# Chapter 3

# STATE SPACE ANALYSIS USING WAVELET PACKETS

---

## 3.1 Introduction

The very first step in the analytical study of systems is to setup mathematical equations to describe the system. Because of different analytical methods used, we may often setup different mathematical models to describe the same system. When the analysis in the time domain is to be preferred, the use of state space approach will offer a great deal of convenience conceptually, notationally and sometimes analytically. The notational and analytical conveniences come through the use of vector matrix representation which allows the system equations and the form of solutions to be written compactly.The adaption of the state space representation to the numerical solution is an added advantage, particularly when the system to be investigated contains time-varying and non-linear elements.

In 1989, P. N. Paraskevopoulos had introduced a new orthogonal series approach to the state space analysis of linear time invariant system. In this approach the key idea was based on the orthogonal basis vector and the $r \times r$ constant matrix P, called the operational matrix of integration. The matrix P has been determined for Walsh series, block pulse series, Laguarre series etc.

This approach yields explicit expressions for the state input and output vector coefficient matrices X and Y. No algebraic system of equations needs to be solved and therefore no inversion of large matrices is required here, as compared to known techniques using the Kronecker product approach.Hence, our method reduces the computational effort involved and improves the accuracy due to round-off errors.

In this chapter we have discussed the following two problems:

1. State space analysis using wavelet packets

2. Wavelet packet series approach to state space analysis using bilinear systems

For this, we have introduced the operational matrices for Walsh wavelet packets using Haar bases and Walsh bases respectively (refer [29]). Using the operational matrices for different values of m and $r$ we have obtained the state space matrices with the procedure given by P. N. Paraskevopoulos. Also, comparative study of the results obtained using the Haar bases and Walsh bases with different values of m and $r$ has been made.

The bilinear systems may be considered as a specialization of non-linear systems, under the assumption of linearity in the control or respectively in the state but not in both jointly.

P. N. Paraskevoulos, A. S. Tsirikos and K. G. Arvanitis [60] presented a new orthogonal series approach for the solution of the state- space equation of bilinear time invariant systems. This approach involves product of matrices of small dimensions. In this chapter, we are using the operational matrices of Walsh type wavelet packet series as defined by Glabisz [29] for solving the state space equation of bilinear time invariant systems.The chapter also includes, the comparative study of the exact solution and the solution obtained using Walsh wavelet packet series.

## 3.2 Preliminaries

**The Concept of State :**

The classical design methods for control system analysis suffer from certain limitations due to the fact that the transfer function model is applicable only to linear time invariant systems and is restricted to Single Input Single Output (SISO) systems, as it becomes highly cumbersome for use in Multi-Input Multi-Output (MIMO) systems.

Another limitation of the transfer function is that it reveals only the system output for a given input and provides no information about the internal behavior of the system.

The limitations of classical methods based on transfer function models have led to the development of state variable approach of analysis and design. It is a powerful technique for the analysis and design of the linear-nonlinear, time invariant or time varying MIMO systems. It is easily amendable to solution through digital computers.

**State Space :** The n-dimensional space whose co-ordinate axes consists of $x_1 - axis,\ x_2 - axis, ....\ x_n - axis$ is called state space.

State equations are arranged as a set of first order differential equations and have the following form for a linear time invariant system

$$\dot{x} = Ax + Bu$$

where $x(t)$ is the n-dimensional state vector, $\dot{x}$ is the derivative of x with respect to time and $u(t)$ is the m-dimensional input vector A and B are matrices of dimensions $n \times n$ and $n \times m$ respectively with constant elements. The output of the system is given by :

$$y(t) = Cx(t) + Du(t)$$

where $y(t)$ is a vector of dimension $p$, the numbers of outputs C and D are constants matrices of dimensions $p \times n$ and $p \times m$ respectively.

Here we restrict ourselves to the case of single input single output unforced systems for which $m = 1$ and $p = 1$.

In this chapter, we use the wavelet packets generated from the Haar filter for which, $h_0 = h_1 = g_0 = -g_1 = \frac{1}{\sqrt{2}}$ as defined in Chapter 2.

Let us assume that we have an arbitrary Walsh wavelet packet bases represented by matrix $H(x)$ with size $m \times m$ where parameter $m = 2^J$ is further referred as a degree of approximation. Matrix $H(x)$ can be represented by Haar basis, Walsh basis or wavelet packet basis defined for $x \in [0, 1)$.

## 3.3 Walsh wavelet packet bases coefficient determination

A function $f(t)$ which is absolutely integrable in $[0, 1)$ may be expanded into Walsh wavelet packet bases

$$f(t) = c_0 W_0(t) + c_1 W_1(t) + ........ + c_n W_n(t) + ..... \tag{3.3.1}$$

where $c_n$ are the coefficients of Walsh wavelet packet bases of $f(t)$.

It is desirable to determine the coefficients $c_n$ such that the integral square error

$$\int_0^1 \left[ f(t) - \Sigma_{n=0}^{N} c_n W_n(t) \right]^2 dt = \varepsilon \tag{3.3.2}$$

is minimized.

Taking the partial derivative of $\varepsilon$ with respect to $c_n$, yields

$$\frac{\partial \varepsilon}{\partial c_n} = 2c_n - 2 \int_0^1 f(t) W_n(t) dt$$

and setting it equal to zero we have

$$c_n = \int_0^1 W_n(t).f(t) dt \tag{3.3.3}$$

This simple result is due to the orthonormality property of wavelet packet bases.

## 3.4 Operational Matrix

Let us assume that $\int_0^x W(x)dx \cong PW(x)$, where $P$ stands for the operational matrix (refer [29]) which is a matrix of coefficients of expansion in basis $W(x)$, of the integral from $W(x)$. Let us assume that the position of the representative point for each of the m intervals of $x$ is given by the parameter $r = 0$, the interval's representative point is located at it's beginning, whereas if $r = 1$, the point lies at the interval's end.

As given by Glabisz [29], the operational matrices for the Walsh wavelet packet bases is defined as follows :

**For Haar basis:**

$$P_m = \frac{1}{2m}\begin{bmatrix} 2\left[mP_{\frac{m}{2}} - (r - \frac{1}{2})I\right] & -H_{\frac{m}{2}} \\ H_{\frac{m}{2}}^{-1} & 2(r - \frac{1}{2})I \end{bmatrix} \tag{3.4.1}$$

$$P_1 = |r|$$

**For Walsh basis:**

$$P_m = \frac{1}{2m}\begin{bmatrix} 2\left[mP_{\frac{m}{2}} - (r - \frac{1}{2})I\right] & -I \\ I & 2(r - \frac{1}{2})I \end{bmatrix} \tag{3.4.2}$$

$$P_1 = |r|$$

For any Walsh wavelet packet bases it is easy to obtain a modified operational matrix numerically using $\int_0^x WH(x)dx \cong PW(x)$.

