CHAPTER III

DERIVATIONS OF A FOURIER-BESSEL SERIES AND CONVERGENCE

3.1 Let, for $\nu \geqslant -1/2$,

$$c_\nu(\alpha,\beta) = J_\nu(\alpha)\, Y_\nu(\beta) - J_\nu(\beta)\, Y_\nu(\alpha).$$

For an arbitrary function $f \in L^1[0,1]$, let the series

$$(3.1.1) \qquad f(x) \sim \sum_{n=1}^{\infty} a_n\, J_\nu(x j_n), \qquad 0 \leq x \leq 1,$$

where $j_1 < j_2 < j_3 < \ldots$ are the positive zeros of $J_\nu(t)$ arranged in the ascending order of magnitude and

$$(3.1.2) \qquad a_n = \frac{2}{J_{\nu+1}^2(j_n)} \int_0^1 t\, f(t)\, J_\nu(t j_n)\, dt,$$

be the Fourier-Bessel series of the first type (FB-I) and for $f \in L^1[a,b]$, $0 < a < b$, let the series

$$(3.1.3) \qquad f(x) \sim \sum_{n=1}^{\infty} d_n\, c_\nu(x\gamma_n, b\gamma_n), \qquad a \leq x \leq b,$$

where $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ are the positive zeros of $c_\nu(at, bt)$ and

$$(3.1.4) \quad d_n = \frac{\pi^2 \gamma_n^2 J_\nu^2(a\gamma_n)}{2\left\{J_\nu^2(a\gamma_n) - J_\nu^2(b\gamma_n)\right\}} \int_a^b t\, f(t)\, c_\nu(t\gamma_n, b\gamma_n)\, dt,$$

be the Fourier-Bessel series of third type (FB-III).

Let us denote by $s_n(x,f)$ and $S_n(x,f)$ the n-th partial sums of the series (3.1.1) and (3.1.3) respectively, so that

$$(3.1.5) \quad s_n(x,f) = \sum_{m=1}^{n} a_m J_\nu(xj_m) = \int_0^1 t\, f(t)\, T_n(t,x)\, dt,$$

where

$$T_n(t,x) = \sum_{m=1}^{n} \frac{2 J_\nu(xj_m) J_\nu(tj_m)}{J_{\nu+1}^2(j_m)};$$

and

$$S_n(x,f) = \sum_{m=1}^{n} d_m c_\nu(x\gamma_m, b\gamma_m)$$

$$(3.1.6) \qquad = \int_a^b t\, f(t)\, R_n(t,x)\, dt,$$

where

$$(3.1.7) \quad R_n(t,x) = \sum_{m=1}^{n} \frac{\pi^2 \gamma_m^2 J_\nu^2(a\gamma_m)}{2\left\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\right\}} \times$$

$$\times\, c_\nu(x\gamma_m, b\gamma_m)\, c_\nu(t\gamma_m, b\gamma_m).$$

L.C.Young[1] has established certain boundedness properties for the integrated Fourier-Bessel kernel given by

$$\int_0^t \sqrt{xt}\, K_n(t,x)\, dt,$$

which is a function of t and which also depends on the parameters x and n. He has also proved a theorem concerning the convergence of series FB-I to f(x).

In this chapter we establish theorems similar to the theorems of Young concerning the series FB-III.[2] We also modify the convergence theorem and establish a theorem similar to Theorem 2.1 for series FB-III.

Our theorems are as follows (K_1, K_2, K_3, ..., etc. are suitable positive constants independent of n, x, t.):

THEOREM 3.1. Let $\nu \geqslant -1/2, \nu \neq 0$, and let

$$(3.1.8) \quad h_n(t,x) = \int_a^t \sqrt{xt}\, K_n(t,x)\, dt, \quad a \leqslant t \leqslant b, \; a \leqslant x \leqslant b.$$

Then for a given $\delta > 0$, we have uniformly in n, t and x,

$$(3.1.9) \qquad \left| h_n'(t,x) \right| < K_1(\delta),$$

when $|t-x| \geqslant \delta$, where h_n' denotes the partial derivative

---

1) Young [111]. 2) Agrawal and Patel [7].

of h_n with respect to the first variable and $K_1(\delta)$ is depending only on δ. Moreover,

$$(3.1.10)\qquad \underset{t}{\text{osc.}}\left\{h_n(t,x):\ a \le t \le x-\delta,\ x+\delta \le t \le b\right\} < \frac{K_1(\delta)}{n}.$$

