CHAPTER II

UNIFORM CONVERGENCE OF FOURIER-BESSEL SERIES.

2.1 Let $L^p[a,b]$ be the class of all functions which are defined and integrable in the Lebesgue sense with p-th power over $[a,b]$, $0 < p < \infty$. Denote by $C[a,b]$, the class of continuous functions over $[a,b]$.

Denote by $j_1 < j_2 < j_3 < \cdots$ the successive positive zeros of $J_\nu(t)$ and by $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ those of $c_\nu(at,bt)$, where, $0 < a < b$, and

$$c_\nu(\alpha,\beta) = J_\nu(\alpha)\,Y_\nu(\beta) - J_\nu(\beta)\,Y_\nu(\alpha).$$

Define

$$(2.1.1) \qquad \varphi_\nu(t) = \begin{cases} \sqrt{t}\,J_\nu(t), & t > 0, \\ \lim\limits_{t \to 0+} \varphi_\nu(t), & t = 0; \end{cases}$$

and

$$(2.1.2) \qquad C_n^{(\nu)}(t) = \sqrt{t}\,c_\nu(t\gamma_n, b\gamma_n), \qquad a \le t \le b.$$

For any function $f \in L^1[0,1]$, the series

$$(2.1.3) \qquad f(x) \sim \sum_{m=1}^{\infty} a_m \, \varphi_\nu(x j_m), \qquad 0 \le x \le 1,$$

where

$$(2.1.4) \qquad a_m = \frac{2}{\varphi_{\nu+1}^2(j_m)} \int_0^1 f(t)\, \varphi_\nu(t j_m)\, dt, \quad m = 1, 2, \ldots,$$

is called the Fourier-Bessel series of the first type (FB-I) corresponding to f.

Also, if $f \in L^1[a,b]$, $0 < a < b$, the series

$$(2.1.5) \qquad f(x) \sim \sum_{m=1}^{\infty} d_m \, C_m^{(\nu)}(x), \qquad a \le x \le b,$$

where

$$(2.1.6) \quad d_m = \frac{\pi^2 \gamma_m^2 J_\nu^2(a\gamma_m)}{2\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\}} \int_a^b f(t)\, C_m^{(\nu)}(t)\, dt,$$

is called the Fourier-Bessel series of third type (FB-III) for f.

The series (2.1.5) was first studied by Titchmarsh[1] and subsequently by Khoti[2].

2.2. The problem of uniform convergence of FB-I has been studied by C.N.Moore[3], L.C.Young[4], R.Taberski[5] etc.

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1) Titchmarsh [98]. 2) Khoti [53]; [55]. 3) Moore [67].
4) Young [111]. 5) Taberski [96].

L.C.Young[1] has proved in his paper a theorem on the uniform convergence of the series FB-I corresponding to a continuous function of bounded p-th power variation, vanishing at 0 and 1. As a particular case of the said theorem, he has mentioned the following theorem:

THEOREM 2A. If f satisfies uniformly in x the Lipschitz condition

$$\left| f(x+h) - f(x) \right| < K \left[\log (1/h) \right]^{-(2+\epsilon)},$$

where $\epsilon > 0$, K is a constant, then it is uniformly in x the sum of series (2.1.3).

Young has, in the above paper, also conjectured that the exponent $-(2+\epsilon)$ in the above theorem may be replaced by -1.

In this Chapter, we prove the following theorem in this direction, which answers Young's conjecture to some extent:

THEOREM 2.1. Let $S_n(x,f)$ be the n-th partial sum of the series (2.1.3) corresponding to a function $f \in L[0,1]$, such that $f(0) = f(1) = 0$. Then

$$(2.2.1) \qquad \left| f(x) - S_n(x,f) \right| < K\, \omega(1/n,f) \log n, \quad 0 \le x \le 1,$$

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[1] Young [111].

where K *is a constant, independent of* x, n *and* ν.

Hence, if $f \in \Lambda_\alpha[0,1]$, $0 < \alpha \le 1$, *then*

(2.2.2) $$\left| f(x) - S_n(x,f) \right| \le \frac{K \log n}{n^\alpha}, \quad 0 \le x \le 1,$$

i.e. the series (2.1.3) *converges uniformly in* x, *for* $0 \le x \le 1$, *to* f(x). *Also, if*

(2.2.3) $$\omega(\delta, f) = o\left(\frac{1}{\log \frac{1}{\delta}}\right),$$

uniformly for $0 \le x \le 1$, *then the series* (2.1.3) *converges uniformly, for* $0 \le x \le 1$, *to* f(x).

The following is a theorem for the series (2.1.3), which is similar to Lebesgue's theorem[1]:

THEOREM 2.2. If f(x) *is a bounded function, then for the partial sum* $S_n(x,f)$ *of* (2.1.3), *we have*

$$|S_n(x,f)| \le KM \log n, \quad 0 \le x \le 1, \; n = 1, 2, \ldots,$$

where $|f(x)| \le M$, K *is an absolute constant.*

Taberski[2] has studied the series (2.1.3) with coefficients a_n tending to zero. He has generalised certain theorems on trigonometric series[3].

Some of the theorems proved by him are as follows:

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[1] Zary [11], p. 110. [2] Taberski [90].

[3] Zary [11], pp. 87-91; [12], pp. 200-214; Ul'yanov [101].