Operational matrices for Haar bases with $m = 4$ and $m = 8$ have the following forms :

For $r = 1$

$$
\begin{aligned}
m &= 1; \\
P_1 &= [r] = 1 \\
m &= 2; \\
P_2 &= \frac{1}{4}\begin{bmatrix} 2\left[2P_1 - (1-\frac{1}{2})I\right] & -[1] \\ [1] & 2(1-\frac{1}{2})I \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\
m &= 4; \\
P_4 &= \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & -0.1250 \\ 0.2500 & 0.1250 & -0.1250 & 0.1250 \\ 0.0625 & 0.0625 & 0.1250 & 0 \\ 0.0625 & -0.0625 & 0 & 0.1250 \end{bmatrix} \\
m &= 8;
\end{aligned}
$$

$$
P_8 = \begin{bmatrix}
0.5625 & -0.2500 & -0.1250 & -0.1250 & -0.0625 & -0.0625 & -0.0625 & -0.0625 \\
0.2500 & 0.0625 & -0.1250 & 0.1250 & -0.0625 & -0.0625 & 0.0625 & 0.0625 \\
0.0625 & 0.0625 & 0.0625 & 0 & -0.0884 & 0.0884 & 0 & 0 \\
0.0625 & -0.0625 & 0 & 0.0625 & 0 & 0 & -0.0884 & 0.0884 \\
0.0156 & 0.0156 & 0.0221 & 0 & 0.0625 & 0 & 0 & 0 \\
0.0156 & 0.0156 & -0.0221 & 0 & 0 & 0.0625 & 0 & 0 \\
0.0156 & -0.0156 & 0 & 0.0221 & 0 & 0 & 0.0625 & 0 \\
0.0156 & -0.0156 & 0 & -0.0221 & 0 & 0 & 0 & 0.0625
\end{bmatrix}
$$

Similarly we can calculate the operational matrices for different values of $m$ and $r$. On the same line, operational matrices using Walsh bases for $m = 4$ and $m = 8$ are as follows :

$For \quad r = 1$

$$m = 1;$$

$$P_1 = [r] = 1$$

$$m = 2;$$

$$P_2 = \frac{1}{4}\begin{bmatrix} 2\left[2P_1 - (1-\frac{1}{2})I\right] & -[I] \\ [I] & 2(1-\frac{1}{2})I \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$m = 4;$$

$$P_4 = \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & 0 \\ 0.2500 & 0.1250 & 0 & -0.1250 \\ 0.1250 & 0 & 0.1250 & 0 \\ 0 & 0.1250 & 0 & 0.1250 \end{bmatrix}$$

$m = 8;$

$$P_8 = \begin{bmatrix} 0.5625 & -0.2500 & -0.1250 & 0 & -0.0625 & 0 & 0 & 0 \\ 0.2500 & 0.0625 & 0 & -0.1250 & 0 & -0.0625 & 0 & 0 \\ 0.1250 & 0 & 0.0625 & 0 & 0 & 0 & -0.0625 & 0 \\ 0 & 0.1250 & 0 & 0.0625 & 0 & 0 & 0 & -0.0625 \\ 0.0625 & 0 & 0 & 0 & 0.0625 & 0 & 0 & 0 \\ 0 & 0.0625 & 0 & 0 & 0 & 0.0625 & 0 & 0 \\ 0 & 0 & 0.0625 & 0 & 0 & 0 & 0.0625 & 0 \\ 0 & 0 & 0 & 0.0625 & 0 & 0 & 0 & 0.0625 \end{bmatrix}$$

**Bilinear Systems:**

Bilinear systems is a specialization of non linear systems, under the assumption of linearity in control or in state but not jointly. A bilinear time invariant system is described by :

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t)$$

$$x(t = 0) = x(0)$$

where $x(t) \in R^n, u(t) \in R^m$ and A, B and C are constant matrices of appropriate dimensions.

## 3.5 State Space Analysis using Wavelet Packets

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.5.1}$$

$$y(t) = Cx(t) \tag{3.5.2}$$

$$x(0) = x_0 \tag{3.5.3}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ and A,B and C are constant matrices of appropriate dimensions.

The closed form solution of (3.5.1) is given by

$$x(t) = exp(At).x(0) + \int_0^t exp[A(t-\sigma)]Bu(\sigma)d\sigma \tag{3.5.4}$$

Using the new orthogonal series approach for state-space analysis method, P. N. Paraskevopoulos [58] in 1989 has derived the solution of equation (3.5.1) as

$$X = \begin{bmatrix} x(0) & Ax(0) & .. & A^{k-1}x(0) \end{bmatrix} \begin{bmatrix} e^T \\ e^T.P \\ . \\ . \\ . \\ e^T.P^{k-1} \end{bmatrix} + \begin{bmatrix} B & AB & .. & A^{k-1}B \end{bmatrix} \begin{bmatrix} UP \\ UP^2 \\ . \\ . \\ . \\ UP^k \end{bmatrix}$$

where $e^T = [1, 0, 0...., 0]$

For k=3,

$$X = x(0)e^T + Ax(0)e^T P + A^2x(0)e^T P^2 + BUP + ABUP^2 + A^2BUP^3 \quad (3.5.5)$$

To apply the proposed method, he has solved the single input system by using Laguarre orthogonal polynomial where the approximate (k,m)-solution is identical to the exact solution if k=3 and m=5.

In this paper, using the Walsh wavelet packets we are solving the single input system given by

$$\dot{x} = Ax + bu, \qquad x(0) = 0$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and $u(t) = 1 + t$.

Then the exact solution is given by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 2t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 \\ 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 \\ t + \frac{1}{2}t^2 \end{bmatrix} \quad (3.5.6)$$

The Exact solution using Haar bases for m=4 and m=8 is given by

$$X = \begin{bmatrix} 1.2167 & -0.6693 & -0.2024 & -0.2749 \\ 1.2083 & -0.1615 & -0.0253 & -0.0916 \\ 0.6666 & -0.375 & -0.1105 & -0.1547 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.2167 & -0.6693 & -0.2024 & -0.2749 & -0.0666 & -0.077 & -0.090 & -0.0105 \\ 1.2083 & -0.1615 & -0.0253 & -0.0916 & 0.004 & -0.050 & -0.0210 & -0.0393 \\ 0.6666 & -0.375 & -0.1105 & -0.1547 & -0.035 & -0.043 & -0.051 & -0.059 \end{bmatrix}$$

Similarly Exact solution using Walsh bases with m=4 and m=8 is given by

$$X = \begin{bmatrix} 1.2166 & -0.6693 & -0.3376 & 0.0513 \\ 1.2084 & -0.1615 & -0.0827 & 0.0469 \\ 0.6666 & -0.375 & -0.1875 & 0.0313 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.2166 & -0.6693 & -0.3376 & 0.0513 & -0.1696 & 0.0257 & 0.0131 & -0.0029 \\ 1.2084 & -0.1615 & -0.0827 & 0.0469 & -0.0416 & 0.0234 & 0.0118 & -0.0020 \\ 0.6666 & -0.375 & -0.1875 & 0.0313 & -0.0938 & 0.0156 & 0.0078 & 0 \end{bmatrix}$$

### 3.5.1 Solution using Haar bases

Now we shall find the solution of (3.5.1) using the method as given by Paraskevopoulos first using Haar bases and then using Walsh bases with $m = 4$, $k = 3$ and $r = 1$.