THEOREM 3.2. *For $a \le x \le b$, $\nu \geq -1/2$, $\nu \neq 0$, we have,*

$$\left|h_n(t,x)\right| \le K_2.$$

THEOREM 3.3. *Let $\nu \geq -1/2$, $\nu \neq 0$ and let f be a function of bounded variation vanishing at a and b. Then (i) as $n \to \infty$, we have for $a < x < b$,*

$$(3.1.11)\qquad S_n(x,f) \longrightarrow \tfrac{1}{2}\left\{f(x+0)+f(x-0)\right\}$$

boundedly, and (ii) if f is continuous, we have uniformly in x for $a \le x \le b$,

$$(3.1.12)\qquad S_n(x,f) \longrightarrow f(x), \quad \text{as } n \to \infty.$$

THEOREM 3.4. *Let $f \in L^1[a,b]$, $f(a)=f(b)=0$. Define $F(x)=x^{1/2}f(x)$. If $\nu \geq -1/2$, $\nu \neq 0$, then for $a \le x \le b$,*

$$(3.1.13)\qquad \left|S_n(x,f)-f(x)\right| \le K_3\, \omega(1/n,F)\log n.$$

Hence, if $F \in \Lambda_\alpha[a,b]$, $0<\alpha\le 1$, then

$$(3.1.14)\qquad \left|S_n(x,f)-f(x)\right| \le K_4 \frac{\log n}{n^{\alpha}}, \quad a \le x \le b,$$

i.e., the series (3.1.3) converges uniformly to f in $[a,b]$.

If, further, $\omega(\delta,F) = o\left(\frac{1}{\log 1/\delta}\right)$, uniformly for

$a \leq x \leq b$, *the series* (3.1.3) *converges uniformly, for* $x \in [a,b]$, *to* $f(x)$.

3.2. We need the following lemmas to prove the above theorems:

LEMMA 3.1. (Khoti[1]). For $\nu \geqslant -1/2$, sufficiently large $\lambda > 0$ and fixed $x > 0$,

$$\lambda\sqrt{x}\; c_\nu(x\lambda, b\lambda) = M_\nu \sin(b\lambda - x\lambda) + \frac{\gamma'}{\lambda x},$$

where γ' remains bounded as $\lambda \to \infty$.

LEMMA 3.2. Let $\gamma_n < B_n < \gamma_{n+1}$. *Then on the rectangle* Γ, *whose vertices are at* $\pm Bi$, $B_n \pm Bi$ *in the w-plane, where* B *is to be made to tend to infinity and* $w = u + iv$,

$$\left|\frac{w\, c_\nu(xw,aw)\, c_\nu(tw,bw)}{c_\nu(aw,bw)}\right| = O\left(\frac{e^{-(t-x)|v|}}{\sqrt{xt}}\right),$$

for $a < x < t < b$ *and* n *sufficiently large.*

PROOF. Using the relation (2.3.14) and inequalities (2.3.15), we obtain,

$$\left|\frac{w\, c_\nu(xw,aw)\, c_\nu(tw,bw)}{c_\nu(aw,bw)}\right| \leqslant \left|\frac{w}{\sqrt{xt}}\,\frac{\sin(x-a)w\,\sin(b-t)w}{\pi w\,\sin(b-a)w}\right| + O\left(\frac{e^{-(t-x)|v|}}{|w|^{2}}\right) =$$

$$= O\left(\frac{e^{-(t-x)|v|}}{\sqrt{xt}}\right).$$

---

[1] Khoti [53], Lemma 2.

LEMMA 3.3. For $a < x < b$, $\nu \geq -1/2$, $\nu \neq 0$,

$$(3.2.1) \qquad \lim_{n \to \infty} \int_a^x t^{\nu+1/2} \sqrt{xt}\, R_n(t,x)\, dt = \frac{x^{\nu+1/2}}{2},$$

and

$$(3.2.2) \qquad \lim_{n \to \infty} \int_a^b t^{\nu+1/2} \sqrt{xt}\, R_n(t,x)\, dt = x^{\nu+1/2},$$

uniformly for $a+c \leq x \leq b-c$, $c > 0$.

PROOF. The limit (3.2.1) has been proved by Titchmarsh[1]. The limit (3.2.2), for each x, $a < x < b$, has also been proved by Titchmarsh. To prove the uniformity of this limit, let us consider the integral[1]:

$$\int_a^b t^{\nu+1} R_n(t,x)\, dt = x^{\nu} + \frac{1}{\pi i} \int_{R_n - \infty i}^{R_n + \infty i} \times$$

$$\times \frac{b^{\nu}\, c_{\nu}(xw, aw) - a^{\nu}\, c_{\nu}(xw, bw)}{w\, c_{\nu}(aw, bw)}\, dw$$

$$= x^{\nu} + I, \text{ say}; \qquad \nu \neq 0.$$

By Lemma 2.5,

$$|I| \leq \frac{K_6\, b^{\nu+1/2}}{\sqrt{x}} \int_0^{\infty} \frac{e^{-(b-x)v}}{\sqrt{R_n^2 + v^2}}\, dv +$$

$$+ \frac{K_6\, a^{\nu+1/2}}{\sqrt{x}} \int_0^{\infty} \frac{e^{-(x-a)v}}{\sqrt{R_n^2 + v^2}}\, dv$$

---

1) Titchmarsh [96], p. xv.

$$\leq \frac{K_7}{\sqrt{x}\, B_n} \left\{ \frac{1}{b-x} + \frac{1}{x-a} \right\}.$$

(3.2.2), now, follows from this.

