## CHAPTER IV

## MEAN CONVERGENCE OF FOURIER-BESSEL SERIES.

4.1. Let $L_\beta^p[0,1]$, $p \geq 1$, $\beta$ any real number, be the class of all measurable functions $f$ defined on $[0,1]$, for which,

$$(4.1.1) \qquad \| f \|_{p,\beta} = \left\{ \int_0^1 |f(x)|^p \, x^\beta \, dx \right\}^{1/p} < \infty.$$

It is known[1], that $L_\beta^p$ is a Banach space under the norm defined by (4.1.1). It is easy to verify that $L_\alpha^p \subseteq L_\beta^p$, for $\alpha \leq \beta$; $L_0^p = L^p$ and for $0 < a < b$,

$$L_\beta^p[a,b] = L^p[a,b].$$

For $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\beta' = -\frac{\beta}{p-1}$, $L_{\beta'}^{p'}[0,1]$ denotes the conjugate space for $L_\beta^p[0,1]$. If $f \in L_\beta^p[0,1]$,

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[1] Alexits [9].

$g \in L_{\beta'}^{p'}[0,1]$; then $fg \in L^1[0,1]$ and the following Hölder type inequality holds:

$$(4.1.2) \qquad \left| \int_0^1 f(x)\, g(x)\, dx \right| \leqslant \|f\|_{p,\beta} \|g\|_{p',\beta'} .$$

Let $J_\nu(t)$ and $Y_\nu(t)$ denote Bessel functions of the first and second kinds respectively, each of order $\nu \geqslant -1/2$. Denote by $j_1 < j_2 < j_3 < \dots$ the positive zeros of $J_\nu(t)$, arranged in ascending order of magnitude, by $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ of $\ell t J_\nu'(t) + h J_\nu(t)$ and by $\gamma_1 < \gamma_2 < \gamma_3 < \dots$ those of $c_\nu(at, bt)$, where $0 < a < b$, and

$$c_\nu(\alpha, \beta) = J_\nu(\alpha)\, Y_\nu(\beta) - J_\nu(\beta)\, Y_\nu(\alpha).$$

Let us define, for $m = 1, 2, 3, \dots$,

$$(4.1.3) \qquad \begin{cases} u_m(t) = \dfrac{\sqrt{2t}\, J_\nu(t j_m)}{J_{\nu+1}(j_m)}, & t > 0, \\ u_m(0) = \lim\limits_{t \to 0+} u_m(t); \end{cases}$$

$$(4.1.4) \qquad \begin{cases} v_m(t) = \dfrac{\sqrt{t}\, J_\nu(t\lambda_m)}{f_m}, & t > 0, \\ v_m(0) = \lim\limits_{t \to 0+} v_m(t), \end{cases}$$

where

$$(4.1.5) \qquad 2f_m^2 = (1 - \nu^2/\lambda_m^2)\, J_\nu^2(\lambda_m) + J_\nu'^2(\lambda_m);$$

and

$$(4.1.6)\quad G_m(t) = \frac{\sqrt{t}\; c_\nu(t\gamma_m, b\gamma_m)}{c_m}, \quad a \le x \le b,$$

where

$$(4.1.7)\qquad c_m^2 = \frac{2\left\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\right\}}{\pi^2\, \gamma_m^2\, J_\nu^2(a\gamma_m)}.$$

It is known[1)] that, each of the sequences $\{u_m(t)\}$ and $\{v_m(t)\}$ is an orthonormal sequence on $[0,1]$ and the sequence $\{G_m(t)\}$ is an orthonormal sequence on $[a,b]$, $0 < a < b$.

Since $J_\nu(t) \sim \frac{t^\nu}{2\Gamma(\nu+1)}$, near zero, $u_m(t)$, $v_m(t)$ $\in L_\beta^p[0,1]$, if

$$(4.1.8)\qquad \begin{cases} \nu + 1/2 > 0, \text{ for } \beta > -1 \text{ and} \\ (\nu+1/2)p + \beta + 1 > 0, \text{ for } \beta \le -1. \end{cases}$$

Mean convergence of Fourier-Bessel series has been studied by Wing[2)], Stendich[3)], Hochstadt[4)], Gol'dman[5)], Benedek and Panzone[6)] etc. For Fourier series also, this concept is widely known[7)].

Gol'dman[5)] has shown that the system $\{u_m(t)\}$, forms

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1) Bowman [18]; Tolstov [99]; Titchmarsh [96].
2) Wing [110]. 3) Stendich [84]. 4) Hochstadt [50].
5) Gol'dman [36]. 6) Benedek and Panzone [13]; [14].
7) Titchmarsh [98]; Edwards [29].

an orthonormal basis in the space $L_\beta^p[0,1]$, $p > 1$, $-1 < \beta < p-1$. He has also proved that if $\beta$ does not lie in this range, then there exists a function in $L_\beta^p[0,1]$, whose Fourier-Bessel expansion corresponding to this system diverges.

In this chapter, we establish similar results for the orthonormal sequences $\{v_m(t)\}$ and $\{C_m(t)\}$. We also prove the convergence in the $L_\beta^p$-norm of the sequence of partial sums of the Dini series (FB-II),

$$(4.1.9) \qquad f(x) \sim \sum_{m=1}^{\infty} b_m v_m(x), \qquad 0 \leq x \leq 1,$$

where $h/\ell + \nu > 0$ or $\ell = 0$, and

$$b_m = \int_0^1 f(t)\, v_m(t)\, dt,$$

corresponding to $f \in L_\beta^p[0,1]$, [the integral on the right makes sense on account of the estimate (4.6.3), which is proved later in this chapter].

For the Fourier-Bessel series of third type (FB-III),

$$(4.1.10) \qquad f(x) \sim \sum_{m=1}^{\infty} d_m C_m(x), \qquad 0 < a \leq x \leq b,$$

where

$$d_m = \int_a^b f(t)\, C_m(t)\, dt,$$

corresponding to any function $f \in L^p[a,b]$, $p > 1$, it has been derived that the series (4.1.10) converges in the $L^p$-norm[1] to f, in case $1 < p \leq 2$.

We prove the following theorems:

THEOREM 4.1. The system $\{v_n(t)\}$ forms a basis in the Banach space $L_\beta^p[0,1]$, where $p > 1$, $-1 < \beta < p-1$, $\nu \geq -1/2$.