The corresponding operational matrix is given by

$$P_4 = \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & -0.1250 \\ 0.2500 & 0.1250 & -0.1250 & 0.1250 \\ 0.0625 & 0.0625 & 0.1250 & 0 \\ 0.0625 & -0.0625 & 0 & 0.1250 \end{bmatrix}$$

Now,

$$x(0)e^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ax(0)e^T P = \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & -0.1250 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2x(0)e^TP^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we will express $u(t) = 1 + t$ in terms of Walsh wavelet packets using Haar bases

$$c_0 = \int_0^1 (1+t).H_0(x)dx = \frac{3}{2}, \qquad c_1 = \int_0^1 (1+t).H_1(x)dx = \frac{-1}{4}$$
$$c_2 = \int_0^1 (1+t).H_2(x)dx = \frac{-\sqrt{2}}{16}, \quad c_3 = \int_0^1 (1+t).H_3(x)dx = \frac{-\sqrt{2}}{16}$$

Hence

$$U = \left[\frac{3}{2}, \frac{-1}{4}, \frac{-\sqrt{2}}{16}, \frac{-\sqrt{2}}{16}\right]$$

$$BUP = \begin{bmatrix} 0.8642 & -0.4063 & -0.1673 & -0.2298 \\ 0 & 0 & 0 & 0 \\ 0.8640 & -0.4063 & -0.1673 & -0.2298 \end{bmatrix}$$

$$ABUP^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4163 & -0.2629 & -0.0781 & -0.1875 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2BUP^3 = \begin{bmatrix} 0.1762 & -0.1294 & -0.0286 & -0.1080 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence using (3.5.5)

$$X = \begin{bmatrix} 1.6651 & -0.7857 & -0.3209 & -0.4628 \\ 1.4136 & -0.2629 & -0.0781 & -0.1875 \\ 0.8640 & -0.4063 & -0.1673 & -0.2298 \end{bmatrix}$$

On the same line solution using haar bases with m=8 and $r = 1$ is given by

$$X = \begin{bmatrix} 1.4322 & -0.7220 & -0.3013 & -0.4228 & -0.1364 & -0.164 & -0.188 & -0.234 \\ 1.3044 & -0.2116 & -0.0570 & -0.1585 & -0.0138 & -0.043 & -0.059 & -0.099 \\ 0.7682 & -0.3906 & -0.1618 & -0.2243 & -0.0723 & -0.088 & -0.103 & -0.119 \end{bmatrix}$$

Similarly the solution of the input system can be obtained for different values of $r$ and different values of m.

### 3.5.2 Solution using Walsh bases

With $m = 4$, $k = 3$ and $r = 1$.

The corresponding operational matrix is given by

$$P_4 = \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & 0 \\ 0.2500 & 0.1250 & 0 & -0.1250 \\ 0.1250 & 0 & 0.1250 & 0 \\ 0 & 0.1250 & 0 & 0.1250 \end{bmatrix}$$

Now,

$$x(0)e^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Ax(0)e^T P = \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2x(0)e^T P^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we will express $u(t) = 1+t$ in terms of Walsh wavelet packets using Walsh bases

$$c_0 = \int_0^1 (1+t).W_0(x)dx = \frac{3}{2}, \qquad c_1 = \int_0^1 (1+t).W_1(x)dx = \frac{-1}{4}$$

$$c_2 = \int_0^1 (1+t).W_2(x)dx = \frac{-1}{8}, \quad c_3 = \int_0^1 (1+t).W_3(x)dx = 0$$

Hence

$$U = \left[\frac{3}{2}, \frac{-1}{4}, \frac{-1}{8}, 0\right]$$

$$BUP = \begin{bmatrix} 0.8594 & -0.4063 & -0.2031 & -0.0313 \\ 0 & 0 & 0 & 0 \\ 0.8594 & -0.4063 & -0.2031 & -0.0313 \end{bmatrix}$$

$$ABUP^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4102 & -0.2617 & -0.1328 & 0.0547 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2BUP^3 = \begin{bmatrix} 0.1743 & -0.1284 & -0.0679 & 0.0396 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, using (3.5.5)

$$X = \begin{bmatrix} 1.6587 & -0.7847 & -0.3960 & -0.0708 \\ 1.4102 & -0.2617 & -0.1328 & 0.0547 \\ 0.8594 & -0.4063 & -0.2031 & 0.0313 \end{bmatrix}$$

On the same line solution using Haar bases with m=8 and $r = 1$ is given by

$$X = \begin{bmatrix} 1.4239 & -0.7209 & -0.3637 & 0.0605 & -0.1823 & 0.0303 & 0.0153 & -0.0033 \\ 1.3003 & -0.2100 & -0.1069 & 0.0508 & -0.0537 & 0.0254 & 0.0127 & -0.0020 \\ 0.7617 & -0.3906 & -0.1953 & 0.0313 & -0.0977 & 0.0156 & 0.0078 & 0 \end{bmatrix}$$

Similarly the solution of the input system can be obtained for different values of $r$ and different values of m.

Solution for Haar bases with m=4 and different values of $r$ for

$$x_1(t) = 2t + (1/2)t^2 + (1/6)t^3 + (1/24)t^4$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 0.2612 | 0.8336 | 1.4972 | 2.2748 |
| r=0.4 | 0.0999 | 0.9017 | 1.2758 | 2.3518 |
| r=0.5 | 0.1531 | 0.9683 | 1.3551 | 2.4639 |
| r=0.8 | 0.3149 | 1.1803 | 1.6103 | 2.8321 |
| r=1.0 | 0.4256 | 1.3332 | 1.7963 | 3.1053 |

Solution for Haar bases with m=4 and different values of $r$ for

$$x_2(t) = 1 + (1/2)t^2 + (1/6)t^3$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 1.0110 | 1.0826 | 1.2403 | 1.4993 |
| r=0.4 | 0.9993 | 1.0839 | 1.1644 | 1.5052 |
| r=0.5 | 1.0025 | 1.1091 | 1.2004 | 1.5720 |
| r=0.8 | 1.0209 | 1.1951 | 1.3209 | 1.7867 |
| r=1.0 | 1.0402 | 1.2612 | 1.4113 | 1.9417 |

Solution for Haar bases with m=4 and different values of $r$ for

$$x_3(t) = t + (1/2)t^2$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 0.1347 | 0.4473 | 0.8222 | 1.2598 |
| r=0.4 | 0.0524 | 0.4880 | 0.7016 | 1.3140 |
| r=0.5 | 0.0805 | 0.5225 | 0.7421 | 1.3609 |
| r=0.8 | 0.1649 | 0.6257 | 0.8639 | 1.5015 |
| r=1.0 | 0.2211 | 0.6943 | 0.9453 | 1.5953 |

