a given infinite series. Denote by[3]

---

1) Gupta and Moti [39]. 2) Titchmarsh [96], pp. xv-xvi.

3) Cesàro [21]; Knopp [59]; Knopp [60]; Chapman [22]; Hardy [43], p. 96.

(1.10.1) $S_n^{\alpha} = \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k, \quad A_n^{\alpha} = \binom{n+\alpha}{\alpha},$

where $\binom{n+\alpha}{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}$ and $\alpha > -1$.

Then

(1.10.2) $\sigma_n^{\alpha} = \frac{S_n^{\alpha}}{A_n^{\alpha}}$,

is called the n-th Cesàro mean of order α, or (C,α) mean of the given series. The series $\sum a_n$ is called summable (C,α) to the sum S if

(1.10.3) $\lim_{n \to \infty} \sigma_n^{\alpha} = S$.

When a series is summable $(C,1)$, it is also called summable by Fejér's method.[1)]

Let $\{\lambda_n\}$ be a strictly increasing sequence of positive numbers, such that $\lambda_0=0$ and $\lambda_n \to \infty$, as $n \to \infty$. Then $\sum a_n$ is said to be summable by Riesz method of order 1 or summable $(R,\lambda_n,1)$ to a sum S, if S is the limit of Riesz-means given by[2)]

(1.10.4) $\sum_{m=0}^{n} \left(1 - \frac{\lambda_m}{\lambda_{n+1}}\right) a_m,$

as $n \to \infty$.

If $\lambda_n = n$, the summability $(R,\lambda_n,1)$ reduces to

---

1) Bary [11], p. 153. 2) Hardy [45], p. 86.

summability (C,1). If $\lambda_n = \log n$, the Riesz-summability is known as Riesz-logarithmic summability.

The Cesaro summability of FB-II was discussed by Moore[1)] in case $\alpha = 1/2$ at the origin. Moore[2)] has also constructed a continuous function for which the series FB-II, for $\nu = 0$, is not summable (C,α) for $0 \leq \alpha < 1/2$ at $x = 0$. This function is analogous to the function given by Fejér[3)], whose Fourier Series diverges at $x = 0$.

From the work of Moore, it is clear that the discussion of (C,α) summability by direct methods involves very complicated analysis. However, many results regarding this summability can be inferred from Riesz-summability. Recently, Taberski[4)] has applied this method in case of series FB-I.

Watson[5)] has discussed the Riesz-summability of FB-I for a function f, for which the integral

$$\int_0^1 t^{1/2} f(t)\, dt$$

exists and is absolutely convergent.

Taberski[4)] has estimated the orders of Riesz and

---

1) Moore [68]; [69]. 2) Moore [70]. 3) Fejér [31].
4) Taberski [88]. 5) Watson [103], pp. 606-616.

Fejér means of series FB-I corresponding to a function f, such that $\frac{f(t)}{t^{\nu+1/2}} \in \Lambda_\alpha[0,1]$,[1] $0<\alpha\leq 1$. Š.M.Nafarov[2] has also studied the order of approximation of certain functions by Riesz and Fejér means of FB-I.

Taberski[3], in a series of papers, has investigated the series FB-I and FB-II for certain higher order Riesz-means. D.P.Gupta and B.P.Khoti[4] have also studied certain problems on Riesz summability of series FB-III.

We have proved, in Chapter VIII, several results on Riesz means parallel to Taberski's work for series FB-III. We have considered the following higher order Riesz means:

$$(1.10.5) \quad R_n^r(x,f) = \sum_{m=1}^{n} \left(1-\frac{\gamma_m^2}{B_n^2}\right)^r a_m\, c_m^{(\nu)}(x), \quad r>0,$$

and

$$(1.10.6) \quad Q_n^\alpha(x,f) = \sum_{m=1}^{n} \left(1-\frac{\gamma_m^\alpha}{B_n^\alpha}\right) a_m\, c_m^{(\nu)}(x), \quad \alpha>0,$$

where $a \leq x \leq b$, corresponding to series FB-III taken in the form (1.9.9), $\gamma_n < B_n < \gamma_{n+1}$, $n = 1, 2, \ldots$.

---

1) A continuous function f defined on $[a,b]$ belongs to the class $\Lambda_\alpha[a,b]$, if $\omega(\delta,f) = O(\delta^\alpha)$, as $\delta \to 0+$, $0<\alpha\leq 1$. Refer Bary [11], p. 36.

2) Nafarov [52]. 3) Taberski [87], [89], [91] to [94].

4) Gupta and Khoti [40]; Khoti [95], Ch. VIII.

Some of the theorems proved are as follows:[1]

THEOREM 17. *Let r be a positive integer,* $r \geq \nu + 5/2$. *If f is continuous and* $f(a) = f(b) = 0$, *then*

$$\left| P_n^r(x,f) - f(x) \right| \leq K.\,\omega(1/n,f).$$

THEOREM 18. *If f is continuous and is of bounded variation over* $[a,b]$, *such that* $f(a) = f(b) = 0$, *then the condition*

$$\sum_{n=1}^{\infty} \frac{1}{n} \{\omega(1/n,f)\}^{1/2} < \infty$$

implies that the series FB-III for f converges uniformly and absolutely in $[a,b]$.

THEOREM 19. *Let* $f \in \Lambda_\alpha[a,b]$, $\frac{1}{p} - \frac{1}{2} < \alpha < 1$, $1 \leq p \leq 2$. *If* $f(a) = f(b) = 0$, *then*

$$\left\{ \sum_{m=n+1}^{\infty} |a_m|^p \right\}^{1/p} = O\left(\frac{1}{n^{\alpha + 1/2 - 1/p}} \right), \quad \text{as } n \to \infty.$$

THEOREM 20. *If* $f \in \Lambda_\alpha[a,b]$, $1/2 < \alpha < 1$, $1 \leq p \leq 2$, $f(a) = f(b) = 0$, *then the series FB-III converges absolutely and uniformly in* $[a,b]$.

THEOREM 21. *A necessary and sufficient condition for the series (1.5.9) to be the Fourier-Bessel series of a function* $f \in C[a,b]$, $f(a) = f(b) = 0$, *is that*

---

1) Agrawal [1]; Agrawal and Patel [6].

$$\lim_{n\to\infty} Q_n^{\alpha}(x) = f(x).$$

<u>THEOREM 22. The series (1.5.9) is the Fourier-Bessel series of a function</u> $f \in L^1$ <u>or</u> $f \in L^p$, $1 < p < \infty$, <u>if and only if</u>

$$\lim_{n\to\infty} \int_a^b \left| Q_n^{\alpha}(x) - f(x)\right| dx = 0,$$

<u>or</u>

$$\sup_{n \geq 1} \int_a^b \left| Q_n^{\alpha}(x)\right|^p dx < \infty,$$

<u>respectively.</u>

Apart from these theorems, many other interesting properties regarding uniform and absolute convergence and the orders of terms, under various conditions, for the series (1.5.9), have been proved in this chapter.

In chapter IX, we have proved the Riesz summability of series FB-IV. Our theorem is as follows:

<u>THEOREM 23. Let</u> $f \in L[a,b]$, $\nu > -1/2$, $\nu \neq 0$ <u>and let for</u> $a < x < b$, <u>the limits</u> $f(x \pm 0)$ <u>exist. Then the series FB-IV is Riesz summable to the sum</u>

$$\frac{1}{2}\left\{ f(x+0) + f(x-0)\right\}.$$

***