THEOREM 4.2. For $\nu \geq -1/2$ and $f \in L_\beta^p[0,1]$, $p > 1$, $-1 < \beta < p-1$,

$$(4.1.11) \qquad \lim_{n \to \infty} \int_0^1 |f(x) - S_n(x,f)|^p x^\beta \, dx = 0,$$

where $S_n(x,f)$ is the n-th partial sum of (4.1.9).

THEOREM 4.3. There exists a function $f_0 \in L_{p-1}^p$, whose Dini series diverges in the $L_{p-1}^p$-norm.

THEOREM 4.4. For $p > 1$, if $\beta > p-1$ or if $\beta < -1$, then there is a function in $L_\beta^p[0,1]$, whose Dini series diverges.

Remark 4.1. It may be noted that the system (4.1.3) is a particular case of (4.1.4), when $\ell = 0$; and when $\nu = 1/2$ or $-1/2$, (4.1.4) reduces to trigonometric systems. Hence our theorems 4.1 to 4.4 generalize the results on FB-I and Fourier-trigonometric series.

THEOREM 4.5. The system $\{c_n(t)\}$ forms an orthonormal basis in the space $L^p[a,b]$, $0 < a < b$, $p > 1$, $\nu \geq -1/2$.

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1) Sansone [SO], Ch. I.

THEOREM 4.6. *If* $1 < p \leq 2$, $f \in L^p[a,b]$, $0 < a < b$, *and* $\nu \geq -1/2$, *then*

$$\lim_{n \to \infty} \int_a^b |f(x) - T_n(x,f)|^p \, dx = 0,$$

*where* $T_n(x,f)$ *is the n-th partial sum of the series* (4.1.10).

4.2. The partial sums $S_n(x,f)$ of (4.1.9) are given by

$$S_n(x,f) = \sum_{m=1}^{n} b_m \, v_m(x) = \int_0^1 f(t)\, U_n(t,x)\, dt, \tag{4.2.1}$$

where

$$U_n(t,x) = \sum_{m=1}^{n} \frac{2\sqrt{xt}\,\lambda_m^2\, J_\nu(t\lambda_m)\, J_\nu(x\lambda_m)}{(\lambda_m^2-\nu^2)J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m)}. \tag{4.2.2}$$

If $\lambda_n < D_n < \lambda_{n+1}$, then[1)]

$$U_n(t,x) = \frac{1}{2i}\int_{D_n - i\infty}^{D_n + i\infty} \frac{\sqrt{tx}\, w\, \phi(w,x)\, J_\nu(tw)}{\ell w J_\nu'(w) + h J_\nu(w)}\, dw, \tag{4.2.3}$$

when $0 < t < x < 1$, and

$$U_n(t,x) = \frac{1}{2i}\int_{D_n - i\infty}^{D_n + i\infty} \frac{\sqrt{xt}\, w\, \phi(w,t)\, J_\nu(xw)}{\ell w J_\nu'(w) + h J_\nu(w)}\, dw, \tag{4.2.4}$$

when $0 < x < t < 1$, where

$$\begin{cases} \phi(w,x) = (h + \ell\nu)\,\phi_1(w,x) - w\,\phi_2(w,x), \\ \phi_1(w,x) = J_\nu(w)\, Y_\nu(xw) - J_\nu(xw)\, Y_\nu(w), \text{ and} \\ \phi_2(w,x) = J_{\nu+1}(w)\, Y_\nu(xw) - J_\nu(xw)\, Y_{\nu+1}(w). \end{cases} \tag{4.2.5}$$

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1) Watson [103], p. 602.

Moore[1] has shown that

(4.2.6) $$\lambda_n = n\pi + q + M(\lambda_n)/n,$$

where

(4.2.7) $$q = \begin{cases} k\pi + (2\nu + 1)\pi/4, & \text{if } \ell \neq 0, \\ k\pi + (2\nu - 1)\pi/4, & \text{if } \ell = 0, \end{cases}$$

$k$ is an integer, positive, negative or zero and $M(\lambda_n)$ remains bounded for sufficiently large $n$.

We, may therefore, choose for large $n$,

(4.2.8) $$B_n = n\pi + (2\nu + 3)\pi/4.$$

The n-th partial sum $S_n(x,f)$ of series FB-III, i.e. series (4.1.10), is given by

(4.2.9) $$S_n(t,x) = \int_a^b f(t)\, R_n(t,x)\, dt,$$

where

(4.2.10) $$R_n(t,x) = \sum_{m=1}^{n} \frac{\pi^2 \sqrt{xt}\, \gamma_m^2 J_\nu^2(a\gamma_m) c_\nu(x\gamma_m, b\gamma_m) c_\nu(t\gamma_m, b\gamma_m)}{2\left\{J_\nu^2(a\gamma_m) - J_\nu^2(b\gamma_m)\right\}}.$$

By contour integration[2]

(4.2.11) $$R_n(t,x) = \frac{\sqrt{tx}}{2i} \int_{B_n - i\infty}^{B_n + i\infty} \frac{w\, c_\nu(xw, aw)\, c_\nu(tw, bw)}{c_\nu(aw, bw)}\, dw$$

where $0 < a < x < t < b$ and $\gamma_n < B_n < \gamma_{n+1}$.

If $0 < a < t < x < b$, then $x$ and $t$ are to be interchanged in the integrand on the right hand side in (4.2.11).

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[1] Moore [67]. [2] Titchmarsh [96].

In view of Naylor[1], we may take, for large n,

$$B_n = \frac{(n+1/2)\pi}{b-a}. \qquad (4.2.12)$$

4.3. The following lemmas are required to complete the proofs of our theorems [$K_1, K_2, K_3, \ldots$ etc. denote suitable positive constants depending upon p, $\beta$ and $\nu$ only]:

LEMMA 4.1. (Gol'dman[2]). Let $f \in L_\beta^p[0,1]$, $p>1$, $\beta < p-1$; and let $F(x) = \int_0^x |f(t)|\,dt$. Then $\frac{F(x)}{x} \in L_\beta^p[0,1]$, and

$$\left\| \frac{F(x)}{x} \right\|_{p,\beta} \leq K_1 \|f\|_{p,\beta}. \qquad (4.3.1)$$

LEMMA 4.2. (Gol'dman[2]). Let $f \in L_\beta^p[0,1]$, $p>1$, $-1<\beta<p-1$; and let $G(x) = \int_0^1 \frac{|f(t)|}{x+t}\,dt$. Then $G \in L_\beta^p[0,1]$ and

$$\|G\|_{p,\beta} \leq K_2 \|f\|_{p,\beta}. \qquad (4.3.2)$$

LEMMA 4.3. (Gol'dman[2]). If $f \in L^p[0,1]$, $p>1$ and $g(x) = \int_0^1 \frac{|f(t)|}{2-x-t}\,dt$, then $g \in L^p[0,1]$ and $\|g\|_p \leq K_3 \|f\|_p$.