<u>LEMMA 3.4. The following inequalities hold true for</u> $a \leq x \leq b$, $a \leq t \leq b$:

(3.2.3) $$\sqrt{xt}\, \left| R_n(t,x) \right| \leq \frac{K_8}{|t-x|}, \text{ if } x \neq t;$$

(3.2.4) $$\sqrt{xt}\, \left| R_n(t,x) \right| \leq K_8\, B_n;$$

(3.2.5) $$\left| \int_a^t t^{\nu+1}\, R_n(t,x)\, (t^2 - x^2)\, dt \right| \leq \frac{K_9\, (t+x)\, b^{\nu+1}}{B_n\, \sqrt{xt}}.$$

<u>PROOF.</u> Inequality (3.2.3) has been proved by Titchmarsh[1]. In order to prove (3.2.4), using rectangle Γ' in the w-plane with vertices at $\pm B_n i$, $B_n \pm B_n i$ as the contour of integration, we have[1],

$$\left| R_n(t,x) \right| = \left| \frac{1}{2\pi i} \left\{ \int_{-B_n i}^{B_n - B_n i} + \int_{B_n - B_n i}^{B_n + B_n i} + \int_{B_n + B_n i}^{B_n i} \right\} G(w)\, dw \right|$$

where

$$G(w) = \pi w \frac{c_\nu(xw, aw)\, c_\nu(tw, bw)}{c_\nu(aw, bw)}, \quad a \leq x \leq t \leq b.$$

Using Lemma 3.2, we obtain,

$$\left| R_n(t,x) \right| \leq \frac{K_{10}\, B_n}{\sqrt{xt}}.$$

---

[1] Titchmarsh [96], p. xiv.

The case $a \le t \le x \le b$ can be dealt with by interchanging x and t.

Now, in case $a \le x < t \le b$, we have[1)]

$$\int_a^t t^{\nu+1} R_n(t,x)\,(t^2-x^2)\,dt =$$

$$(3.2.6) \qquad = \frac{1}{2i}\int_{B_n-i\infty}^{B_n+i\infty} \frac{w\,c_\nu(xw,aw)}{c_\nu(aw,bw)}\,dw \;\times$$

$$\times \int_a^t t^{\nu+1}\,(t^2-x^2)\,c_\nu(tw,bw)\,dt.$$

Also,

$$\int_a^t t^{\nu+1}\,(t^2-x^2)\,c_\nu(tw,bw)\,dt =$$

$$= \left[\frac{t^{\nu+1}(t^2-x^2)}{w}\left\{J_{\nu+1}(tw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+1}(tw)\right\}\right]_a^t -$$

$$-\int_a^t \frac{2t}{w}t^{\nu+1}\left\{J_{\nu+1}(tw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+1}(tw)\right\}dt$$

$$(3.2.7) \qquad \begin{aligned} &= \frac{t^{\nu+1}(t^2-x^2)}{w}\left\{J_{\nu+1}(tw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+1}(tw)\right\} + \\ &+ \frac{a^{\nu+1}(x^2-a^2)}{w}\left\{J_{\nu+1}(aw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+1}(aw)\right\} - \\ &- \frac{2t^{\nu+2}}{w^2}\left\{J_{\nu+2}(tw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+2}(tw)\right\} + \\ &+ \frac{2a^{\nu+2}}{w^2}\left\{J_{\nu+2}(aw)Y_\nu(bw)-J_\nu(bw)Y_{\nu+2}(aw)\right\}. \end{aligned}$$

Since[2)], for large values of $|z|$,

---

1) Titchmarsh [96], p. xiv. 2) Watson [103], p. 199.

$$(3.2.8)\quad J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\left\{\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+\frac{1-4\nu^2}{8z}\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+ O(|z|^{-2})\right\},$$

$$(3.2.9)\quad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\left\{\sin\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+\frac{1-4\nu^2}{8z}\cos\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+ O(|z|^{-2})\right\};$$

it follows that,

$$(3.2.10)\quad J_{\nu+1}(tw)Y_\nu(bw) - J_\nu(bw)Y_{\nu+1}(tw) \sim O(|w|^{-2}) + \frac{4}{\pi w\sqrt{tb}}\cos(bw-tw),$$

and

$$(3.2.11)\quad J_{\nu+2}(tw)Y_\nu(bw) - J_\nu(bw)Y_{\nu+2}(tw) \sim O(|w|^{-2}) + \frac{4}{\pi w\sqrt{tb}}\sin(tw-bw).$$

