Chapter 4

VARIATIONAL PROBLEMS AND WAVELET PACKETS

4.1 Introduction

During 1970's the theory of Walsh functions was applied to various fields of engineering and science. For example, Communication, Spectroscopy, ECG,etc ([21], [28], [31], [45], [63], [68]). Later on this theory of Walsh functions is applied to mathematical problems eg. Variational problems,ordinary differential equation (ODE), partial differential equation (PDE),Integral equations etc (refer [14]).

The basic idea of a direct method for solving variational problems is to convert the problem of extremization of a functional into one which involves a finite number of variables. Ritz's method is well known in this area. In 1976, Chen and Hsiao [14] gave the Walsh series direct method for solving variational problems. In this chapter we first introduce the Walsh wavelet packet bases and then present a direct method for solving variational problems via Walsh wavelet packet bases i.e. using Haar bases and Walsh bases. Because of the orthonormality property of wavelet packet bases, the new direct method is simpler in reasoning as well as in calculation.

Now a days the wavelets are applied to technical problems whose mathematical models have form of boundary value problems described by ODE or PDE and integral equations. Such type of problems is discussed in the books by Goswami and Chen [30], Resnikoff and Wells [65], Fang and Thews [24]. Our aim in this chapter is to study Poisson's problem in one variable using Haar bases and Walsh bases.

4.2 Direct method for solving Variational Problems

The regular method for solving the extremization problem of a functional :

$$J = \int_0^1 F[t, y(t), \dot{y}(t)]dt \tag{4.2.1}$$

is through the Euler equation

$$F_y - \frac{d}{dt}F_{\dot{y}} = 0$$

In this chapter, we first assume the variable $\dot{y}(t)$ as a Walsh wavelet packet bases whose coefficients are to be determined.

$$\dot{y}(t) = \Sigma_{i=0}^{\infty} c_i W_i \tag{4.2.2}$$

Taking finite terms as an approximation we have,

$$\dot{y}(t) \cong c_0 W_0 + c_1 W_1 + ... + c_{m-1} W_{m-1} \cong c' W(x) \tag{4.2.3}$$

and as we know that,

$$\int_0^1 W(x)dx \cong PW(x) \tag{4.2.4}$$

Then the variable $y(t)$ can be written as

$$y(t) = \int_0^t \dot{y}(\lambda)d\lambda + y(0) = c'PW(t) + y(0) \tag{4.2.5}$$

The other functions of (4.2.1) can be expanded in terms of Walsh wavelet packet bases and we finally have,

$$J = J(c_0, c_1, ...c_{m-1}) \tag{4.2.6}$$

The original extremization of a functional problem (4.2.1) becomes the extremization of a function of a finite set of derivatives of J with respect to c_1 and setting them equal to zero, we get

$$\frac{\partial J}{\partial c_i} = 0 \qquad i = 0, 1, 2, 3...., m-1$$

Solving for c_i and substituting in to (4.2.5) we have the result.

4.3 Example

We shall apply above method to solve the following example.

$$\ddot{y} = -x^2, \quad y(0) = y(1) = 0 \tag{4.3.1}$$

Equation (4.3.1) is called a Poisson's problem in one variable but in fact requiring only two integration to discover the solution

$$y(x) = \frac{x(1-x^3)}{12} \tag{4.3.2}$$

which can be converted to variational problem:

$$J(y) = \int_0^1 \left[\frac{1}{2}(\dot{y})^2 - x^2 y\right] dx \tag{4.3.3}$$

The boundary conditions are

$$y(0) = 0 \quad y(1) = 0$$

The exact solution (4.3.2) is given by:

$$\begin{aligned} y\left(\frac{1}{4}\right) &= 0.02050, & y\left(\frac{1}{2}\right) &= 0.03645 \\ y\left(\frac{3}{4}\right) &= 0.03613, & y\left(1\right) &= 0 \end{aligned} \tag{4.3.4}$$

We shall first obtain the solution of the above problem using Haar bases and then using Walsh bases.