LEMMA 4.4. Let $f \in L^p[a,b]$ and let

$$\varphi(x) = \int_a^b \frac{|f(t)|}{t+x-2a}\,dt, \quad a \leq x \leq b.$$

Then $\varphi \in L^p[a,b]$ and $\|\varphi\|_p \leq \|f\|_p$.

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[1] Naylor [72], p. 69; Also refer Lemma 2.4 of this thesis.
[2] Gol'dman [26].

PROOF. We have,

$$\varphi(x+a) = \int_0^{b-a} \frac{|f(t+a)|}{t+x}\,dt, \quad 0 \leq x \leq b-a.$$

Now, the lemma follows from Lemma 4.2.

To prove Theorems 4.1 and 4.2, we need to study the behavior of $U_n(t,x)$ in the rectangle $0 < x < 1$, $0 < t < 1$.

We choose, for any $x, t \in (0,1)$, $x \neq t$, a positive integer n, such that

$$D_n > K_4/(x-t)_. \tag{4.3.3}$$

LEMMA 4.5. Let $0 < x < 2/D_n$, $0 < t < 4/D_n$. Then

$$|U_n(t,x)| \leq K_5\, D_n. \tag{4.3.4}$$

PROOF. Let $t < x$. For $w$ lying on $D_n - i\infty$ to $D_n + i\infty$,[1]

$$|\phi_1(w,x)| = \frac{1}{2}\left|H_\nu^{(1)}(w)\, H_\nu^{(2)}(xw) - H_\nu^{(1)}(xw)\, H_\nu^{(2)}(w)\right|$$

$$\leq \frac{K_6\, e^{(1-x)|v|}}{|w|\ \sqrt{x}}, \tag{4.3.5}$$

and similarly,

$$|\phi_2(w,x)| \leq \frac{K_6\, e^{(1-x)|v|}}{|w|\ \sqrt{x}}, \tag{4.3.6}$$

where $w = D_n + iv$, $-\infty < v < \infty$.

From (4.2.5), (4.3.5) and (4.3.6), we obtain,

$$|\phi(w,x)| \leq \frac{K_7\, e^{(1-x)|v|}}{\sqrt{x}}, \tag{4.3.7}$$

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1) Watson [105], §§ 3.61, 18.51.

for $w = D_n + iv$, $-\infty < v < \infty$.

Again, from the inequalities[1],

(4.3.8) $$|J_\nu(wt)| \leq \frac{K_8\, e^{|tv|}}{\sqrt{|wt|}},$$

and

(4.3.9) $$|J_\nu(w)| \geq \frac{K_9\, e^{|v|}}{\sqrt{w}},$$

for $w = D_n + iv$, $|v| \to \infty$, we have,

(4.3.10) $$\left| \frac{w}{h J_\nu(w) + \ell w J_\nu'(w)} \right| \leq K_{10} \sqrt{|w|}\, e^{-|v|}.$$

By (4.2.3), (4.3.7), (4.3.8) and (4.3.10), we now have,

$$|U_n(t,x)| \leq K_7 K_8 K_{10} \int_0^\infty e^{-(x-t)v}\, dv$$

$$< K_5 D_n,$$

by (4.3.5). If $x < t$, then (4.2.4) is used instead of (4.2.3) and (4.3.4) follows.

LEMMA 4.6. For $2/D_n < x < 1$, $1/D_n < t < 1$,

(4.3.11) $$U_n(t,x) = \left\{1 + O(D_n^{-1})\right\} \overline{U}_n(t,x),$$

where

(4.3.12) $$\overline{U}_n(t,x) = \frac{1}{2}\left\{\frac{\sin(x-t)D_n}{\sin(x-t)\pi/2} + \frac{\sin(2-x-t)D_n}{\sin(x+t)\pi/2}\right\} + O\left(\frac{1}{D_n xt(2-x-t)}\right) + O(1).$$

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[1] Watson [103], p. 584.

<u>PROOF.</u> Using asymptotic expansions[1],

$$\frac{1}{h\,J_\nu(w) + \ell w\,J_\nu'(w)} = \frac{1}{-\ell\sqrt{\frac{2w}{\pi}}\cos\left(w-\frac{\nu\pi}{2}-\frac{3\pi}{4}\right) + O(|w|^{-1})}.$$

Therefore, if $t < x$, then by (4.2.5), (4.3.11) is true, for

$$(4.3.13)\quad \bar{U}_n(t,x) = \frac{1}{2i\ell}\int_{D_n-\infty i}^{D_n+i\infty} \sqrt{\frac{\pi wtx}{2}}\,\frac{\phi(w,x)\,J_\nu(tw)\,dw}{\cos\left(w-\frac{\nu\pi}{2}+\frac{\pi}{4}\right)},$$

where $\ell \neq 0$.

Again, using the following asymptotic expansions[2],

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{P_z\cos(z-\alpha) - Q_z\sin(z-\alpha)\right\},$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\left\{P_z\sin(z-\alpha) + Q_z\cos(z-\alpha)\right\},$$

where $|z| > 1$, $\alpha = (2\nu+1)\pi/4$ and

$$(4.3.14)\quad P_z = 1 + O(|z|^{-2}),\quad Q_z = \frac{K_{11}}{z}\left\{1 + O(|z|^{-2})\right\};$$

we obtain from (4.2.5),

$$(4.3.15)\quad \phi_1(w,x) = \frac{2}{\pi w\sqrt{x}}\left\{A\cos(1-x)w - B\sin(1-x)w\right\},$$

and

$$(4.3.16)\quad \phi_2(w,x) = \frac{2}{\pi w\sqrt{x}}\left\{A\sin(1-x)w + B\cos(1-x)w\right\},$$

where

$$(4.3.17)\quad A = P_w Q_{xw} - Q_w P_{xw},\ \text{and}\ B = P_w P_{xw} + Q_w Q_{xw}.$$

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[1] Watson [103], § 7.21. [2] Gol'dman [56].