Using Lemma 3.1, Lemma 2.5, (3.2.6), (3.2.7), (3.2.10) and (3.2.11), we get,

$$\left|\int_a^t t^{\nu+1} F_n(t,x)\,(t^2-x^2)\,dt\right| \leq \frac{2(t^2-x^2)\,t^{\nu+1}K_{11}}{\pi\sqrt{xt}} \times$$

$$\times \int_{-\infty}^{\infty} \frac{e^{-(b-x)|v|}}{\sqrt{B_n^2+v^2}}\left|\cos\{(b-t)(B_n+iv)\}\right|dv +$$

$$+ \frac{2(x^2-a^2)\,a^{\nu+1}K_{11}}{\pi\sqrt{ax}}\int_{-\infty}^{\infty}\frac{e^{-(b-x)|v|}}{\sqrt{B_n^2+v^2}}\left|\cos\{(b-a)(B_n+iv)\}\right|dv +$$

$$+ \frac{4t^{\nu+2}K_{11}}{\sqrt{xt}}\int_{-\infty}^{\infty}\frac{e^{-(b-x)|v|}}{B_n^2+v^2}\left|\sin\{(t-b)(B_n+iv)\}\right|dv +$$

$$+ \frac{4\, a^{\nu+2} K_{11}}{\sqrt{ax}} \int_{-\infty}^{\infty} \frac{e^{-(b-x)|v|}}{B_n^2 + v^2} \left|\sin\{(a-b)(B_n+iv)\}\right| dv +$$

$$+ K_{12} \int_{-\infty}^{\infty} \frac{e^{-(b-x)|v|}}{(B_n^2+v^2)^{5/2}}\, dv,$$

$$= I_1 + I_2 + I_3 + I_4 + I_5, \text{ say.}$$

Now,

$$I_1 \leq \frac{2(t^2-x^2)\, t^{\nu+1} K_{11}}{\pi \sqrt{xt}\, B_n} \int_{-\infty}^{\infty} e^{-(t-x)|v|}\, dv$$

$$\leq \frac{4(t+x)\, b^{\nu+1} K_{11}}{\pi \sqrt{xt}\, B_n}.$$

Similarly,

$$I_2 \leq \frac{4(x+a)\, a^{\nu+1} K_{11}}{\pi \sqrt{xt}\, B_n} < \frac{4(t+x)\, b^{\nu+1} K_{11}}{\pi \sqrt{xt}\, B_n}.$$

I_3, I_4 and I_5 can be estimated in the same way and each is found to be bounded and sufficiently small for large values of n. Thus (5.2.5) is proved in case $x < t$. The case $t < x$ follows in a similar way.

<u>LEMMA 5.3. For all sufficiently large</u> n <u>and all</u> $x \in [a,b]$,

$$\int_a^b \left|\sqrt{xt}\, R_n(t,x)\right| dt = O(\log n).$$

PROOF. Let $a \leq x \leq a+1/n$. Then, by Lemma 3.4,

$$\int_a^b \left|\sqrt{xt}\, R_n(t,x)\right| dt \leq K_8\, R_n\, (x+\tfrac{1}{n}-a) + K_8 \log \frac{b-x}{1/n}$$

$$= O(\log n),$$

by using Lemma 2.4.

In the same way, the estimation holds for $a+1/n < x < b-1/n$ and $b-1/n \leq x \leq b$.

LEMMA 3.6. (Titchmarsh[1]). If $a \leq A < B \leq b$, x lies outside $[A,B]$, and $\int_A^B t^{1/2} f(t)\, dt$ exists and is absolutely convergent, then

$$\lim_{n\to\infty} \int_A^B t\, f(t)\, R_n(t,x)\, dt = 0.$$

LEMMA 3.7. For $a+1/n < x < b-1/n$, and $\nu \geq -1/2$, $\nu \neq 0$,

$$\left|\int_a^b \sqrt{xt}\, R_n(t,x)\, dt - 1\right| \leq \frac{K_{12}}{n}\left[\frac{1}{x} + \frac{1}{b-x} + \frac{1}{x-a}\right].$$

PROOF. Let $a < d \leq x-1/n$. Define,

$$f_1(t) = \sqrt{x}\, t^{\nu}, \quad a \leq t \leq d,$$

$$= \sqrt{\tfrac{x}{t}}\, d^{\nu+1/2}, \quad d < t \leq b.$$

Then,

$$S_n(x,f_1) - f_1(x) = \left(\int_a^d + \int_d^b\right) \left\{\frac{f_1(t)}{t^{\nu}} - \frac{f_1(x)}{x^{\nu}}\right\} \times$$

1) Titchmarsh [96], p. xv.