THEOREM 2B. Suppose that

$$\sum_{m=1}^{\infty} \left| b_m^{(\nu)} - b_{m+1}^{(\nu)} \right| < \infty,$$

where $b_m^{(\nu)} = a_m \varphi_{\nu+1}^{2}(j_m)$, $m = 1, 2, \ldots$. Then the series (2.1.3) converges to a function f continuous on $(0,1]$, such that $f(x) = O(1/x)$, as $x \to 0+$,

$$\text{mes.}\, \{x : f(x) \geqslant n\} = o(1/n), \text{ as } n \to \infty.$$

THEOREM 2C. If

$$\sum_{m=2}^{\infty} \left| b_m^{(\nu)} - b_{m+1}^{(\nu)} \right| \log m < \infty,$$

then series (2.1.3) converges to a function f of class L^1, whose Fourier-Bessel series is identical with (2.1.3). If, moreover, $b_m^{(\nu)} \downarrow 0$, we have

$$\int_0^1 \left| f(x) - S_n(x,f) \right| dx \leqslant 2C \sum_{m=n}^{\infty} \left(b_m^{(\nu)} - b_{m+1}^{(\nu)} \right) \log m,$$

where C is some constant.

For

(2.2.4) $$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx,$$

and

(2.2.5) $$F(x) = \sum_{m=1}^{\infty} b_m \sin mx,$$

Ul'yanov[1] proved certain theorems on the L^p-convergence, $0<p<1$, of the partial sums of the series (2.2.4) and (2.2.5) to f and $\bar{F}$ respectively.

In the present chapter, we prove theorems concerning series (2.1.5), which are similar to the theorems of Zaborski and Ul'yanov.

We denote by $S_n^{(\nu)}(x)$, the n-th partial sum of (2.1.5),

$$D_n^{(\nu)}(x) = \sum_{m=1}^{n} \frac{C_m^{(\nu)}(x)}{e_m}, \quad n = 1, 2, \ldots, \tag{2.2.6}$$

where

$$e_m = \frac{2\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\}}{\pi^2 \gamma_m J_\nu^2(a\gamma_m)}, \tag{2.2.7}$$

and

$$f_m^{(\nu)} = d_m e_m, \quad m = 1, 2, \ldots. \tag{2.2.8}$$

Our theorems are as follows:[2]

<u>THEOREM 2.3. Suppose that</u>

$$f_n^{(\nu)} \to 0, \text{ as } n \to \infty, \text{ and } \sum_{n=1}^{\infty} \left| f_n^{(\nu)} - f_{n+1}^{(\nu)} \right| < \infty. \tag{2.2.9}$$

<u>Then series (2.1.5) converges to a function f continuous in [a,b), such that $f(x) = O(\frac{1}{b-x})$, as $x \to b-0$ and</u>

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1) Ul'yanov [102]; Bary [12], p. 215. 2) Agrawal & Patel [3].

mes. $\{x: f(x) \geq n\} = o(1/n)$, *as* $n \to \infty$. *Moreover, if* $\delta > 0$, *then series* (2.1.5) *converges uniformly for* $a \leq x \leq b-\delta$.

THEOREM 2.4. If

$$(2.2.10)\quad \begin{cases} \sum_{m=2}^{\infty} \left| f_m^{(\nu)} - f_{m+1}^{(\nu)} \right| \log m < \infty, \quad \text{and} \\ f_m^{(\nu)} \to 0, \quad \text{as } m \to \infty, \end{cases}$$

then the series (2.1.5) *converges to a function* $f \in L^1[a,b]$, *whose Fourier-Bessel series of third type is given by* (2.1.5).

If, moreover, $f_m^{(\nu)} \downarrow 0$, *we have*

$$(2.2.11)\qquad \int_a^b \left| f(x) - S_n^{(\nu)}(x) \right| dx \leq 2K \sum_{m=n}^{\infty} \left(f_m^{(\nu)} - f_{m+1}^{(\nu)} \right) \log m,$$

where K is some constant.

THEOREM 2.5. If conditions (2.2.9) *are satisfied, then for any function f, given by*

$$(2.2.12)\qquad f(x) = \sum_{m=1}^{\infty} a_m C_m^{(\nu)}(x), \qquad a \leq x \leq b,$$

we have, for $0 < p < 1$,

$$(2.2.13)\qquad \lim_{n \to \infty} \int_a^b \left| f(x) - S_n^{(\nu)}(x) \right|^p dx = 0.$$

THEOREM 2.6. If conditions (2.2.9) *are satisfied, then any function f given by* (2.2.12) *is of class* $L^p[a,b]$, *for any* p, $0 < p < 1$.

Kolmogoroff[1] has proved for trigonometric series

$$(2.2.14) \qquad \sum_{m=1}^{\infty} a_m \cos mx,$$

the following theorem:

<u>THEOREM 2D.</u> If $a_n \to 0$ and the sequence $\{a_m\}$ is quasi-convex, then series (2.2.14) converges, save for $x=0$, to an integrable function f, and is the Fourier series of f.

Gupta[2] has extended this theorem to the series of ultraspherical functions and Khoti[3] has further proved a similar theorem for series FB-I.