4.3.1 Solution using Haar bases

First of all we shall express $\dot{y}(x)$ in terms of Walsh wavelet packet bases using Haar bases with four terms as

$$\dot{y}(x) = \Sigma_{i=0}^{\infty} c_i H_i(x) \cong c_0 H_0(x) + c_1 H_1(x) + c_2 H_2(x) + c_3 H_3(x) \cong c' H(x) \tag{4.3.5}$$

Integrating $\dot{y}(x)$ and using the operational matrix P, we obtain

$$y(x) = \int \dot{y}(x) = c' P H(x) \tag{4.3.6}$$

Now the Walsh wavelet packet bases expansion of x^2 using Haar bases can be shown as follows:

$$\begin{aligned} c_0 &= \int_0^1 x^2.H_0(x)dx = \frac{1}{3}, & c_1 &= \int_0^1 x^2.H_1(x)dx = \frac{-1}{4} \\ c_2 &= \int_0^1 x^2.H_2(x)dx = \frac{-\sqrt{2}}{32}, & c_3 &= \int_0^1 x^2.H_3(x)dx = \frac{-\sqrt{2}.3}{32} \end{aligned}$$

Hence, $f(x) = x^2$ can be expressed in terms of Walsh wavelet packet bases using Haar bases as

$$x^2 = \frac{1}{3}H_0 - \frac{1}{4}H_1 - \frac{\sqrt{2}}{32}H_2 - \frac{3.\sqrt{2}}{32}H_3 = h'H(x) \qquad (4.3.7)$$

Now substituting (4.3.5),(4.3.6),(4.3.7) in (4.3.3) we have,

$$J = \int_0^1 \left[\frac{1}{2}c'H(x).H'(x).c - c'P.H(x).H'(x)h\right] dx$$

Using the orthonormality of Walsh wavelet packet bases, we get

$$J = \frac{1}{2}c'c - c'Ph = \frac{1}{2}\left[c_0^2 + c_1^2 + c_2^2 + c_3^2\right] - c'Ph$$

Now,

$$c'Ph = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & -0.1250 \\ 0.2500 & 0.1250 & -0.1250 & 0.1250 \\ 0.0625 & 0.0625 & 0.1250 & 0 \\ 0.0625 & -0.0625 & 0 & 0.1250 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{4} \\ \frac{-\sqrt{2}}{32} \\ \frac{-3.\sqrt{2}}{32} \end{bmatrix}$$

Hence

$$J = \frac{1}{2}c_0^2 + \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 + \frac{1}{2}c_3^2 - 0.29293c_0 - 0.041034c_1 + 0.0003c_2 - 0.0199c_3$$

Now using the Boundary conditions and the orthonormality of Walsh wavelet packet bases,

$$y(1) = \int_0^1 d\lambda + y(0)\dot{y}(\lambda) = c' \int_0^1 H(\lambda)\, d\lambda = c' \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $y(1) = 0 \Rightarrow c_0 = 0$.

Also,

$$\begin{aligned}
\frac{\partial J}{\partial c_1} &= 0 \quad \Rightarrow \quad c_1 = 0.0410 \\
\frac{\partial J}{\partial c_2} &= 0 \quad \Rightarrow \quad c_2 = -0.0003 \\
\frac{\partial J}{\partial c_3} &= 0 \quad \Rightarrow \quad c_3 = 0.0199
\end{aligned}$$

and

$$\begin{aligned}
y(x) &= c'PH(x) \\
&= 0.0115H_0 + 0.0039H_1 - 0.0052H_2 + 0.0076H_3 \qquad (4.3.8)
\end{aligned}$$

Hence using (4.3.8), we have

$$\begin{aligned}
y\left(\frac{1}{4}\right) &= 0.0080, \qquad y\left(\frac{1}{2}\right) = 0.0228 \\
y\left(\frac{3}{4}\right) &= 0.0183, \qquad y\left(1\right) = -0.0031 \qquad (4.3.9)
\end{aligned}$$

4.3.2 Solution using Walsh bases

First of all we shall express $\dot{y}(x)$ in terms of Walsh wavelet packet bases using Walsh bases with four terms as

$$\dot{y}(x) = \Sigma_{i=0}^{\infty} c_i W_i(x) \cong c_0 W_0(x) + c_1 W_1(x) + c_2 W_2(x) + c_3 W_3(x) \cong c' W(x) \quad (4.3.10)$$

Integrating $\dot{y}(x)$ and using the operational matrix P, we obtain

$$y(x) = \int \dot{y}(x) = c' P W(x) \quad (4.3.11)$$

Now the Walsh wavelet packet bases expansion of x^2 using Walsh bases can be shown as follows:

$$\begin{aligned} c_0 &= \int_0^1 x^2.W_0(x)dx = \frac{1}{3}, \qquad c_1 = \int_0^1 x^2.W_1(x)dx = \frac{-1}{4} \\ c_2 &= \int_0^1 x^2.W_2(x)dx = \frac{-1}{8}, \quad c_3 = \int_0^1 x^2.W_3(x)dx = \frac{1}{16} \end{aligned}$$