$$\times\ t^{\nu+1} R_n(t,x)\,dt + \frac{d^{\nu+1/2}}{x^{\nu+1/2}}\left\{\int_a^b t^{\nu+1}\sqrt{x}\,R_n(t,x)\,dt - x^{\nu+\frac{1}{2}}\right\},$$

$$(3.2.12)\qquad = I_1 + I_2 + I_3, \text{ say.}$$

By Lemma 3.4,

$$|I_1| \leqslant \left\{1 - (d/x)^{\nu+1/2}\right\} K_8 \int_a^d \frac{t^{\nu+1/2}}{x-t}\,dt$$

$$(3.2.13)\qquad \leqslant K_8 \frac{d^{\nu+3/2} - a^{\nu+3/2}}{(\nu+3/2)(x-d)}\,.$$

$$|I_2| \leqslant d^{\nu+1/2}\left\{\int_a^{x-1/n} + \int_{x-1/n}^{x} + \int_x^{x+1/n} + \int_{x+1/n}^{b}\right\} \times$$

$$\times \left|1 - (t/x)^{\nu+1/2}\right|\sqrt{xt}\,\left|R_n(t,x)\right|dt =$$

$$= d^{\nu+1/2}\left\{I_{21} + I_{22} + I_{23} + I_{24}\right\}, \text{ say.}$$

By Lemma 3.6,

$$I_{21} = o(1), \qquad I_{24} = o(1), \quad \text{as } n \to \infty.$$

By Lemma 3.4,

$$|I_{22}| \leqslant K_8\, x_n \left\{\frac{1}{n} - \frac{x^{\nu+3/2} - (x-1/n)^{\nu+3/2}}{(\nu+3/2)\, x^{\nu+1/2}}\right\} \leqslant \frac{K_{14}}{nx}\,.$$

Similarly,

$$|I_{23}| \leqslant \frac{K_{14}}{nx}.$$

Therefore,

$$(3.2.14)\qquad |I_2| \leqslant d^{\nu+1/2}\left[\frac{2K_{13}}{nx} + o(1)\right].$$

Also, as in the proof of Lemma 3.3,

$$(3.2.15) \quad |I_3| \leq \frac{d^{\nu+1/2}}{x^{\nu+1/2}} \cdot \frac{K_7}{K_n} \left\{ \frac{1}{b-x} + \frac{1}{x-a} \right\}.$$

Collecting (3.2.12) to (3.2.15), we obtain,

$$\left| S_n(x,f_1) - f_1(x) \right| \leq \frac{d^{\nu+1/2}\, K_{13}}{n} \left\{ \frac{1}{x} + \frac{1}{x-a} + \frac{1}{b-x} \right\} + $$
$$+ K_8 \frac{d^{\nu+3/2} - a^{\nu+3/2}}{(\nu+3/2)(x-d)}.$$

Also, we have,

$$S_n(x,f_1) - f_1(x) = \int_a^d t^{\nu+1/2} \sqrt{xt}\, R_n(t,x)\, dt +$$
$$+ d^{\nu+1/2} \left\{ \int_d^b \sqrt{xt}\, R_n(t,x)\, dt - 1 \right\},$$

so that

$$\left| \int_a^b \sqrt{xt}\, R_n(t,x)\, dt - 1 \right| = \left| \int_a^d \sqrt{xt}\, R_n(t,x)\, dt + \right.$$
$$+ d^{-\nu-1/2} \left\{ S_n(x,f_1) - f_1(x) \right\} -$$
$$\left. - d^{-\nu-1/2} \int_a^d t^{\nu+1/2} \sqrt{xt}\, R_n(t,x)\, dt \right|$$
$$\leq \frac{K_{13}}{n} \left\{ \frac{1}{x} + \frac{1}{x-a} + \frac{1}{b-x} \right\} + K_8 d\, \frac{1-(a/d)^{\nu+3/2}}{(\nu+3/2)(x-d)} + \frac{2 K_6 (d-a)}{x-d}.$$

Taking limit as $d \to a$, the lemma follows.

3.3. <u>PROOF OF THEOREM 3.1.</u> By (3.1.8),

$$h_n'(t,x) = \sqrt{xt}\, R_n(t,x),$$

so that by (3.2.3),

$$\left|h_n'(t,x)\right| \le \frac{K_8}{|t-x|} \le \frac{K_8}{\delta}, \qquad \text{when } |t-x| \ge \delta.$$

This proves (3.1.9). In order to prove (3.1.10), we observe that $\frac{t^{-\nu-1/2}}{t^2-x^2}$ is a monotone non-increasing nonnegative function in the interval $x+\delta \le t \le b$. Hence, by the second mean value theorem, for any two numbers t_1, t_2, $x+\delta \le t_1 < t_2 \le b$,

$$h_n(t_2,x) - h_n(t_1,x) = \sqrt{\pi} \int_{t_1}^{t_2} \frac{t^{\nu+1}\, R_n(t,x)\, (t^2-x^2)\, dt}{t^{\nu+1/2}\, (t^2-x^2)}$$

$$= \frac{\sqrt{\pi}}{t_1^{\nu+1/2}\,(t_1^2-x^2)} \times$$

$$\times \int_{t_1}^{t'} t^{\nu+1}\, R_n(t,x)\, (t^2-x^2)\, dt,$$

for some t', $t_1 \le t' \le t_2$.