We prove a similar theorem for the series:

$$(2.2.15) \qquad \sum_{m=1}^{\infty} d_m\, F_\nu(\gamma_m, x),$$

where $\{d_m\}$ is an arbitrary sequence of real numbers and

$$(2.2.16) \qquad F_\nu(\gamma, x) = \gamma^{-\nu}\, c_\nu(x\gamma, b\gamma), \quad \gamma > 0, \quad \nu > 0.$$

Our theorem is as follows[4]:

<u>THEOREM 2.7. Let</u> $\{d_n\}$ <u>be such that</u> $d_n \to 0$ <u>as</u> $n \to \infty$ <u>and for</u> $0 < \varepsilon < \nu$, <u>the sequence</u> $\{d_n/\gamma_n^{\varepsilon}\}$ <u>is of bounded variation. Then the series</u> (2.2.15) <u>converges uniformly throughout any closed interval lying inside</u>

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1) Kolmogoroff [61]. 2) Gupta [38]. 3) Khoti [55], Ch. II.
4) Agrawal and Patel [8-a].

$[\delta, b]$, $\delta > a$, and not including a point of discontinuity of f.

2.3. In order to prove the above theorems, we need certain lemmas. Throughout the remaining part of the chapter, K_i, $i = 1, 2, \ldots$, denote constants depending at most upon ν.

The n-th partial sum of series (2.1.3) is given by

$$(2.3.1) \quad S_n(x,f) = \sum_{m=1}^{n} c_m \, \varphi_\nu(x j_m) = \int_0^1 f(t)\, T_n(t,x)\, dt,$$

where

$$T_n(t,x) = \sum_{m=1}^{n} \frac{2\, \varphi_\nu(t j_m)\, \varphi_\nu(x j_m)}{\varphi_{\nu+1}^2(j_m)}$$

$$(2.3.2) \qquad = \sum_{m=1}^{n} \frac{2\sqrt{xt}\, J_\nu(t j_m)\, J_\nu(x j_m)}{J_{\nu+1}^2(j_m)}.$$

The needed lemmas are as follows:

LEMMA 2.1. (Young[1]). The following estimates are true for $\nu \geq -1/2$,

$$(2.3.3) \qquad |T_n(t,x)| \leq K_1 n, \quad 0 \leq x \leq 1, \; 0 \leq t \leq 1;$$

$$(2.3.4) \qquad |T_n(t,x)| \leq \frac{K_1}{|t-x|}, \quad 0 \leq x \leq 1, \; 0 \leq t \leq 1, \; t \neq x.$$

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[1] Young [111].

LEMMA 2.2. *For all sufficiently large n, and* $x \in [0,1]$,

$$\int_0^1 |T_n(t,x)|\, dt = O(\log n).$$

PROOF. Let $0 \leq x \leq 1/n$. Then using Lemma 2.1,

$$\int_0^1 |T_n(t,x)|\, dt = \left\{\int_0^{x+1/n} + \int_{x+1/n}^1\right\} |T_n(t,x)|\, dt$$

$$\leq K_1 n(x+1/n) + K_1 \int_{x+1/n}^1 \frac{dt}{t-x}$$

$$\leq 2K_1 + K_1 \log \frac{1-x}{1/n}$$

$$= O(\log n).$$

Similarly, this estimate is true for $1-1/n \leq x \leq 1$.

If $1/n < x < 1-1/n$, we again have, by Lemma 2.1,

$$\int_0^1 |T_n(t,x)|\, dt = \left\{\int_0^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^1\right\} |T_n(t,x)|\, dt$$

$$\leq K_1 \left[\log \frac{x}{1/n} + 2 + \log \frac{1-x}{1/n}\right]$$

$$= O(\log n).$$

The lemma is, now, proved.

LEMMA 2.3. *For* $1/n < x < 1-1/n$,

$$\left|\int_0^1 T_n(t,x)\, dt - 1\right| \leq \frac{K_2}{nx}.$$

PROOF. Let $0 < a \leq x-1/n$. Define,

$$g(x) = t^{\nu+1/2}, \quad 0 \leq t \leq a,$$

$$= a^{\nu+1/2}, \quad a < t \leq 1.$$

Then

$$Q_n(x,g) - g(x) = \left(\int_0^a + \int_a^1\right)\left\{\frac{g(t)}{t^{\nu+1/2}} - \frac{g(x)}{x^{\nu+1/2}}\right\} t^{\nu+1/2}\, T_n(t,x)\,dt \, +$$

$$+ \frac{a^{\nu+1/2}}{x^{\nu+1/2}}\left\{\int_0^1 t^{\nu+1/2}\, T_n(t,x)\,dt - x^{\nu+1/2}\right\}$$

(2.3.5) $$= I_1 + I_2 + I_3, \text{ say.}$$

Using (2.3.4),

$$|I_1| \leq K_1\left\{1 - (a/x)^{\nu+1/2}\right\}\int_0^a \frac{t^{\nu+1/2}}{x-t}\,dt$$

(2.3.6) $$\leq \frac{K_3\, a^{\nu+3/2}}{x-a}.$$

Also,

$$I_2 = a^{\nu+1/2}\left\{\int_a^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{1}\right\} \times$$

$$\times \left\{1 - (t/x)^{\nu+1/2}\right\}\; T_n(t,x)\,dt$$

(2.3.7) $$= a^{\nu+1/2}\,(\,I_{21} + I_{22} + I_{23} + I_{24}), \text{ say.}$$

By the analogue of Riemann-Lebesgue Lemma[1),

(2.3.8) $$I_{21} = o(1), \quad I_{24} = o(1), \quad \text{as } n \to \infty.$$

Again, by (2.3.3),

$$|I_{22}| \leq K_1 n\left[\frac{1}{n} - \frac{x^{\nu+3/2} - (x-1/n)^{\nu+3/2}}{\{\nu+(3/2)\}x^{\nu+1/2}}\right] =$$

$$= K_1 n\left[\frac{\nu+1/2}{2n^2 x} - \frac{(\nu+1/2)(\nu-1/2)}{6n^3 x^2} + \cdots\right]$$

(2.3.9) $$\leq K_4/nx.$$

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1) Watson [103], § 18.23.