From (4.3.14) to (4.3.17), we get,

(4.3.18)
$$\begin{cases} A\,P_{tw} = \dfrac{K_{12}}{xw} - \dfrac{K_{12}}{w} + O(|tw|^{-2}), \\ B\,P_{tw} = 1 + O(|tw|^{-2}), \\ A\,Q_{tw} = O(|tw|^{-2}), \text{ and} \\ B\,Q_{tw} = \dfrac{K_{13}}{tw} + O(|tw|^{-2}). \end{cases}$$

Therefore, from (4.2.5), (4.3.15), (4.3.16) and (4.3.18),

$$\begin{aligned} \phi(w,x)\, J_\nu(tw) = \left(\frac{2}{\pi w}\right)^{3/2} \frac{1}{\sqrt{xt}} \Big[ & -\ell w \cos(1-x)w \cos(tw-\alpha) - \\ & - \ell \left\{ K_{12} \frac{\sin(1-x)w \cos(tw-\alpha)}{x} - \right. \\ & \left. - K_{13} \frac{\cos(1-x)w \sin(tw-\alpha)}{t} \right\} - \\ & - (h+\ell\nu) \sin(1-x)w \cos(tw-\alpha) + \\ & + K_{14} \frac{\mu(t,x,w)}{(tw)^{5/2} \sqrt{x}} \Big], \end{aligned}$$

where

(4.3.19) $$|\mu(t,x,w)| \le e^{-(x-t)|v|},$$

for $w = D_n + iv$, $-\infty < v < \infty$.

Therefore, from (4.3.13),

(4.3.20) $$\overline{U}_n(t,x) = I_1 + I_2 + I_3 + I_4,$$

where

$$(4.3.21)\quad \begin{cases} I_1 = \dfrac{1}{\pi i}\displaystyle\int_{D_n - i\infty}^{D_n + i\infty} \dfrac{\cos(1-x)w\,\cos(tw-\alpha)}{\sin(w-\alpha)}\,dw, \\[2ex] I_2 = \dfrac{K_{15}}{\pi i}\displaystyle\int_{D_n - i\infty}^{D_n + i\infty} \left\{\dfrac{\sin(1-x)w\,\cos(tw-\alpha)}{x\,w\,\sin(w-\alpha)} - \right. \\[2ex] \qquad\qquad \left. - \dfrac{\cos(1-x)w\,\sin(tw-\alpha)}{tw\,\sin(w-\alpha)}\right\}dw, \\[2ex] I_3 = \dfrac{K_{16}}{\pi i}\displaystyle\int_{D_n - i\infty}^{D_n + i\infty} \dfrac{\sin(1-x)w\,\cos(tw-\alpha)}{w\,\sin(w-\alpha)}\,dw, \text{ and} \\[2ex] I_4 = \dfrac{K_{17}}{\pi i}\displaystyle\int_{D_n - i\infty}^{D_n + i\infty} \dfrac{\mu(t,x,w)}{t^2\,w^2}\,dw. \end{cases}$$

Now, if $1/D_n < t < x/2$, then $x-t > x/2$, so that from (4.3.19),

$$|I_4| \leq \frac{K_{18}}{t^2 D_n^{\,2}} \cdot \frac{1}{x-t} < \frac{K_{18}}{D_n x t}.$$

If $x/2 \leq t < x$, then from (4.3.3),

$$|I_4| \leq \frac{K_{18}}{D_n x t}.$$

Hence, for $1/D_n < t < x < 1$,

$$(4.3.22)\qquad |I_4| < \frac{K_{18}}{D_n x t}.$$

Further,

$$I_2 = \frac{K_{15}(x+t)}{2\pi i x t} \int_{D_n - i\infty}^{D_n + i\infty} \frac{\sin\{(1-x-t)w+\alpha\}}{w \sin(w-\alpha)}\, dw -$$

$$- \frac{K_{15}(x-t)}{2\pi i x t} \int_{D_n - i\infty}^{D_n + i\infty} \frac{\sin\{(1-x+t)w-\alpha\}}{w \sin(w-\alpha)}\, dw$$

$$= I_2' + I_2'', \text{ say.}$$

$$|I_2'| \leqslant \frac{K_{15}(x+t)}{\pi x t} \int_0^{\infty} \frac{e^{(|1-x-t|-1)v}}{\sqrt{D_n^2+v^2}}\, dv$$

$$\leqslant \frac{K_{15}}{\pi x t D_n}, \text{ if } 1-x-t \geqslant 0,$$

and $|I_2'| \leqslant \dfrac{K_{15}}{x t D_n (2-x-t)}$, if $1-x-t < 0$.

Therefore, treating $I_2''$ similarly, we get,

(4.3.23) $I_2 = O\left(\dfrac{1}{x t D_n (2-x-t)}\right)$.

$$I_3 = \frac{K_{16}}{2\pi i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{\sin\{(1-x+t)w-\alpha\}}{w \sin(w-\alpha)}\, dw +$$

$$+ \frac{K_{16}}{2\pi i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{\sin\{(1-x-t)w+\alpha\}}{w \sin(w-\alpha)}\, dw$$

$$= I_3' + I_3'', \text{ say.}$$

$I_3''$ is treated like $I_2'$, and we obtain,

$$I_3'' = O\left(\frac{1}{x t D_n (x+t)(2-x-t)}\right).$$

$$I_3' = \frac{K_{16}}{2\pi i}\int_{D_n - i\infty}^{D_n + i\infty} \frac{\cos\left(\lambda w - \frac{\nu\pi}{2} - \frac{3\pi}{4}\right) dw}{w\cos\left(w - \frac{\nu\pi}{2} - \frac{3\pi}{4}\right)},$$

[where $\lambda = 1-x+t$],

$$= O(1/D_n),$$

by Watson[1], since $-1 < \lambda < 1$.