Solution for Walsh bases with m=4 and different values of $r$ for

$$x_1(t) = 2t + (1/2)t^2 + (1/6)t^3 + (1/24)t^4$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 0.2610 | 0.8336 | 1.4970 | 2.2748 |
| r=0.4 | 0.2136 | 0.7786 | 1.4290 | 2.1872 |
| r=0.5 | 0.2679 | 0.8439 | 1.5119 | 2.2951 |
| r=0.8 | 0.4340 | 1.0510 | 1.7802 | 2.6484 |
| r=1.0 | 0.5488 | 1.1992 | 1.9766 | 2.9102 |

Solution for Walsh bases with m=4 and different values of $r$ for

$$x_2(t) = 1 + (1/2)t^2 + (1/6)t^3$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 1.0111 | 1.0827 | 1.2403 | 1.4995 |
| r=0.4 | 1.0112 | 1.0700 | 1.2116 | 1.4516 |
| r=0.5 | 1.0176 | 1.0918 | 1.2520 | 1.5138 |
| r=0.8 | 1.0451 | 1.1675 | 1.4885 | 1.7141 |
| r=1.0 | 1.0704 | 1.2266 | 1.4844 | 1.8594 |

Solution for Walsh bases with m=4 and different values of $r$

$$x_3(t) = t + (1/2)t^2$$

| t | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
|---|---|---|---|---|
| Exact solution | 0.1355 | 0.4479 | 0.8229 | 1.2605 |
| r=0.4 | 0.1125 | 0.4187 | 0.7875 | 1.2189 |
| r=0.5 | 0.1407 | 0.4531 | 0.8281 | 1.2657 |
| r=0.8 | 0.2251 | 0.5563 | 0.9499 | 1.4063 |
| r=1.0 | 0.2813 | 0.6249 | 1.0313 | 1.5001 |

Now we represent the exact solution and the solution by Haar as well as Walsh bases graphically.