Similarly,

(2.3.10) $$\left|I_{23}\right| \leq K_4/nx.$$

Now,[1)]

$$\lim_{n\to\infty}\left[\int_0^1 t^{\nu+1/2}\, T_n(t,x)\,dt - x^{\nu+1/2}\right] = 0,\quad 0<x<1.$$

Hence,

(2.3.11) $$I_3 = a^{\nu+1/2}\, o(1),\quad \text{as } n\to\infty,\quad 0<x<1.$$

Combining (2.3.9) to (2.3.11), we get,

(2.3.12) $$\left|S_n(x,g) - g(x)\right| \leq a^{\nu+1/2}\left\{\frac{K_3\, a}{x-a} + \frac{2\,K_4}{nx} + o(1)\right\}.$$

Also,

$$S_n(x,g) - g(x) = \int_0^a t^{\nu+1/2}\, T_n(t,x)\,dt + $$

$$+\, a^{\nu+1/2}\left\{\int_a^1 T_n(t,x)\,dt - 1\right\},$$

so that we have by (2.3.4) and (2.3.12),

$$\left|\int_0^1 T_n(t,x)\,dt - 1\right| \leq \left|\int_0^a T_n(t,x)\,dt\right| + a^{-\nu-1/2}\left|S_n(x,g) - g(x)\right| +$$

$$+\, a^{-\nu-1/2}\left|\int_0^a t^{\nu+1/2}\, T_n(t,x)\,dt\right|$$

$$\leq \frac{2\,K_1\, a}{x-a} + \frac{K_3\, a}{x-a} + \frac{2\,K_4}{nx} + o(1).$$

Taking limit as $a\to 0$, the lemma follows.

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1) Watson [105], § 10.22.

LEMMA 2.4. Let $\gamma_n < B_n < \gamma_{n+1}$. Then for large n,

$$B_n \sim n.$$

PROOF. By Taylor[1],

$$\gamma_n = \frac{n\pi}{b-a} + \frac{(4\nu^2-1)(b-a)}{8n\pi ab} + O(n^{-3}), \tag{2.3.13}$$

when n is sufficiently large.

Now, $B_n < \gamma_{n+1} = O(n)$, as $n \to \infty$, and $B_n > \gamma_n = O(n)$, as $n \to \infty$. This proves the lemma.

LEMMA 2.5. Let Γ denote the rectangle in the w-plane with vertices at $\pm Bi$, $B_n \pm Bi$, where B is made to tend to infinity. Then on this rectangle,

$$\left|\frac{c_\nu(xw,aw)}{c_\nu(aw,bw)}\right| = O\left(\sqrt{\frac{b}{x}}\; e^{-(b-x)|v|}\right),$$

for $0 < a \leq x < b$, $w = u+iv$.

PROOF. By Titchmarsh[2],

$$c_\nu(xw,aw) = \frac{2\sin(x-a)w}{\sqrt{ax}\;\pi w} + O\left(\frac{e^{(x-a)|v|}}{|w|^2}\right), \tag{2.3.14}$$

for $0 < a < x$ and $|w| \to \infty$.

Also, for $w = u+iv$, and any real number p, when $v \geq \frac{1}{|p|}$,

$$|\sin pw| \leq e^{|pv|}, \text{ and } |\sin pw| \geq (1/4)\, e^{|pv|}, \tag{2.3.15}$$

Using (2.3.14) and the inequalities (2.3.15), we

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1) Taylor [72], p. 69, formula (50). 2) Titchmarsh [57], p.73.

obtain, for $a \le x < b$,

$$\left|\frac{c_\nu(xv,av)}{c_\nu(av,bv)}\right| \le \frac{\dfrac{e^{(x-a)|v|}}{\sqrt{ax}\,\pi|v|} + O\left(\dfrac{e^{(x-a)|v|}}{|v|^2}\right)}{\left|\dfrac{e^{(b-a)|v|}}{4\sqrt{ab}\;|v|\pi} - O\left(\dfrac{e^{(b-a)|v|}}{|v|^2}\right)\right|}$$

$$= O\left(\sqrt{\frac{b}{x}}\;e^{-(b-x)|v|}\right),$$

whence the lemma follows.