Therefore,

$$I_3 = O\left(\frac{1}{x\,t\,D_n(x+t)(2-x-t)}\right) + O(1). \tag{4.3.24}$$

Finally, using the integral[2],

$$\int_{-\infty}^{\infty} \frac{\cosh \lambda v}{\cosh v}\, dv = \frac{\pi}{\sin\frac{\pi}{2}(1-\lambda)}, \quad -1 < \lambda < 1, \tag{4.3.25}$$

we obtain, in view of (4.2.8),

$$I_1 = \frac{1}{2\pi i}\int_{D_n - i\infty}^{D_n + i\infty} \Big[\cos\{(1-x+t)w - a\} +$$

$$+ \cos\{(1-x-t)w + a\}\Big] \frac{dw}{\sin(w-a)}$$

$$= \frac{\sin(2-x-t)D_n}{2\pi}\int_{-\infty}^{+\infty} \frac{\cosh(1-x-t)v}{\cosh v}\, dv +$$

$$+ \frac{\sin(x-t)D_n}{2\pi}\int_{-\infty}^{\infty} \frac{\cosh(1-x+t)v}{\cosh v}\, dv$$

$$= \frac{\sin(x-t)D_n}{2\sin(x-t)\frac{\pi}{2}} + \frac{\sin(2-x-t)D_n}{2\sin(x+t)\pi/2}. \tag{4.2.26}$$

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[1] Watson [105], p. 587. [2] Whittaker and Watson [105], p. 261, Ex. 24.

From (4.3.20) to (4.3.24) and (4.3.26), the lemma follows for $t < x$. In case $x < t$, $x$ and $t$ are merely to be interchanged.

LEMMA 4.7. For $0 < x < 2/D_n$, $4/D_n < t < 1$,

$$|U_n(t,x)| \leqslant K_{19}\, x^{\nu+1/2}\, t^{-1}\, D_n^{\,\nu+1/2}, \tag{4.3.27}$$

and for $2/D_n < x < 1$, $0 < t < 1/D_n$,

$$|U_n(t,x)| \leqslant K_{20}\, t^{\nu+1/2}\, x^{-1}\, D_n^{\,\nu+1/2} < \frac{K_{20}}{x}. \tag{4.3.28}$$

PROOF. We prove (4.3.27). In this case, for $x < t$ and $w = D_n + iv$,

$$|J_\nu(xw)| \leqslant \frac{|(1/2)xw|^{\nu}\; e^{x|v|}}{\Gamma(\nu+1)}.$$ [1]

Hence, by (4.2.4), (4.3.7) and (4.3.10),

$$|U_n(t,x)| \leqslant K_{21}\, x^{\nu+1/2} \int_{-\infty}^{\infty} (D_n^{\,2} + v^2)^{(2\nu+1)/4} \times$$

$$\times\; e^{-(t-x)|v|}\, dv$$

$$\leqslant 2^{(2\nu+5)/4} K_{21}\, x^{\nu+1/2} \Big[ D_n^{\,\nu+1/2} \times \tag{4.3.29}$$

$$\times \int_0^{D_n} e^{-(t-x)v}\, dv + \int_{D_n}^{\infty} v^{\nu+1/2}\, e^{-(t-x)v}\, dv \Big].$$

Now, since $t-x > t/2$, we have

$$\int_0^{D_n} e^{-(t-x)v}\, dv < \frac{1}{t-x} < \frac{2}{t}; \tag{4.3.30}$$

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[1] Watson [103], § 3.31.

and

$$\int_{D_n}^{\infty} v^{\nu+1/2} e^{-(t-x)v}\,dv \leq \int_{D_n}^{\infty} v^{\nu+1/2} e^{-tv/2}\,dv$$

(4.3.31) $$\leq K_{22}\, t^{-\nu-3/2}.$$

(4.3.27) follows from (4.3.29) to (4.3.31). (4.3.28) can be proved in a similar manner.

4.4. PROOF OF THEOREM 4.1. By Pollard[1], it is enough to show that

(4.4.1) $$\|S_n(x,f)\|_{p,\beta} \leq K_{23}\,\|f\|_{p,\beta}$$

for all $f \in L_\beta^p[0,1]$.

Let $f \in L_\beta^p$, and let $0 < x < 2/D_n$. Then by (4.2.1),

$$|S_n(x,f)| \leq \left(\int_0^{4/D_n} + \int_{4/D_n}^{1}\right) |f(t)|\,|J_n(t,x)|\,dt$$

(4.4.2) $$= \sigma_1 + \sigma_2, \text{ say.}$$

By (4.3.4) and (4.1.2),

$$\sigma_1 \leq K_5\, D_n\, \|f\|_{p,\beta} \left\{\int_0^{4/D_n} t^{\beta'}\,dt\right\}^{1/p'}$$

(4.4.3) $$\leq K_{24}\, D_n^{(1+\beta)/p}\, \|f\|_{p,\beta}.$$

Again, by (4.3.27) and (4.1.2), we similarly have,

$$\sigma_2 \leq K_{25}\, x^{\nu+1/2}\, D_n^{\nu+1/2}\, \|f\|_{p,\beta}\, D_n^{(\beta+1)/p}$$

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1) Pollard [77], p. 361.

(4.4.4) $$< K_{25}\, 2^{\nu+1/2}\, \|f\|_{p,\beta}\, D_n^{(\beta+1)/p}.$$

By (4.4.2) to (4.4.4), we obtain,

(4.4.5) $$\int_0^{2/D_n} \left|S_n(x,f)\right|^p x^\beta\, dx \leqslant K_{26}\, \|f\|_{p,\beta}^p.$$

Now, let $2/D_n < x < 1$. Then,

$$S_n(x,f) = \left(\int_0^{1/D_n} + \int_{1/D_n}^1\right) f(t)\, U_n(t,x)\, dt$$

(4.4.6) $$= \sigma_3(x) + \sigma_4(x), \text{ say.}$$

By (4.3.28) and Lemma 4.1,

(4.4.7) $$\left\{\int_{2/D_n}^1 \left|\sigma_3(x)\right|^p x^\beta\, dx\right\}^{1/p} \leqslant K_{20}\, K_1\, \|f\|_{p,\beta}.$$