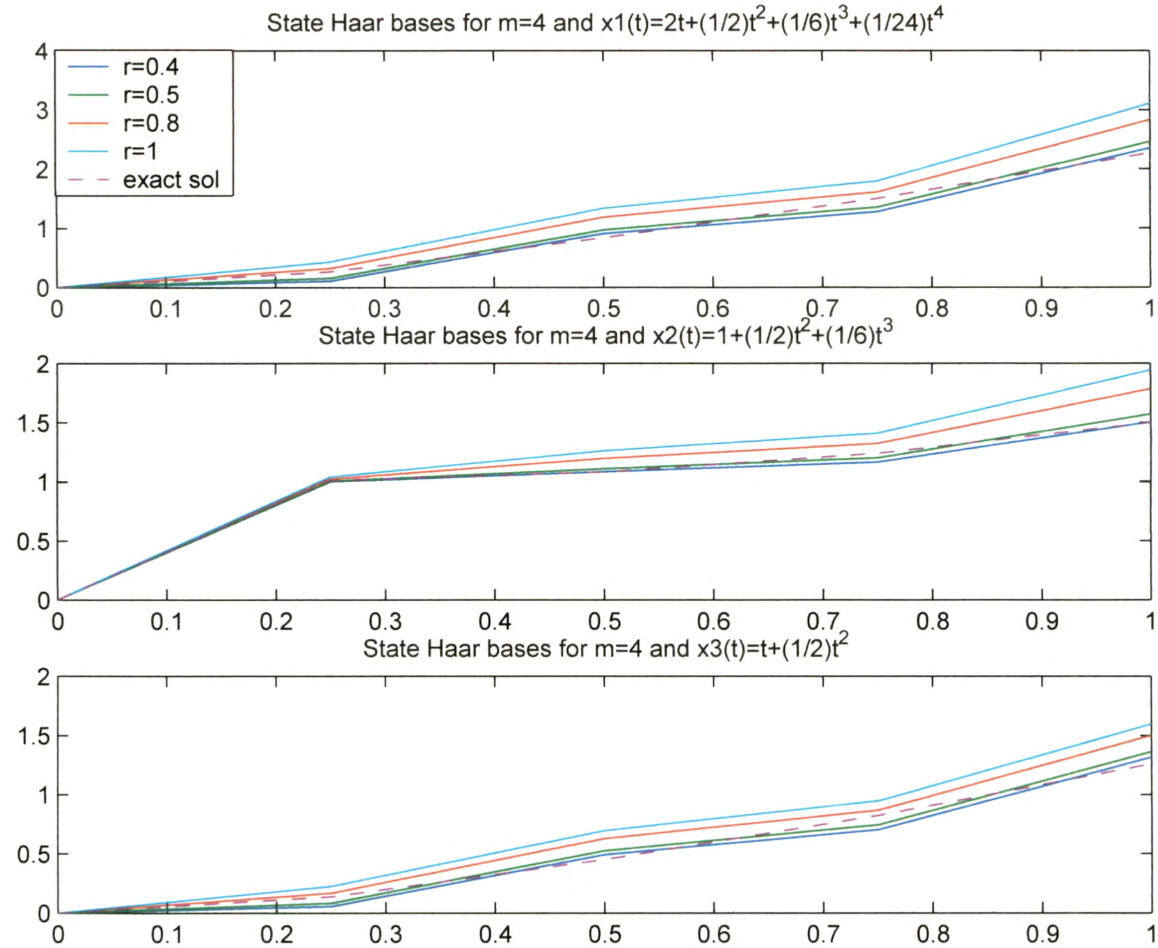


Figure 3.1: Solution Using Haar Bases

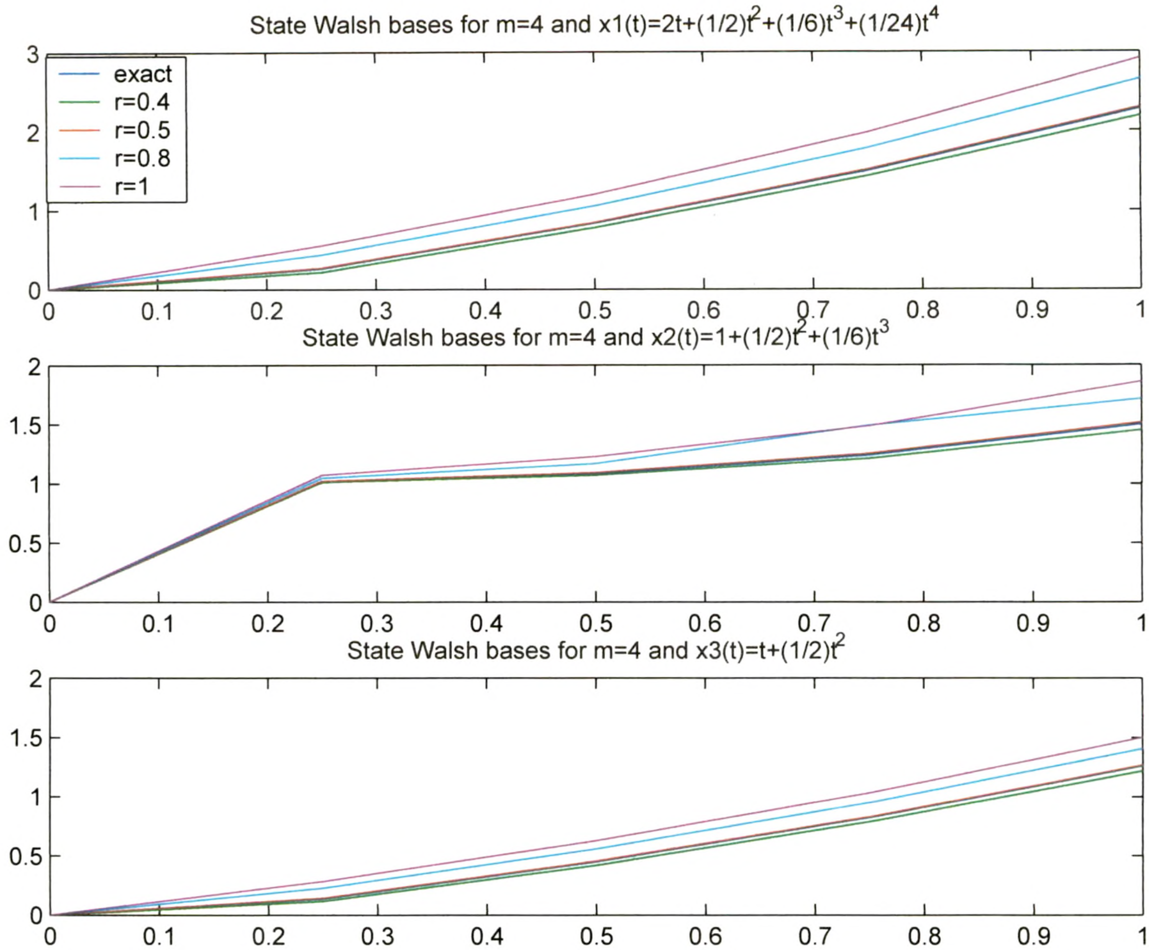


Figure 3.2: Solution Using Walsh Bases

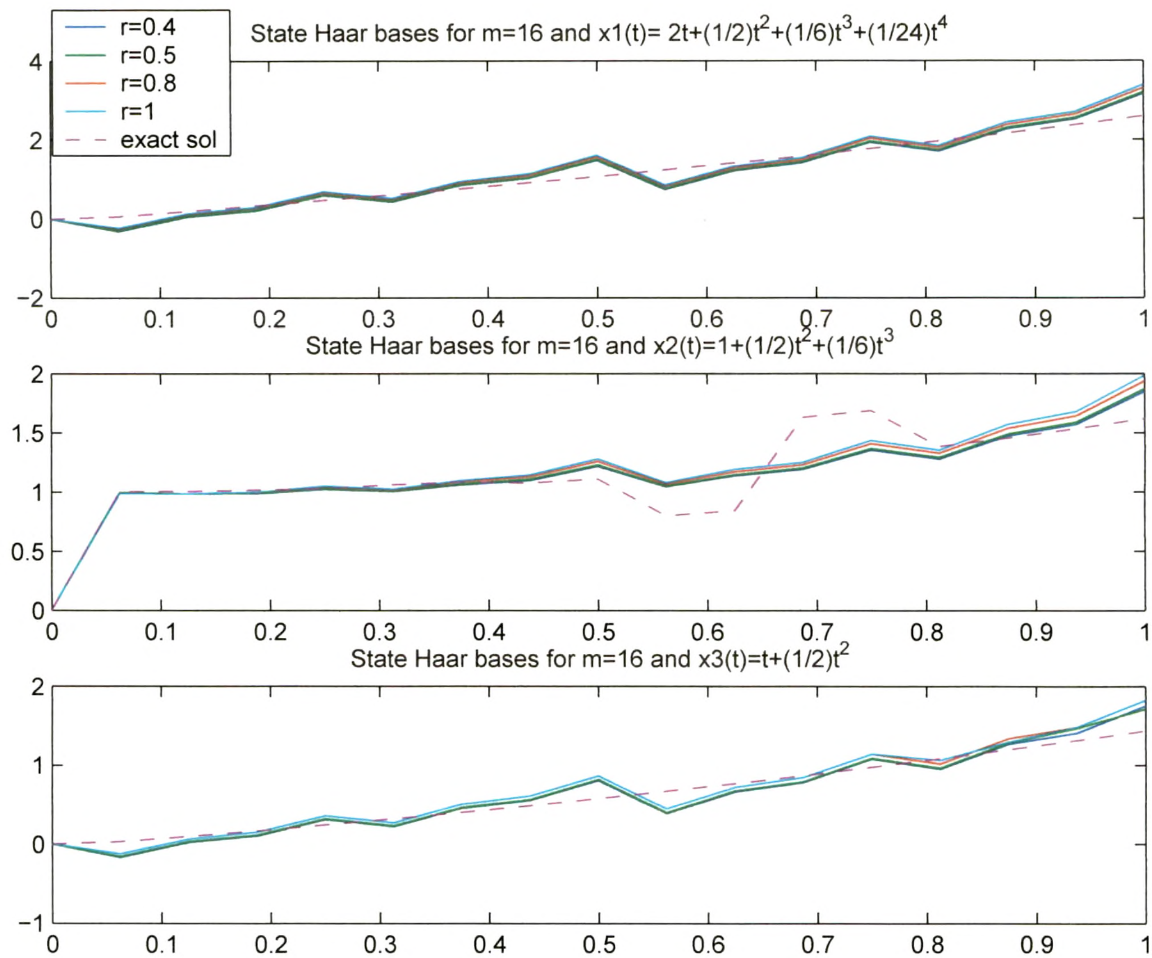


Figure 3.3: Solution Using Haar Bases

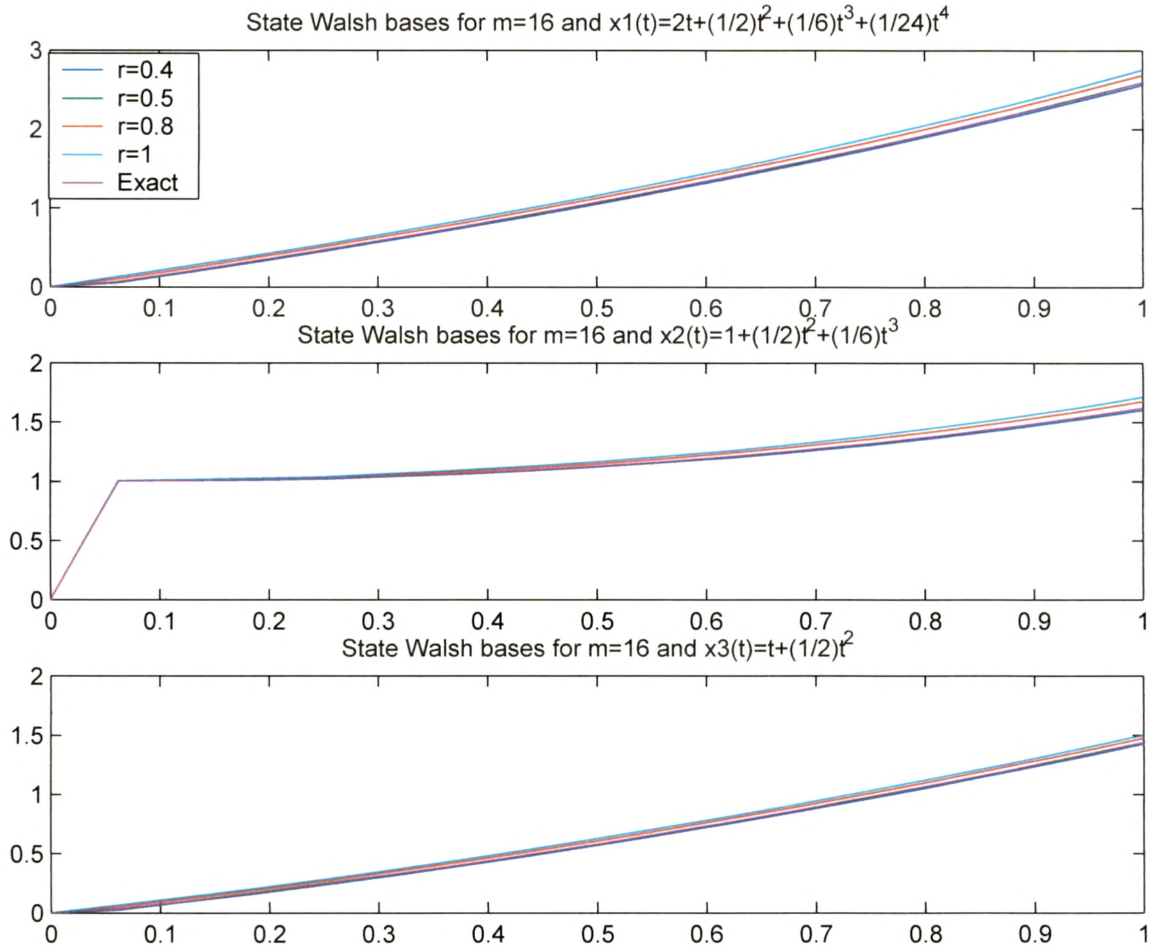


Figure 3.4: Solution Using Walsh Bases

## 3.6 Conclusion

1. Solution of the input system using Walsh bases with $r = 0.5$ coincide with that obtained using Walsh functions.

2. For larger values of m and different values of $r$ we gent the solution which is very much closer to the exact solution.

3. Solution using Walsh bases is very much smoother than that obtained using Haar bases.

4. As observed by P. N. Paraskevopoulos the solution with Laguarre Polynomials agrees with the exact solution with m=5 but in wavelet packets we require higher values of m due to the fact that the functions used in the bases are not polynomials.

## 3.7 Wavelet Packet series approach to State Space Analysis of Bilinear Systems

Consider a bilinear time invariant system described in state space by the following equation

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t) \tag{3.7.1}$$

$$x(t = 0) = x(0)$$

where $x(t) \in R^n, u(t) \in R^m$ and A and B are constant matrices of appropriate dimensions.The term $Nx(t)u(t)$ is a bilinear form in variable $x(t)$ and $u(t)$.

The solution of [3.7.1] is given by

$$x(t) = x_L(t) + x_B(t) \tag{3.7.2}$$

where $x_L(t)$ is the solution for the state vector $x(t)$ for a linear time invariant system given by $\dot{x}(t)Ax(t) + Bu(t)$ and $x_B(t)$ is due to the bilinear term $Nx(t)u(t)$.

The term $x_L(t)$ can be approximated via orthogonal series as

$$x_L(t) \cong G_0 f_r(t) \qquad G_0 \in R^{n\times r}$$

where

$$f_r(t) = [f_0(t)\ f_1(t)\ \text{--------}\ f_{r-1}(t)]^T$$

is the orthogonal basis vector.