LEMMA 2.6. If $\nu \geq -1/2$, then

(2.3.16) $$\left|D_n^{(\nu)}(x)\right| \le \frac{K_5}{b-x}, \quad a \le x < b, \; n = 1, 2, \dots,$$

and

(2.3.17) $$\left|E_n^{(\nu)}(x)\right| \le K_6\, n, \quad a \le x \le b, \; n = 1, 2, \dots.$$

PROOF. Let

$$F(v) = \frac{\pi\sqrt{x}\; c_\nu(xv,av)}{c_\nu(av,bv)}, \quad a \le x < b.$$

The residue of $F(v)$ at $v = \gamma_n$ is given by

$$\lim_{v \to \gamma_n} \frac{(v-\gamma_n)\pi\sqrt{x}\; c_\nu(xv,av)}{c_\nu(av,bv)}$$

$$= \pi\sqrt{x}\; c_\nu(x\gamma_m, a\gamma_m) \Big/ \Big[a\left\{J_\nu'(a\gamma_n)\,Y_\nu(b\gamma_n) - J_\nu(b\gamma_n)Y_\nu'(a\gamma_n)\right\} +$$

$$+\, b\left\{J_\nu(a\gamma_n)\,Y_\nu'(b\gamma_n) - J_\nu'(b\gamma_n)\,Y_\nu(a\gamma_n)\right\}\Big]$$

$$= \frac{\pi\sqrt{x}\; c_\nu(x\gamma_n, a\gamma_n)}{-\dfrac{a}{u}\cdot\dfrac{2}{\pi a\gamma_n} + b\,u\cdot\dfrac{2}{\pi b\,\gamma_n}},$$

where[1)]

(2.5.18) $\quad \eta = \frac{J_\nu(a\gamma_n)}{J_\nu(b\gamma_n)} = \frac{Y_\nu(a\gamma_n)}{Y_\nu(b\gamma_n)}$, and $\begin{vmatrix} J_\nu(w) & Y_\nu(w) \\ J'_\nu(w) & Y'_\nu(w) \end{vmatrix} = \frac{2}{\pi w}$

Hence, the residue of $F(w)$ at γ_n is $C_n^{(\nu)}(x)/e_n$.

Taking the rectangle Γ, given in Lemma 2.5, as the contour of integration, we therefore, have

(2.5.19) $$D_n^{(\nu)}(x) = \frac{1}{2\pi i} \int_\Gamma F(w)\, dw.$$

Using Lemma 2.5, we have for $a < x < b$,

$$\left| \int_{-Bi}^{Bi} F(w)\, dw \right| \leq 2 K_\eta \int_0^B e^{-v(b-x)}\, dv = \frac{2 K_\eta \left\{1 - e^{-B(b-x)}\right\}}{b-x}$$

(2.5.20) $$\longrightarrow \frac{2 K_\eta}{b-x}, \text{ as } B \to \infty;$$

$$\left| \int_{\pm Bi}^{B_n \pm Bi} F(w)\, dw \right| \leq K_\eta \int_0^{B_n} e^{-B(b-x)}\, du = K_\eta B_n e^{-B(b-x)}$$

(2.5.21) $$\longrightarrow 0, \text{ as } B \to \infty, \text{ for each } n;$$

and similarly,

(2.5.22) $$\int_{B_n - Bi}^{B_n + Bi} F(w)\, dw \longrightarrow \frac{2 K_\eta}{b-x}, \quad \text{as } B \to \infty.$$

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1) Watson [105], p. 76.

By (2.3.19) to (2.3.22), we obtain,

$$\left|D_n^{(\nu)}(x)\right| \leq \frac{4K_7}{b-x}, \quad a \leq x < b, \; n=1,2,\ldots .$$

This proves (2.3.16).

To prove (2.3.17), we consider the contour of integration to be the rectangle Γ' with vertices at $\pm B_n i$, $B_n \pm B_n i$. Then we have by Lemma 2.5,

$$|F(w)| \leq K_8, \qquad w \in \Gamma',$$

and the inequalities corresponding to (2.3.20) - (2.3.22) become:

$$\left|\int_{-B_n i}^{B_n i} F(w)\, dw\right| \leq 2K_8 B_n,$$

$$\left|\int_{\pm B_n i}^{B_n \pm B_n i} F(w)\, dw\right| \leq K_8 B_n,$$

and $$\left|\int_{B_n - B_n i}^{B_n + B_n i} F(w)\, dw\right| \leq 2K_8 B_n.$$

From this and Lemma 2.4, (2.3.17) follows.

<u>LEMMA 2.7. If</u> $\nu \geq -1/2$, <u>then</u>

$$\int_a^b \left|D_n^{(\nu)}(x)\right| dx \leq K_9 \log n, \quad n > 1.$$

<u>PROOF.</u> We have, by Lemma 2.6,

$$\int_a^b \left|D_n^{(\nu)}(x)\right| dx \leq \int_a^{b-1/n} \frac{K_7\, dx}{b-x} + \int_{b-1/n}^{b} K_C\, n\, dx =$$

$$= K_5 \log n + K_5 \log (b-a) + K_6$$

$$\leqslant K_9 \log n.$$

LEMMA 2.8. For $\nu \geqslant -1/2$,

$$\left|F_\nu(\gamma, x)\right| \leqq \frac{K_{10}}{\gamma^{\nu+1}\sqrt{xb}}, \qquad \text{as } \gamma \to \infty.$$

PROOF. By Khoti[1],

$$\left|c_\nu(x\gamma, b\gamma)\right| \leqslant \frac{K_{10}}{\gamma\sqrt{xb}},$$

for sufficiently large $\gamma > 0$ and $0 < a < x < b$.