Therefore, by (3.2.5),

$$\left|h_n(t_2,x) - h_n(t_1,x)\right| \le \frac{\sqrt{b}}{(a+\delta)^{\nu+1/2}(\delta^2+2a\delta)} \cdot \frac{2\, b^{\nu+5/2}}{B_n\sqrt{a(a+\delta)}}$$

$$\le \frac{K_1(\delta)}{n}, \qquad \text{by Lemma 2.4.}$$

Similar conclusion can be drawn for $a \le t \le x-\delta$ also.

3.4. <u>Proof of Theorem 3.2.</u> It is enough to show that

(3.4.1) $\qquad \left|h_n(t,x)\right| \le K_{15}, \quad a \le t \le \frac{a}{3} + \frac{2x}{3};$

$$(3.4.2) \qquad \underset{t}{\text{osc.}} \left\{ h_n(t,x) : \frac{a}{3} + \frac{2x}{3} \leq t \leq \frac{2x}{3} + \frac{b}{3} \right\} \leq K_{15};$$

and

$$(3.4.3) \qquad \left| h_n(t,x) \right| \leq K_{15}, \quad \frac{2x}{3} + \frac{b}{3} \leq t \leq b.$$

In (3.4.1), since $t < x$, by (3.2.3),

$$\left| h_n(t,x) \right| \leq \int_a^t \frac{K_8}{x-t}\,dt$$

$$\leq K_8 \log 3.$$

(3.4.3) follows in a similar way. To prove (3.4.2), let us consider

$$(3.4.4) \qquad h_n^*(t,x) = \int_a^t \sqrt{xt}\, K_n(t,x)\,(t-x)\,dt.$$

By (3.2.5), we have,

$$\left| h_n^*(t,x) \right| \leq \frac{\sqrt{x}}{a^{\nu+1/2}(a+x)} \cdot \frac{K_9(t+x)b^{\nu+1}}{D_n\sqrt{xt}}$$

$$(3.4.5) \qquad \leq \frac{K_{16}}{D_n}, \quad \text{for} \quad \frac{a}{3} + \frac{2x}{3} \leq t \leq \frac{2x}{3} + \frac{b}{3}.$$

Now, by the second mean value theorem, for $x-\delta \leq t \leq x-\delta/2$, $\delta \leq (x-a)/3$,

$$(3.4.6) \qquad h_n(t,x) = \frac{1}{x-t} \int_\tau^t \sqrt{xt}\, K_n(t,x)\,(x-t)\,dt,$$

for some τ, $a \leq \tau \leq t$.

Hence, for $x-\delta \leq t \leq t' \leq x-\delta/2$, $\delta \leq (x-a)/3$, by (3.4.5) and (3.4.6),

$$\left|h_n(t,x)-h_n(t',x)\right| \leq \frac{2}{\delta}\Big\{\left|h_n^*(t,x)\right| + \left|h_n^*(\tau,x)\right| + \left|h_n^*(t',x)\right| + \left|h_n^*(\tau',x)\right|\Big\}$$

$$\leq \frac{8K_{16}}{\delta D_n},$$

so that,

$$(3.4.7) \qquad \underset{t}{\text{osc.}}\left\{h_n(t,x):\ x-\delta \leq t \leq x-\delta/2\right\} \leq \frac{8K_{16}}{\delta D_n}.$$

Similarly, for $\eta \leq (b-x)/3$,

$$(3.4.8) \qquad \underset{t}{\text{osc.}}\left\{h_n(t,x):\ x+\eta/2 \leq t \leq x+\eta\right\} \leq \frac{8K_{16}}{\delta D_n}.$$

Let us choose $\delta_r = \dfrac{x-a}{3\cdot 2^r}$ and $\eta_r = \dfrac{b-x}{3.2^r}$, $r=0, 1, 2,\ldots$, so that each of δ_r and η_r exceeds $1/D_n$. Adding the inequalities corresponding to (3.4.7) and (3.4.8) we get

$$\underset{t}{\text{osc.}}\left\{h_n(t,x):\ \frac{a}{3}+\frac{2x}{3} \leq t < x-1/D_n\right\} \leq \sum_{\delta_r > \frac{1}{D_n}} \frac{8K_{16}}{\delta_r D_n} =$$

$$= \frac{24\, K_{16}\,(2^N-1)}{(x-a)\, D_n}, \quad \text{where } \delta_{N-1} > 1/D_n \geq \delta_N,$$

$$(3.4.9) \qquad = K_{17}.$$

Similarly,

$$(3.4.10) \qquad \underset{t}{\text{osc.}}\left\{h_n(t,x):\ x+1/D_n < t \leq \frac{2x}{3}+\frac{b}{3}\right\} \leq K_{17}.$$

Also, by (3.2.4),

$$(3.4.11) \qquad \int_{x-1/D_n}^{x+1/D_n} |h_n'(t,x)| \, dt \leq 2 K_8.$$

The proof of (3.4.2) is now completed by (3.4.9) to (3.4.11).