$$G_0 = \begin{bmatrix} x(0) & Ax(0) & \text{-----} & A^{k-1}x(0) \end{bmatrix} \begin{bmatrix} e^T \\ e^T P_r \\ | \\ | \\ e^T P_r^{k-1} \end{bmatrix} \tag{3.7.3}$$

$$+ \begin{bmatrix} B & AB & \text{--------} & A^{k-1}B \end{bmatrix} \begin{bmatrix} UP_r \\ UP_r^2 \\ | \\ | \\ UP_r^k \end{bmatrix}$$

where $e$ is a constant $r \times 1$ vector, whose form depends on the particular orthogonal series and $P_r$ is the $r \times r$ operational matrix of Walsh type wavelet packet series and $U$ is $m \times r$ coefficient matrix of input $u(t)$.

The second term $x_B(t)$ is defined as

$$x_B(t) = \sum_{i=1}^{l} G_i f_r(t)$$

where

$$G_i = \sum_{j=1}^{m} \left[ N_j G_{i-1} \;\; AN_j G_{i-1} \;\; ------ \;\; A^{k-1} N_j G_{i-1} \right] \begin{bmatrix} \widetilde{U_j} P_r \\ \widetilde{U_j} P_r^2 \\ | \\ | \\ \widetilde{U_j} P_r^k \end{bmatrix} \tag{3.7.4}$$

for $\quad i = 1, 2, 3 ----l.$

Hence using (3.7.2) $\qquad x = G_0 + \sum_{i=1}^{l} G_i.$

This method involves three approximations. The first is for the truncation of power series expansion of $e^{At}$ and $e^{A(t-r)}$ where the first $k$-terms are kept. The second is for the truncation of the series (3.7.4) where the first $l$-terms are kept and the third is for the truncation of the orthogonal series.

## 3.8 Example

Consider the bilinear system (3.7.4) with

$$A = \begin{bmatrix} 0.5 & -2 \\ -2 & 0.5 \end{bmatrix} \qquad B = 0 \qquad N = I_{2\times 2}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u(t) = e^{-t}$$

The exact solution of the given bilinear system has the form

$$x_e(t) = \frac{1}{2}\begin{bmatrix} e^{\lambda_1} + e^{\lambda_2} \\ e^{\lambda_1} - e^{\lambda_2} \end{bmatrix}$$

where $\lambda_1 = -e^{-t} - 1.5t + 1$ and $\lambda_2 = -e^{-t} + 2.5t + 1$

Taking $k = 4$, $l = 3$ and $r = 4$ we are obtaining the solution using Walsh wavelet packet series.