Hence,

$$\left|F_\nu(\gamma, x)\right| \leqslant \frac{K_{10}}{\gamma^{\nu+1}\sqrt{xb}}.$$

2.4. PROOF OF THEOREM 2.1. We have, by (2.3.1),

$$S_n(x,f) - f(x) = \int_0^1 \left\{ f(t) - f(x) \right\} T_n(t,x)\, dt + f(x) \left\{ \int_0^1 T_n(t,x)\, dt - 1 \right\}$$

(2.4.1) $$= U_n + V_n, \text{ say.}$$

For any x, $0 \leqslant x \leqslant 1/n$, by Lemma 2.2 and the analogue of Riemann-Lebesgue Lemma[2],

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[1] Khoti [53], Lemma 1. [2] Watson [103], § 18.25.

$$|U_n| \leq \int_0^{x+1/n} |f(t)-f(x)|\,|T_n(t,x)|\,dt + \left| \int_{x+1/n}^{1} \{f(t)-f(x)\}\, T_n(t,x)\,dt \right|$$

$$\leq \omega(1/n,f) \int_0^{x+1/n} |T_n(t,x)|\,dt + o(1)$$

(2.4.2) $\leq K_{11}\,\omega(1/n,f)\log n + o(1).$

If $1-1/n \leq x \leq 1$, we have by a similar argument,

(2.4.3) $|U_n| \leq K_{11}\,\omega(1/n,f)\log n + o(1).$

For $1/n < x < 1-1/n$, we similarly obtain,

$$|U_n| \leq \left| \left(\int_0^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{1} \right) \times \{f(t)-f(x)\}\, T_n(t,x)\,dt \right|$$

(2.4.4) $\leq \omega(1/n,f).\ 2K_1 + o(1),$ using Lemma 2.1.

Combining (2.4.2) to (2.4.4), we get,

(2.4.5) $|U_n| \leq K_{12}\,\omega(1/n,f)\log n, \qquad 0 \leq x \leq 1.$

Again, for $0 \leq x \leq 1/n$, $1-1/n \leq x \leq 1$, since $f(0) = f(1) = 0$,

$$|V_n| \leq |f(x)| \left\{ \int_0^1 T_n(t,x)\,dt + 1 \right\}$$

$$\leq \omega(1/n,f) \left\{ K_{11}\log n + 1 \right\}$$

(2.4.6) $\leq K_{12}\,\omega(1/n,f)\log n.$

Since, for any $x, t \in [0,1]$,

$$|f(t)-f(x)| \leq \omega(|t-x|, f) \leq \omega(1/n, f)\{n|t-x| + 1\},$$

we have, for $1/n < x < 1-1/n$, by using Lemma 2.3,

$$|V_n| \leq K_2 \frac{|f(x) - f(\xi)|}{nx}$$

$$\leq K_2\, \omega(1/n, f)\, \frac{nx+1}{nx}$$

(2.4.7) $$< 2K_2\, \omega(1/n, f).$$

By (2.4.6) and (2.4.7),

(2.4.8) $$|V_n| \leq 2K_{12}\, \omega(1/n, f) \log n, \qquad 0 \leq x \leq 1.$$

From (2.4.1), (2.4.5) and (2.4.8), (2.2.1) is proved. The estimation (2.2.2) is, now, obvious. The uniform convergence of series (2.1.5) to $f(x)$ in $[0,1]$, follows from (2.2.1) and (2.2.3).

The theorem is, now, completely proved.

Theorem 2.2 follows immediately from Lemma 2.2.

2.5. <u>PROOF OF THEOREM 2.3.</u> We have, for integers n and p,

$$\left| S_n^{(\nu)}(x) - S_{n+p}^{(\nu)}(x) \right| = \left| \sum_{m=n+1}^{n+p} a_m\, c_m^{(\nu)}(x) \right|$$

$$= \left| \sum_{m=n+1}^{n+p} a_m^{(\nu)}\, \frac{c_m^{(\nu)}(x)}{c_m} \right|$$

$$= \left| \sum_{m=n+1}^{n+p-1} \left\{ f_m^{(\nu)} - f_{m+1}^{(\nu)} \right\} D_m^{(\nu)}(x) + f_{n+p}^{(\nu)} D_{n+p}^{(\nu)}(x) - f_{n+1}^{(\nu)} D_n^{(\nu)}(x) \right|,$$

using Abel's transformation[1].

Hence, by Lemma 2.6, for each x, $a \leq x < b$,

$$\left| S_n^{(\nu)}(x) - S_{n+p}^{(\nu)}(x) \right| \leq \frac{K_5}{b-x} \left\{ \sum_{m=n+1}^{n+p-1} \left| f_m^{(\nu)} - f_{m+1}^{(\nu)} \right| + \left| f_{n+p}^{(\nu)} \right| + \left| f_{n+1}^{(\nu)} \right| \right\}. \tag{2.5.1}$$

From (2.2.9) and (2.5.1), the convergence of sequence $\left\{ S_n^{(\nu)}(x) \right\}$ follows.