Also, by Lemma 4.6,

(4.4.8) $$\sigma_4(x) = \frac{1}{2}\int_{1/D_n}^1 \frac{\sin(x-t)D_n}{\sin(x-t)\pi/2}\, f(t)\, dt + \tau(x),$$

where

$$|\tau(x)| \leqslant \frac{1}{2}\left|\int_{1/D_n}^1 \frac{\sin(2-x-t)D_n}{\sin(x+t)\pi/2}\, f(t)\, dt\right| +$$

$$+ K_{27}\int_{1/D_n}^1 \frac{|f(t)|\, dt}{x\, t\, D_n\, (2-x-t)} +$$

$$+ K_{28}\int_{1/D_n}^1 |f(t)|\, dt$$

(4.4.9) $$= \tau_1(x) + \tau_2(x) + \tau_3(x), \text{ say.}$$

Now,

$$\tau_1(x) \leq K_{29} \int_{1/D_n}^{1} \frac{|f(t)|}{x+t}\, dt,$$

hence, by Lemma 4.2,

(4.4.10) $$\int_{2/D_n}^{1} |\tau_1(x)|^p x^\beta\, dx \leq K_{30}^p \|f\|_{p,\beta}^p.$$

$$\tau_2(x) \leq \frac{2K_{27}}{x D_n} \int_{1/D_n}^{1/2} \frac{|f(t)|}{t}\, dt + K_{27} \int_{1/2}^{1} \frac{|f(t)|}{2-x-t}\, dt$$

(4.4.11) $$= \tau_2'(x) + \tau_2''(x), \text{ say.}$$

Making an evaluation similar to (4.4.4), we obtain,

$$\int_{2/D_n}^{1} |\tau_2'(x)|^p x^\beta\, dx \leq K_{30}^p D_n^{\beta-p+1} \|f\|_{p,\beta}^p$$

(4.4.12) $$\leq K_{30}^p \|f\|_{p,\beta}^p, \quad \text{since } \beta < p-1.$$

Also, by Lemma 4.3,

$$\int_{2/D_n}^{1} |\tau_2''(x)|^p x^\beta\, dx = K_{27}^p \left(\int_{2/D_n}^{1/2} + \int_{1/2}^{1}\right) \times$$

$$\times \left\{ \int_{1/2}^{1} \frac{|f(t)|}{2-x-t}\, dt \right\}^p x^\beta\, dx$$

$$\leq K_{27}^p \left[ 2^p \|f\|_1^p \int_{2/D_n}^{1/2} x^\beta\, dx + \right.$$

$$\left. + \int_{1/2}^{1} |g(x)|^p x^\beta\, dx \right].$$

Since, by (4.1.2),

$$\|f\|_1 \leq \|f\|_{p,\beta} \left\{ \int_0^1 x^{\beta'}\, dx \right\}^{1/p'} \leq K_{31} \|f\|_{p,\beta},$$

and by Lemma 4.3,

$$\int_{1/2}^1 |g(x)|^p\, dx \leq K_{32}^p \int_{1/2}^1 |f(x)|^p\, dx$$

$$\leq K_{33}^p \int_{1/2}^1 |f(x)|^p x^\beta\, dx$$

$$\leq K_{33}^p \|f\|_{p,\beta};$$

we obtain,

$$(4.4.13)\quad \int_{2/D_n}^1 |\tau_2''(x)|^p x^\beta\, dx \leq K_{34}^p \|f\|_{p,\beta}^p.$$

Similarly,

$$(4.4.14)\quad \int_{2/D_n}^1 |\tau_3(x)|^p x^\beta\, dx \leq K_{34}^p \|f\|_{p,\beta}^p.$$

From (4.4.9) to (4.4.14), it follows that

$$(4.4.15)\quad \left\{\int_{2/D_n}^1 |\tau(x)|^p x^\beta\, dx\right\}^{1/p} \leq K_{35} \|f\|_{p,\beta}.$$

Further, for $2/D_n < x < 1$, $0 < t < 1/D_n$, we have $x-t > x/2$, so that by Lemma 4.1,

$$\left\{\int_{2/D_n}^1 \left| \frac{1}{2} \int_0^{1/D_n} \frac{\sin(x-t)D_n}{\sin(x-t)\pi/2} f(t)\, dt \right|^p x^\beta\, dx\right\}^{1/p}$$

$$\leq K_{36} \left\{\int_{2/D_n}^1 \left(\int_0^{1/D_n} \frac{|f(t)|}{x-t}\, dt\right)^p x^\beta\, dx\right\}^{1/p}$$

$$\leqslant 2K_{36}\left\{\int_{2/D_n}^{1}\left|\frac{f(x)}{x}\right|^{p} x^{\beta}\, dx\right\}^{1/p}$$

(4.4.16) $\leqslant K_{37}\, \|f\|_{p,\beta}.$

Thus, from (4.4.6) to (4.4.8), (4.4.15) and (4.4.16), it follows that for $2/D_n < x < 1$,

(4.4.17) $$S_n(x,f) = \frac{1}{2}\int_0^1 \frac{\sin(x-t)D_n}{\sin(x-t)\pi/2}\, f(t)\, dt + \overline{E}_n(x),$$

where

(4.4.18) $$\left\{\int_{2/D_n}^{1} |\,\overline{E}_n(x)|^{p} x^{\beta}\, dx\right\}^{1/p} \leqslant K_{38}\, \|f\|_{p,\beta}.$$

Using (4.2.8), if we set $\gamma = (\nu + 1/2)\frac{\pi}{2}$, we obtain,

$$\frac{1}{2}\int_0^1 \frac{\sin(x-t)D_n}{\sin(x-t)\pi/2}\, f(t)\, dt = \cos\gamma x \int_0^1 \triangle_n(x,t)\, f(t) \cos\gamma t\, dt +$$

(4.4.19) $$+ \sin\gamma x \int_0^1 \triangle_n(x,t)\, f(t)\, \sin\gamma t\, dt +$$

$$+\int_0^1 \frac{\sin(x-t)\gamma}{2\sin(x-t)\pi/2}\, f(t)\, \cos\left\{(n+\tfrac{1}{2})\pi(x-t)\right\} dt,$$

where $\triangle_n(x,t)$ denotes the Dirichlet kernel for the Fourier-trigonometric series.