Absolute Error of X1 for m=4

| t | 1/4 | 1/2 | 3/4 | 1 |
|---|---|---|---|---|
| r=0.025 | 0.5852 | 1.4587 | 2.7625 | 2.0805 |
| r=0.05 | 0.5758 | 1.4192 | 2.5822 | 1.1190 |
| r=0.1 | 0.5555 | 1.3312 | 2.1654 | 1.1925 |
| r=0.15 | 0.5327 | 1.2005 | 1.6574 | 4.1486 |
| r=0.2 | 0.5073 | 1.1116 | 1.0339 | 7.9400 |

Absolute Error of X2 for m=4

| t | 1/4 | 1/2 | 3/4 | 1 |
|---|---|---|---|---|
| r=0.025 | 0.7240 | 1.5563 | 2.7731 | 2.018 |
| r=0.05 | 0.7108 | 1.5140 | 2.5922 | 1.0548 |
| r=0.1 | 0.6829 | 1.4203 | 2.1733 | 1.2628 |
| r=0.15 | 0.6528 | 1.2836 | 1.6618 | 4.2276 |
| r=0.2 | 0.6201 | 1.1888 | 1.0336 | 8.0302 |

Absolute Error of X1 for m=8

| t | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 | 1 |
|---|---|---|---|---|---|---|---|---|
| r=0.025 | 0.2302 | 0.3802 | 0.5452 | 0.6114 | 0.2356 | 1.5386 | 7.2520 | 23.3801 |
| r=0.05 | 0.2256 | 0.3697 | 0.5216 | 0.5572 | 0.1051 | 1.8666 | 8.0941 | 25.5414 |
| r=0.1 | 0.2159 | 0.3477 | 0.4719 | 0.4414 | 0.1770 | 2.5823 | 13.9757 | 30.3072 |
| r=0.15 | 0.2055 | 0.3242 | 0.4183 | 0.3149 | 0.4901 | 3.3865 | 12.0403 | 35.7308 |
| r=0.2 | 0.1947 | 0.2992 | 0.3605 | 0.1765 | 0.8379 | 4.2911 | 14.4163 | 41.9019 |

Absolute Error of X2 for m=8

| t | 1/8 | 2/8 | 3/8 | 4/8 | 5/8 | 6/8 | 7/8 | 1 |
|---|---|---|---|---|---|---|---|---|
| r=0.025 | 0.2961 | 0.4415 | 0.5878 | 0.6254 | 0.2161 | 1.5947 | 7.3490 | 23.5264 |
| r=0.05 | 0.2897 | 0.4293 | 0.5627 | 0.5698 | 0.0841 | 1.9246 | 8.2241 | 25.6924 |
| r=0.1 | 0.2763 | 0.4037 | 0.5097 | 0.4513 | 0.2009 | 2.6444 | 14.0827 | 30.4681 |
| r=0.15 | 0.2627 | 0.3768 | 0.4529 | 0.3219 | 0.5172 | 3.4532 | 12.1538 | 35.9024 |
| r=0.2 | 0.2485 | 0.3483 | 0.3919 | 0.1803 | 0.8687 | 4.3628 | 14.5374 | 42.0848 |