Let

$$f(x) = \lim_{n \to \infty} S_n^{(\nu)}(x) = \sum_{m=1}^{\infty} a_m C_m^{(\nu)}(x), \quad a \leq x < b.$$

Then, by a similar reasoning as above,

$$\left| S_n^{(\nu)}(x) \right| \leq \frac{K_5}{b-x} \left\{ \sum_{m=1}^{n-1} \left| f_m^{(\nu)} - f_{m+1}^{(\nu)} \right| + \left| f_n^{(\nu)} \right| \right\}. \tag{2.5.2}$$

Passing with n to infinity, we obtain,

$$f(x) = O\left(\frac{1}{b-x}\right).$$

Further, for any $\delta > 0$, and $a \leq x \leq b-\delta$, from (2.5.1), the sequence $\left\{ S_n^{(\nu)}(x) \right\}$ converges uniformly. Hence, f is continuous in $[a, b-\delta]$. Since δ is arbitrary, f is

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[1] Bary [11], p. 1.

continuous in $[a,b)$.

Again, in view of (2.2.9), for each $\varepsilon > 0$, there is a positive integer N such that

(2.5.3) $$\sum_{n=N+1}^{\infty} \left| f_n^{(\nu)} - f_{n+1}^{(\nu)} \right| < \varepsilon/2, \quad \left| f_n^{(\nu)} \right| < \varepsilon/2, \text{ for } n \geq N.$$

Suppose,

$$f(x) = \sum_{n=1}^{N} a_n C_n^{(\nu)}(x) + \sum_{n=N+1}^{\infty} a_n C_n^{(\nu)}(x)$$

(2.5.4) $$= S_1(x) + S_2(x).$$

Let M be a positive number, such that,

$$|S_1(x)| \leq M.$$

Hence, if $n > 2M$,

(2.5.5) $$\text{mes.} \left\{ x: \; |S_1(x)| \geq n/2 \right\} = 0.$$

Also,

$$S_2(x) = \lim_{n \to \infty} \sum_{n=N+1}^{n} f_n^{(\nu)} \cdot \frac{C_n^{(\nu)}(x)}{c_n}$$

$$= \lim_{n \to \infty} \left\{ \sum_{n=N+1}^{n-1} \left(f_n^{(\nu)} - f_{n+1}^{(\nu)} \right) D_n^{(\nu)}(x) + \right.$$

$$\left. + f_n^{(\nu)} D_n^{(\nu)}(x) - f_{N+1}^{(\nu)} D_N^{(\nu)}(x) \right\} .$$

Therefore, from (2.5.3) and Lemma 2.6, we obtain

$$|S_2(x)| \leq \frac{A_5 \varepsilon}{b-x} .$$

Hence

$$\text{mes.}\left\{x: |g_2(x)| \geq n/2\right\} \leq \text{mes.}\left\{x: \frac{K_5 c}{b-x} \geq n/2\right\} =$$

$$(2.5.6) \qquad = \frac{2 K_5 c}{n} = o(1/n), \text{ as } n \to \infty,$$

since c is arbitrary.

By (2.5.4) to (2.5.6),

$$\text{mes.}\left\{x: |f(x)| \geq n\right\} \leq \text{mes.}\left\{x: |g_1(x)| \geq n/2\right\} + $$
$$+ \text{mes.}\left\{x: |g_2(x)| \geq n/2\right\} =$$
$$= o(1/n), \text{ as } n \to \infty.$$

This proves the theorem completely.

2.6. PROOF OF THEOREM 2.4. Since (2.2.9) follows from (2.2.10), by Theorem 2.3, (2.1.5) converges to a function f(x) for $a \leq x \leq b$. If

$$f_n(x) = \sum_{m=2}^{n} \left\{f_m^{(\nu)} - f_{m+1}^{(\nu)}\right\} D_m^{(\nu)}(x),$$

then we observe that $\lim_{n \to \infty} f_n(x) = f(x)$, for every $x \in [a,b]$ and

$$\int_a^b |f_n(x)|\, dx \leq \sum_{m=2}^{n} \left|f_m^{(\nu)} - f_{m+1}^{(\nu)}\right| \int_a^b \left|D_m^{(\nu)}(x)\right| dx$$

$$\leq K_9 \sum_{m=2}^{n} \left|f_m^{(\nu)} - f_{m+1}^{(\nu)}\right| \log m,$$

by Lemma 2.7. Hence, $f_n \in L^1[a,b]$, for each n. Using

Fatou's Lemma, we obtain by (2.2.10),

$$\int_a^b |f(x)|\,dx \le \underline{\lim}_{n \to \infty} \int_a^b |f_n(x)|\,dx$$

$$\le K_9 \sum_{n=2}^{\infty} \left| f_n^{(\nu)} - f_{n+1}^{(\nu)} \right| \log n$$

$$< \infty.$$

This shows that $f \in L^1[a,b]$.

The fact that (2.1.5) is the Fourier-Bessel series of third type for f, follows from the orthogonality of $C_n^{(\nu)}(x)$.[1]

Finally, if $f_n^{(\nu)} \downarrow 0$, then

$$\left| f(x) - S_n^{(\nu)}(x) \right| = \left| \sum_{m=n+1}^{\infty} f_m^{(\nu)} \frac{C_m^{(\nu)}(x)}{c_m} \right|$$

$$\le \sum_{m=n+1}^{\infty} \left\{ f_m^{(\nu)} - f_{m+1}^{(\nu)} \right\} \left| D_m^{(\nu)}(x) \right| ,$$

for $a \le x \le b$. Hence, by Lemma 2.7,

$$\int_a^b \left| f(x) - S_n^{(\nu)}(x) \right| dx \le K_9 \sum_{m=n+1}^{\infty} \left\{ f_m^{(\nu)} - f_{m+1}^{(\nu)} \right\} \log m.$$

This proves (2.2.11) and completes the proof.