3.5. To prove Theorem 3.3, we need the following lemmas:

LEMMA 3.8. (Localization principle). Given any fixed $\delta > 0$, *the integrals*

$$(3.5.1) \qquad \int_a^{x-\delta} t^{1/2} f(t) \sqrt{xt}\, R_n(t,x) \, dt$$

and

$$(3.5.2) \qquad \int_{x+\delta}^{b} t^{1/2} f(t) \sqrt{xt}\, R_n(t,x) \, dt$$

tend to zero uniformly in x, *as* $n \to \infty$ *for* $a \leq x \leq b$, *provided* $f(t)$ *is Lebesgue integrable.*

PROOF. If we assume

$$\varphi(t,x,n) = \sqrt{xt}\, R_n(t,x),$$

the general convergence theorem[1] applies. In fact, the the conditions (1) and (2) of the theorem are satisfied by (3.1.9) and (3.1.10) of Theorem 3.1.

---

[1] Hobson [49], p. 422.

<u>LEMMA 3.9. Let f be a function of bounded variation in [a,b], vanishing at a and b. Then the expression</u>

$$S_n(x,f) - f(x-0)\,h_n(x,x) - f(x+0)\left\{h_n(b,x) - h_n(x,x)\right\}$$

<u>tends to zero boundedly as</u> $n \to \infty$, <u>for</u> $a \leq x \leq b$.

[Note:- The functions $f(t)$ and $h_n(t,x)$ can be supposed constant for $t \leq a$ and for $t \geq b$, if necessary, so that $f(a-0) = f(b-0) = 0$.]

<u>PROOF.</u> The expression is equal to

$$x^{-1/2}\left|\int_a^b t^{1/2} f(t)\,dh_n(t,x) - \int_a^x x^{1/2} f(x-0)\,dh_n(t,x) - \right.$$

$$\left. - \int_x^b x^{1/2} f(x+0)\,dh_n(t,x)\right| =$$

$$= x^{-1/2}\left|\int_a^x \left\{t^{1/2} f(t) - x^{1/2} f(x-0)\right\} dh_n(t,x) + \right.$$

$$\left. + \int_x^b \left\{t^{1/2} f(t) - x^{1/2} f(x+0)\right\} dh_n(t,x)\right|$$

$$= x^{-1/2}\left|\int_{x-\delta}^x \left\{t^{1/2} f(t) - x^{1/2} f(x-0)\right\} dh_n(t,x) + \right.$$

$$\left. + \int_x^{x+\delta} \left\{t^{1/2} f(t) - x^{1/2} f(x+0)\right\} dh_n(t,x) + o(1)\right|, \tag{3.5.3}$$

by Lemma 3.8, where $o(1)$ is an expression tending to zero uniformly as $n \to \infty$, for fixed $\delta > 0$.

By using the inequality[1)]

---

1) Young [111], p. 301, foot note.

$$\int_a^b F(t)\, dG(t) \leq \left[|F(\alpha)| + V_1 \{F(t): a \leq t \leq b\} \right] \times$$

$$\times \underset{t}{\text{osc.}} \{G(t): a \leq t \leq b\}, \quad a \leq \alpha \leq b,$$

where $V_1\{F(t): a \leq t \leq b\}$ is the total variation of F in $[a,b]$, and Theorem 3.2, if $V(\delta;x)$ is the total variation of $t^{1/2} f(t)$ in $x-\delta \leq t \leq x$ and $x \leq t \leq x+\delta$, we observe that the right hand side of (3.5.3) does not exceed

$$2\, K_2\, V(\delta;x) + o(1).$$

Since, $V(\delta;x)$ is bounded in x and δ, and for a fixed x, it is arbitrarily small with δ, the lemma follows.

LEMMA 3.10. *Let f be a continuous function of bounded variation in $[a,b]$, vanishing at a and b. Then*

$$S_n(x,f) - f(x)\, h_n(b,x)$$

tends to zero uniformly, as $n \to \infty$, for $a \leq x \leq b$.

PROOF. Since f is continuous in $[a,b]$, it is also uniformly continuous there. Hence, $f(x+0) = f(x-0) = f(x)$, for $a \leq x \leq b$. The expression of Lemma 3.9, now, reduces to

$$S_n(x,f) - f(x)\, h_n(b,x),$$

the absolute value of which does not exceed

$$2\, K_2\, V(\delta;x) + o(1),$$

as has already been proved. Since, by the continuity of $x^{1/2} f(x)$ in $[a,b]$, $V(\delta;x)$ converges to zero uniformly as $\delta \to 0$, the conclusion of the lemma is proved.