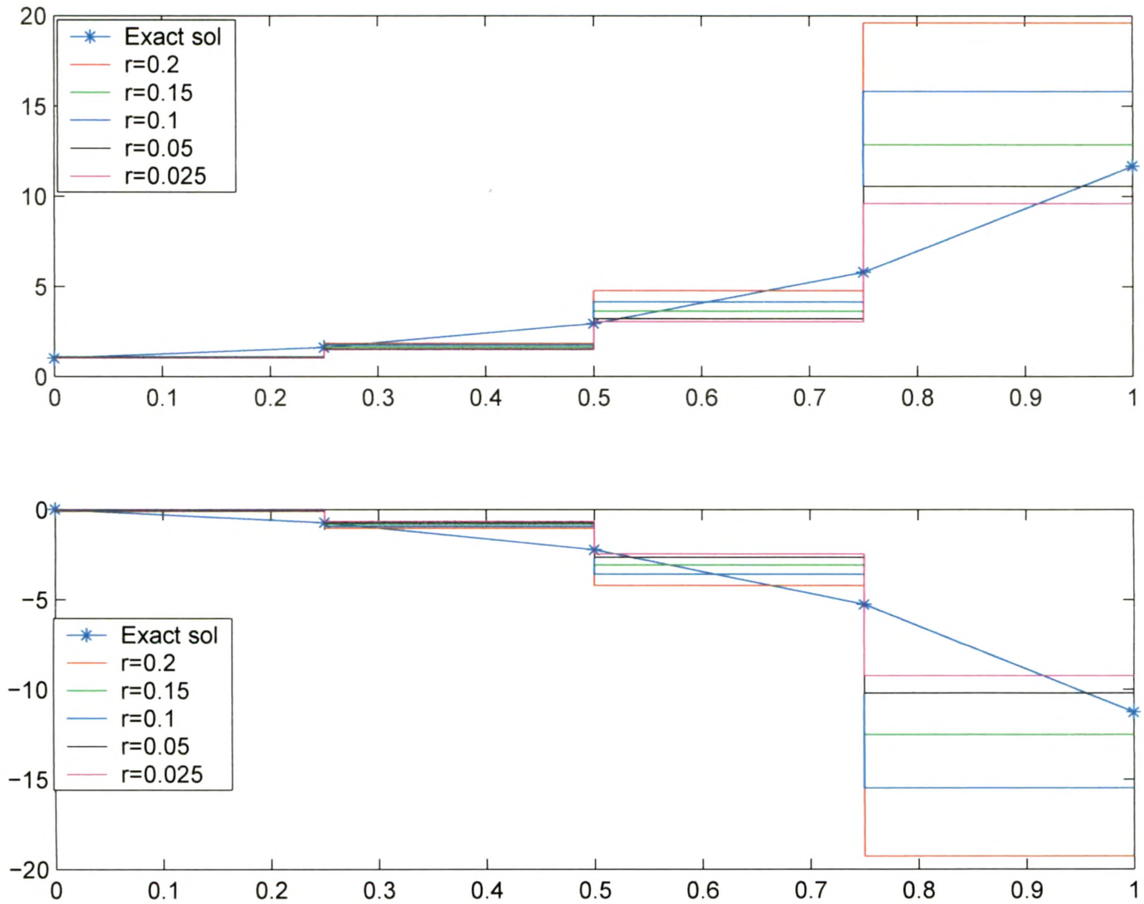


Figure 3.5: Solution Using Wavelet Packets for m=4

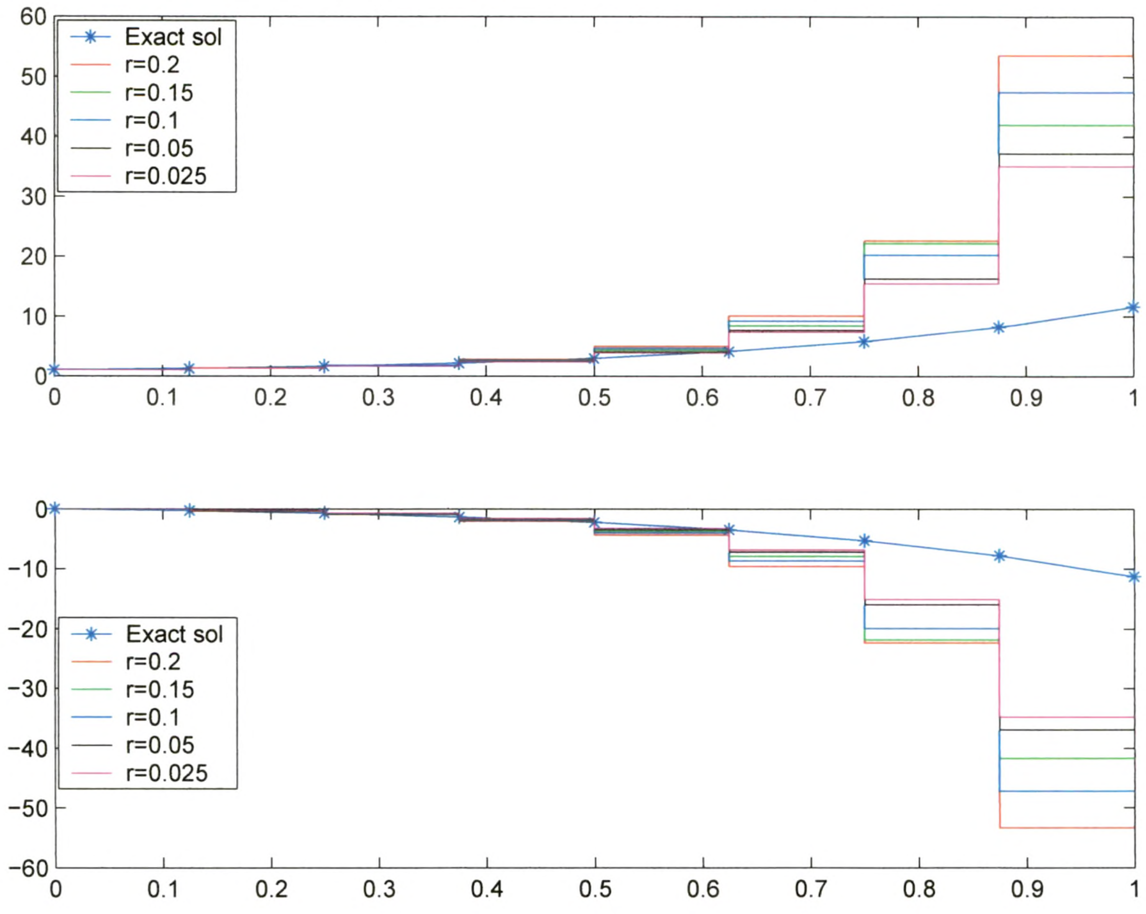


Figure 3.6: Solution Using Wavelet Packets for m=8

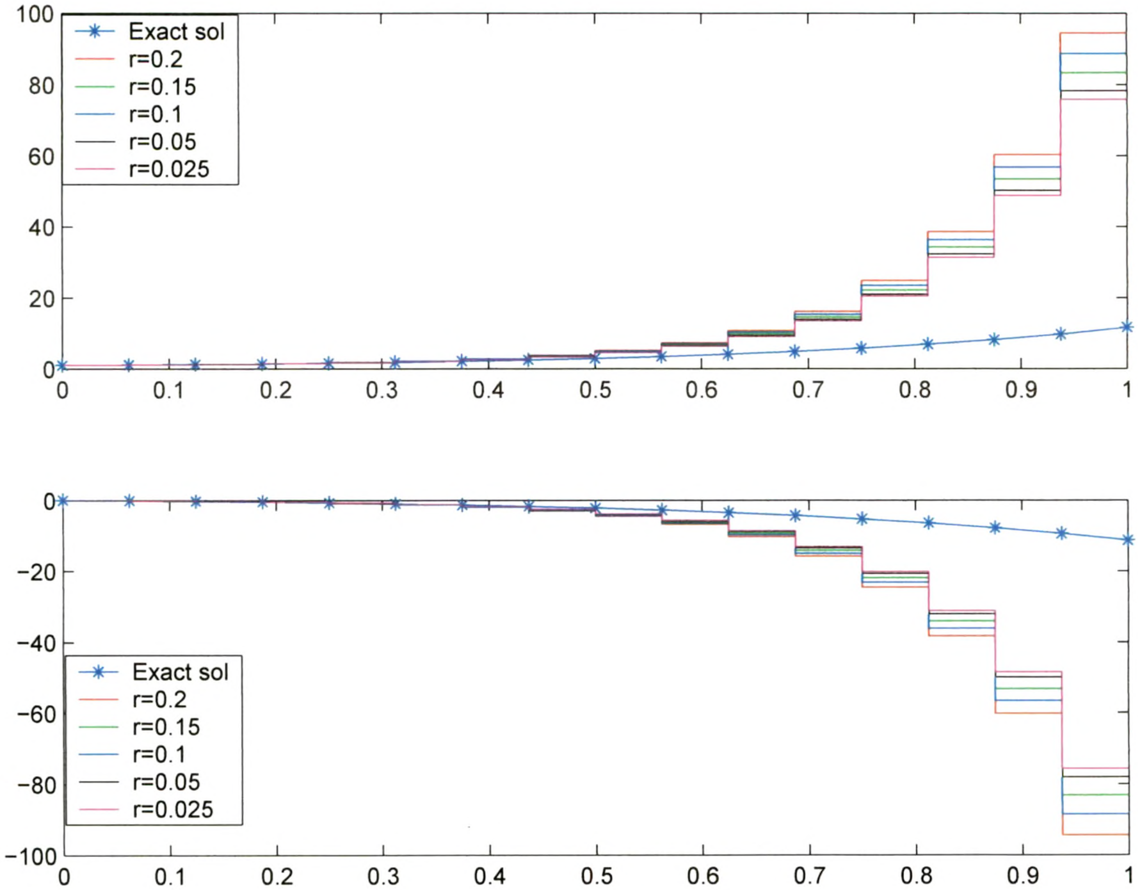


Figure 3.7: Solution Using Wavelet Packets for m=16

## 3.9 Conclusion

1. From the graphical representation we say that we get better solution by increasing the value of m and decreasing the value of r. Also the length of the interval over which exact solution matches with the solution by wavelet packets increases with the increase in the value of m.

2. From the table about absolute error we can say that absolute error goes on decreasing as we increase the value of m and decrease the value of r.