2.7. PROOF OF THEOREM 2.5. As in the proof of

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1) Titchmarsh [90], p. xiv.

Theorem 2.3, we have,

$$\left|f(x)-S_n^{(\nu)}(x)\right| \leq \frac{K_5}{b-x}\left\{\sum_{m=n+1}^{\infty}\left|f_m^{(\nu)}-f_{m+1}^{(\nu)}\right|+\left|f_{n+1}^{(\nu)}\right|\right\}$$

for $a \leq x \leq b$. Hence, if $0<p<1$,

$$\int_a^b\left|f(x)-S_n^{(\nu)}(x)\right|^p dx \leq K_5{}^p\left\{\sum_{m=n+1}^{\infty}\left|f_m^{(\nu)}-f_{m+1}^{(\nu)}\right|+\right.$$

$$\left.+\left|f_{n+1}^{(\nu)}\right|\right\}^p\frac{(b-a)^{-p+1}}{-p+1}.$$

Therefore, (2.2.13) follows from (2.2.9).

2.8. PROOF OF THEOREM 2.6. By the inequality (2.5.2), for $0<p<1$,

$$\int_a^b\left|S_n^{(\nu)}(x)\right|^p dx \leq K_5{}^p\left\{\sum_{m=1}^{n-1}\left|f_m^{(\nu)}-f_{m+1}^{(\nu)}\right|+\left|f_n^{(\nu)}\right|\right\}^p \times$$

$$\times\frac{(b-a)^{-p+1}}{-p+1}$$

$$(2.8.1) \qquad < \infty, \quad \text{for every } n.$$

Hence, by Zygmund[1] and Theorem 2.5,

$$(2.8.2)\quad \int_a^b|f(x)|^p dx \leq \int_a^b\left|S_n^{(\nu)}(x)\right|^p dx+o(1),$$

for sufficiently large n.

The theorem, now, follows from (2.8.1) and (2.8.2).

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[1] Zygmund [11], p. 21, inequality (10.4).

2.9 PROOF OF THEOREM 2.7. Let

$$p_m(x) = \gamma_m^{\nu+1} F_\nu(\gamma_m, x),$$

and

$$P_n(x) = \sum_{m=1}^{n} p_m(x).$$

Then, by Lemma 2.8,

$$(2.9.1) \qquad \left| P_n(x) \right| \leq \frac{K_{10} n}{\sqrt{xb}} .$$

Now,

$$\gamma_m^{\varepsilon} F_\nu(\gamma_m, x) = \gamma_m^{\varepsilon-\nu-1} p_m(x) = h_m p_m(x),$$

where $h_m = \gamma_m^{\varepsilon-\nu-1}$.

Using Abel's transformation, we obtain,

$$\left| \sum_{m=1}^{n} h_m p_m(x) \right| = \left| \sum_{m=1}^{n-1} (h_m - h_{m+1}) P_m(x) + h_n P_n(x) \right|$$

$$(2.9.2) \qquad \leq \frac{K_{10}}{\sqrt{xb}} \sum_{m=1}^{n-1} m \left| \gamma_m^{\varepsilon-\nu-1} - \gamma_{m+1}^{\varepsilon-\nu-1} \right| + \frac{K_{10} n}{\sqrt{xb}} \gamma_n^{\varepsilon-\nu-1}, \text{ by (2.9.1).}$$

By (2.3.15), we now have, for large m,

$$\left| \gamma_m^{\varepsilon-\nu-1} - \gamma_{m+1}^{\varepsilon-\nu-1} \right| = \left(\frac{m\pi}{b-a}\right)^{\varepsilon-\nu-1} \left| \left\{ 1 + \frac{(4\nu^2-1)(b-a)^2}{8m^2\pi^2 ab} + O(m^{-4}) \right\}^{\varepsilon-\nu-1} - \left\{ 1 + \frac{1}{m} + \frac{(4\nu^2-1)(b-a)^2}{8(m+1)m\pi^2 ab} + O(m^{-4}) \right\}^{\varepsilon-\ \ -1} \right|$$

(2.9.3) $$= K_{13}\, n^{c-\nu-2}.$$

Since $0 < c < \nu$, by (2.9.2) and (2.9.3), we obtain,

$$\left| \sum_{n=1}^{n} a_n\, p_n(x) \right| \leq \frac{K_{14}}{\sqrt{x\delta}} \sum_{n=1}^{n-1} n^{c-\nu-1} + O(1)$$

$$< \frac{K_{14}}{\sqrt{x\delta}} \sum_{n=1}^{\infty} n^{c-\nu-1} + O(1)$$

$$= K_{15}, \quad a \leq x \leq b.$$

Here K_{15} is independent of **x**. Hence the series

$$\sum_{n=1}^{\infty} a_n\, p_n(x)$$

has its partial sums uniformly bounded for $a \leq x \leq b$.

Now

$$\sum_{n=1}^{\infty} d_n\, E_\nu(\gamma_n, x) = \sum_{n=1}^{\infty} \frac{d_n}{\gamma_n^{\,c}}\, a_n\, p_n(x),$$

and $\dfrac{d_n}{\gamma_n^{\,c}} \to 0$, as $n \to \infty$ and is of bounded variation.

Hence, by Abel's criterion for uniform convergence, the proof of the theorem is complete.

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