From Hardy-Littlewood[1] and K.I.Babenko[2], it is true that the functions of the class $L_\beta^p[0,1]$ can be

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[1] Hardy-Littlewood [45]. [2] Babenko [10].

expanded by the trigonometric systems $\{\sqrt{2}\sin\pi mx\}$ and $\{1, \sqrt{2}\cos\pi mx\}$. Hence, the partial sums of the Fourier series of such a function with respect to these systems satisfy the inequality (4.4.1).

Thus, from (4.4.5) and (4.4.17) to (4.4.19), (4.4.1) is proved. This proves the theorem.

4.5. PROOF OF THEOREM 4.2. By Young[1], if

$$\int_0^1 f(t)\, v_m(t)\, dt = 0, \qquad m = 1, 2, \dots,$$

for any function f, then f is a null function. Hence, in view of Theorem 4.1, it follows that $\{v_m(t)\}$ is a complete orthonormal basis in $L_\beta^p[0,1]$, hence, it is closed in $L_\beta^p[0,1]$. This proves (4.1.11).

4.6. In order to prove Theorems 4.3 and 4.4, the following lemmas are also required:

LEMMA 4.8. (Gol'dman[2]). Let $\nu \geqslant -1/2$. Then the following relations are true:

$$(4.6.1) \qquad \int_0^1 \left|x^{1/2} J_\nu(x\lambda_m)\right|^p x^\beta\, dx \sim \lambda_m^{-p/2}, \quad \beta > -1,\ p \geqslant 1;$$

and

$$(4.6.2) \qquad \int_0^1 \left|x^{1/2} J_\nu(x\lambda_m)\right|^p x^\beta\, dx \sim \lambda_m^{-p/2} \log \lambda_m,$$

for $\beta = -1$, $p \geqslant 1$.

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1) Young [112], § 9. 2) Gol'dman [36].

LEMMA 4.9. *For $\nu \geq -1/2$ and n sufficiently large, the following estimates are true:*

$$\|v_n\|_{p,\beta} \sim 1, \quad \beta > -1,\ p \geq 1; \tag{4.6.3}$$

*and*

$$\|v_n\|_{p,\beta} \sim (\log \lambda_n)^{1/p}, \quad \beta = -1,\ p \geq 1. \tag{4.6.4}$$

PROOF. We have from (4.1.5), by using asymptotic expansions[1],

$$f_n^2 = \frac{1}{\pi\lambda_n}\Big\{\cos^2(\lambda_n - \alpha) - \frac{\nu^2 - 1/4}{2\lambda_n}\sin 2(\lambda_n - \alpha) +$$

$$+ \sin^2(\lambda_n - \alpha) + \frac{\nu^2 - 1/4}{2\lambda_n}\sin 2(\lambda_n - \alpha) +$$

$$+ O(\lambda_n^{-2})\Big\} - \frac{2\nu}{\pi\lambda_n^2}\Big\{\cos(\lambda_n - \alpha)\sin(\lambda_n - \alpha) +$$

$$+ O(\lambda_n^{-1})\Big\}$$

$$= \frac{1}{\pi\lambda_n}\Big\{1 + O(\lambda_n^{-1})\Big\}. \tag{4.6.5}$$

Now, (4.6.3) follows from (4.6.1) and (4.6.5). Similarly, (4.6.4) follows from (4.6.2) and (4.6.5).

PROOF OF THEOREM 4.3. For each $n = 1, 2, 3, \ldots$, $v_n(t) \in L_{-1}^{p'}$, the space conjugate to $L_{p-1}^{p}$, and hence, defines a linear functional on the space $L_{p-1}^{p}$. Thus, by (4.6.4), $\{v_n\}$ is a sequence of linear functionals on $L_{p-1}^{p}$, whose norms form an unbounded set. By Banach-

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[1] Watson [103], p. 199.

Steinhaus theorem[1], there exists a function $f_0 \in L_{p-1}^p$, such that the Dini-coefficients corresponding to $f_0$ do not form a bounded sequence.

Hence, the general term of the Dini series of $f_0$ does not tend to zero in the $L_{p-1}^p$-norm. This proves the divergence of the Dini series for $f_0$.

Remark 4.2. For $\beta \geq p-1$, to have $v_n \in L_\beta^p[0,1]$, we have $(\nu + 1/2)p + (p-1) - \beta > 0$.

PROOF OF THEOREM 4.4. If $p > 1$, $\beta > p-1$, then $L_{p-1}^p \subseteq L_\beta^p$. Hence, the conclusion of the theorem follows from Theorem 4.3.

For $\beta \leq -1$, either $\{v_n\}$ does not form a basis for $L_\beta^p$, or it forms a basis for $L_{\beta'}^{p'}$. If $\{v_n\}$ forms a basis for $L_{\beta'}^{p'}$, then for $\beta \leq -1$, $\beta' \geq p'-1$. Therefore, the theorem follows from the case $\beta > p-1$ and Theorem 4.3.

4.7. The following Lemmas are used in proving Theorems 4.5 and 4.6:

LEMMA 4.10.[2] For $x, t \in [a,b]$, $0 < a < b$,

(4.7.1) $$|R_n(t,x)| \leq K_{39}\, D_n;$$

and for $x, t \in [a,b]$, $x \neq t$,

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[1] Taylor [95], § 4.4.

[2] This thesis, Ch. III, Lemma 3.4.

(4.7.2) $$\left|R_n(t,x)\right| \leqslant \frac{K_{39}}{|t-x|}.$$

LEMMA 4.11. For $a+2h/B_n < x < b$, $a+h/B_n < t < b$, $h = b-a$, we have,

(4.7.3) $$R_n(t,x) = \left\{1 + O(B_n^{-1})\right\} \bar{R}_n(t,x),$$

where

(4.7.4) $$\bar{R}_n(t,x) = \frac{1}{2}\left[\frac{\sin(t-x)B_n}{(b-a)\sin\frac{t-x}{b-a}\cdot\pi/2} - \frac{\sin(2b-t-x)B_n}{(b-a)\sin\frac{t+x-2a}{b-a}\cdot\pi/2}\right]$$