LEMMA 3.11. For $a < x < b$,

$$\lim_{n\to\infty} \int_a^x \sqrt{xt}\, h_n(t,x)\, dt = 1/2, \tag{3.5.4}$$

and for $a+\varepsilon \le x \le b-\varepsilon$, where $\varepsilon > 0$, we have uniformly,

$$\lim_{n\to\infty} \int_a^b \sqrt{xt}\, h_n(t,x)\, dt = 1. \tag{3.5.5}$$

PROOF. Let us define

$$g(t) = t^{\nu} - a^{\nu+1/2}\, t^{-1/2}, \quad a \le t < x,$$

$$= 0, \qquad x \le t \le b.$$

Then, by Lemma 3.9,

$$\lim_{n\to\infty} S_n(x,g) = \lim_{n\to\infty} (x^{\nu} - a^{\nu+1/2}\, x^{-1/2})\, h_n(x,x). \tag{3.5.6}$$

Also,

$$S_n(x,g) = \int_a^x \sqrt{t}\, (t^{\nu+1/2} - a^{\nu+1/2})\, h_n(t,x)\, dt.$$

Therefore, by (3.2.1),

$$\lim_{n\to\infty} S_n(x,g) = \tfrac{1}{2} x^{\nu} - a^{\nu+1/2}\, x^{-1/2} \lim_{n\to\infty} h_n(x,x). \tag{3.5.7}$$

By (3.5.6) and (3.5.7), we obtain,

$$\lim_{n\to\infty} h_n(x,x) = 1/2.$$

This proves (3.5.4). Again, let

$$k(t) = t^{\nu}, \quad a < t < b,$$

$$= 0, \quad t=a,\ t=b.$$

Then, for any $\varepsilon > 0$, k satisfies the condition of

Lemma 3.10 in $[a+\varepsilon, b-\varepsilon]$, so that

(3.5.8) $$\lim_{n\to\infty} S_n(x,k) = \lim_{n\to\infty} x^{\nu} h_n(b,x),$$

uniformly, for $a+\varepsilon \leq x \leq b-\varepsilon$. Also, by (3.2.2),

(3.5.9) $$\lim_{n\to\infty} S_n(x,k) = x^{\nu},$$

uniformly, for $a+\varepsilon \leq x \leq b-\varepsilon$.

From (3.5.8) and (3.5.9), (3.5.5) is, now, proved.

3.6. PROOF OF THEOREM 3.3. By (3.5.4), for $a < x < b$,

$$f(x-0)\, h_n(x,x) \to \frac{1}{2} f(x-0),$$

and by (3.5.4) and (3.5.5),

$$f(x+0)\left\{h_n(b,x) - h_n(x,x)\right\} \to \frac{1}{2} f(x+0),$$

boundedly, as $n \to \infty$.

Hence, (3.1.11) gets proved by Lemma 3.9.

Again, if f is continuous in $[a,b]$, then by Lemma 3.10,

$$\lim_{n\to\infty} S_n(x,f) = \lim_{n\to\infty} f(x)\, h_n(b,x),$$

uniformly, for $a \leq x \leq b$.

By (3.5.5), the theorem is, now, proved.

3.7. PROOF OF THEOREM 3.4. We have,

$$S_n(x,f) - f(x) = \int_a^b \left\{t^{1/2} f(t) - x^{1/2} f(x)\right\} \sqrt{t} \; \times$$

$$\times\, R_n(t,x)\, dt + f(x)\left\{\int_a^b \sqrt{xt}\, R_n(t,x)\, dt - 1\right\}$$

(3.7.1) $\qquad = U_n(x) + V_n(x)$, say.

As in the proof of Theorem 2.1, by Lemmas 3.5 and 3.6,

(3.7.2) $\quad \left|U_n(x)\right| \leq K_{18}\, \omega(1/n, F) \log n, \quad a \leq x \leq b,$

and

(3.7.3) $\left|V_n(x)\right| \leq K_{19}\, \omega(1/n, F) \log n, \quad a \leq x \leq a+1/n,$

$$b-1/n \leq x \leq b.$$

Also, by Lemma 3.7, for $a+1/n \leq x \leq b-1/n$,

$$\left|V_n(x)\right| \leq \left|F(x)\right| \frac{K_{13}}{n}\left\{\frac{1}{x} + \frac{1}{b-x} + \frac{1}{x-a}\right\}$$

$$\leq K_{13}\left\{\frac{|F(x)-F(a)|}{nx} + \frac{|F(b)-F(x)|}{n(b-x)} + \frac{|F(x)-F(a)|}{n(x-a)}\right\}$$

$$\leq K_{13}\, \omega(1/n, F)\left\{\frac{n(x-a)+1}{nx} + \frac{n(b-x)+1}{n(b-x)} + \frac{n(x-a)+1}{n(x-a)}\right\}$$

(3.7.4) $\qquad \leq 6 K_{13}\, \omega(1/n, F).$

By (3.7.1) to (3.7.4), (3.1.13) is proved. This proves the theorem.

. . .