PROOF. By (2.3.14) and (2.3.15), we have for $a < x < t < b$,

$$\frac{w\, c_\nu(xw,aw)\; c_\nu(tw,bw)}{c_\nu(aw,bw)} = -\frac{2\sin(x-a)w\,\sin(b-t)w}{\pi\sqrt{xt}\,\sin(b-a)w}\times$$
$$\times\left\{1 + O\left(\frac{e^{(x-t)|v|}}{|w|}\right)\right\}.$$

Therefore, from (4.2.11), we get, (4.7.3), where

$$\bar{R}_n(t,x) = -\frac{1}{2\pi i}\int_{B_n-i\infty}^{B_n+i\infty}\frac{2\sin(x-a)w\,\sin(b-t)w}{\sin(b-a)w}\,dw$$

$$= -\frac{1}{2\pi i}\int_{B_n-i\infty}^{B_n+i\infty}\frac{\{\cos(b+a-t-x)w - \cos(b-a-t+x)w\}}{\sin(b-a)w}\,dw$$

(4.7.5) $$= I_1 + I_2,\ \text{say}.$$

By (4.2.12) and (4.3.25),

$$I_2 = \frac{\sin(t-x)B_n}{2\pi}\int_{-\infty}^{\infty}\frac{\cosh(b-a-t+x)v}{\cosh(b-a)v}\,dv$$

$$(4.7.6) \qquad = \frac{1}{2} \frac{\sin (t-x)B_n}{(b-a) \sin \frac{t-x}{b-a} \cdot \pi/2} .$$

Similarly,

$$(4.7.7) \qquad I_1 = -\frac{1}{2} \frac{\sin (2b-t-x)B_n}{(b-a) \sin \frac{t+x-2a}{b-a} \cdot \pi/2} .$$

The lemma, now, follows from (4.7.5) to (4.7.7), in case $a < x < t < b$. The case $a < t < x < b$ can be treated in a similar way by an interchange of $t$ and $x$.

<u>PROOF OF THEOREM 4.5.</u> As in the proof of theorem 4.1, it is enough to prove that [1],

$$(4.7.8) \qquad \| T_n(x,f)\|_p \leq K_{40} \|f\|_p .$$

For $a < x < a+2h/B_n$, $h = b-a$,

$$|T_n(x,f)| \leq \left(\int_a^{a+4h/B_n} + \int_{a+4h/B_n}^b\right) |f(t)| \, |R_n(t,x)| dt$$

$$(4.7.9) \qquad = \sigma_5(x) + \sigma_6(x), \text{ say.}$$

By Lemma 4.10 and Holder's inequality, we get,

$$\sigma_5(x) \leq K_{41} B_n^{1/p} \|f\|_p ,$$

so that,

$$(4.7.10) \qquad \left\{\int_a^{a+2h/B_n} |\sigma_5(x)|^p \, dx\right\}^{1/p} \leq K_{43} \|f\|_p .$$

Also, by (4.7.2),

$$\sigma_6(x) \leq K_{39} \int_{a+4h/B_n}^b \frac{|f(t)|}{t-x} dt .$$

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[1] Pollard [77], p.361; Edwards [29], Vol. II, § 12.10.1.

By the theorem of M. Riesz[1],

$$(4.7.11)\qquad \left\{ \int_a^{a+2h/B_n} |\sigma_6(x)|^p \, dx \right\}^{1/p} \leqslant K_{43} \, \|f\|_p .$$

Therefore, by (4.7.9) to (4.7.11), for $a < x < a+2h/B_n$,

$$(4.7.12)\qquad \left\{ \int_a^{a+2h/B_n} |T_n(x,f)|^p \, dx \right\}^{1/p} \leqslant K_{44} \, \|f\|_p .$$

Now, for $a+2h/B_n < x < b$,

$$T_n(x,f) = \left( \int_a^{a+h/B_n} + \int_{a+h/B_n}^{b} \right) f(t) \, D_n(t,x) \, dt$$

$$(4.7.13)\qquad = \sigma_7(x) + \sigma_8(x), \text{ say.}$$

As in (4.7.11),

$$(4.7.14)\qquad \left\{ \int_{a+2h/B_n}^{b} |\sigma_7(x)|^p \, dx \right\}^{1/p} \leqslant K_{45} \, \|f\|_p .$$

Also, by Lemma 4.11,

$$(4.7.15)\qquad \sigma_8(x) = \frac{1}{2} \int_a^b \frac{\sin (t-x)B_n}{(b-a) \sin \frac{t-x}{b-a} \cdot \pi/2} f(t) \, dt + \tau_n(x),$$

where

$$\tau_n(x) = -\frac{1}{2} \int_a^{a+h/B_n} \frac{\sin (t-x)B_n}{(b-a) \sin \frac{t-x}{b-a} \cdot \pi/2} f(t) \, dt -$$

$$-\frac{1}{2} \int_{a+h/B_n}^{b} \frac{\sin (2b-t-x)B_n}{(b-a) \sin \frac{t+x-2a}{b-a} \cdot \pi/2} f(t) \, dt .$$

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[1] Riesz [79]; Hardy and Littlewood [45], p. 370.

By a similar evaluation, using Lemma 4.4, we obtain,

$$(4.7.16) \qquad \left\{\int_{a+2h/B_n}^{b} |\tau_n(x)|^p \, dx \right\}^{1/p} \leqslant K_{46} \, \|f\|_p.$$

Collecting (4.7.13) to (4.7.16), it follows, for $a+2h/B_n < x < b$, that,

$$(4.7.17) \qquad T_n(x,f) = \frac{1}{2(b-a)} \int_a^b \Delta_n(t,x) \, f(t) \, dt + \overline{T}_n(x),$$

where

$$(4.7.18) \qquad \left\{\int_{a+2h/B_n}^{b} |\overline{T}_n(x)|^p \, dx \right\}^{1/p} \leqslant K_{47} \, \|f\|_p,$$

and $\Delta_n(t,x)$ represents the Dirichlet kernel for the Fourier-trigonometric series in $[a,b]$.

The theorem, now, follows as Theorem 4.1, by (4.7.12), (4.7.17) and (4.7.18).

<u>PROOF OF THEOREM 4.6.</u> The orthonormal sequence $\{G_m(x)\}$ is complete in $L^p$, hence it is closed in $L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$. Now, if $1 < p \leqslant 2$, $p' \geqslant 2$, so that completeness in $L^p$ implies completeness in $L^{p'}$. This proves the theorem.

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