Hence, $f(x) = x^2$ can be expressed in terms of Walsh wavelet packet bases using Haar bases as

$$x^2 = \frac{1}{3}W_0 - \frac{1}{4}W_1 - \frac{1}{8}W_2 + \frac{1}{16}W_3 = h'W(x) \quad (4.3.12)$$

Now substituting (4.3.10),(4.3.11),(4.3.12) in (4.3.3) we have,

$$J = \int_0^1 \left[\frac{1}{2} c' W(x).W'(x).c - c' P.W(x).W'(x)h \right] dx$$

Using the orthonormality of Walsh wavelet packet bases, we get

$$J = \frac{1}{2} c' c - c' P h = \frac{1}{2} \left[c_0^2 + c_1^2 + c_2^2 + c_3^2 \right] - c' P h$$

Now,

$$c'Ph = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0.6250 & -0.2500 & -0.1250 & 0 \\ 0.2500 & 0.1250 & 0 & -0.1250 \\ 0.1250 & 0 & 0.1250 & 0 \\ 0 & 0.1250 & 0 & 0.1250 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{4} \\ \frac{-1}{8} \\ \frac{1}{16} \end{bmatrix}$$

Hence

$$J = \frac{1}{2}c_0^2 + \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 + \frac{1}{2}c_3^2 - 0.2856c_0 - 0.0443c_1 - 0.0260c_2 + 0.0234c_3$$

Now using the Boundary conditions and using the orthonormality of Walsh wavelet packet bases,

$$y(1) = \int_0^1 \dot{y}(x)\,(\lambda)\,d\lambda + y(0) = c' \int_0^1 W\,(\lambda)\,d\lambda = c' \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $y(1) = 0 \Rightarrow c_0 = 0.$

Also,

$$\begin{aligned} \frac{\partial J}{\partial c_1} &= 0 \;\Rightarrow\; c_1 = 0.0443 \\ \frac{\partial J}{\partial c_2} &= 0 \;\Rightarrow\; c_2 = 0.0260 \\ \frac{\partial J}{\partial c_3} &= 0 \;\Rightarrow\; c_3 = -0.0234 \end{aligned}$$

and

$$
\begin{aligned}
y(x) &= c' PW(x) \\
&= 0.0143W_0 + 0.0026W_1 + 0.0032W_2 - 0.0085W_3 \qquad (4.3.13)
\end{aligned}
$$

Hence using (4.3.13), we have

$$
\begin{aligned}
y\left(\frac{1}{4}\right) &= 0.0116, \quad y\left(\frac{1}{2}\right) = 0.0222 \\
y\left(\frac{3}{4}\right) &= 0.0234, \quad y\left(1\right) = 0 \qquad (4.3.14)
\end{aligned}
$$

4.4 Comparative Study of Error Estimates

The following tables gives the absolute error corresponding to $m = 4$, $m = 8$ and $m = 16$ at different points for different values of r. It is clear from the table that at most of the points amount of error is less for $m = 16$. This indicates that for higher values of m the amount of error will still decrease.

Absolute Error for m=4

r	t	Haar Bases	Walsh Bases	Walsh Functions
0.75		0.0123	0.0081	
0.9	0.25	0.0121	0.0082	0.00981
1.0		0.0125	0.0089	
0.75		0.00865	0.00985	
0.9	0.5	0.01165	0.01235	0.00696
1.0		0.01365	0.01425	
0.75		0.00933	0.00593	
0.9	0.75	0.01458	0.01003	0.00118
1.0		0.01783	0.01273	
0.75		0.0016	0.0076	
0.9	1.0	0.0016	0.0027	0.01851
1.0		0.0031	0	

Absolute Error for m=8

r	t	Haar Bases	Walsh Bases	Walsh Functions
0.75		0.0113963	0.0033963	
0.9	0.125	0.0103963	0.0025963	0.0088037
1.0		0.0098963	0.0024963	
0.75		0.0079	0.0045	
0.9	0.25	0.0081	0.0045	0.0013
1.0		0.0064	0.0048	
0.75		0.0116	0.0052	
0.9	0.375	0.0126	0.0058	0.0182
1.0		0.0287	0.0065	
0.75		0.00645	0.00505	
0.9	0.5	0.00825	0.00645	0.00665
1.0		0.0159	0.00755	
0.75		0.01276	0.00436	
0.9	0.625	0.01526	0.00636	0.02736
1.0		0.00885	0.00766	
0.75		0.00713	0.00253	
0.9	0.75	0.00883	0.00493	0.01413
1.0		0.01656	0.00663	
0.75		0.00286	0.00074	
0.9	0.875	0.00616	0.00226	0.02234
1.0		0.00966	0.00396	
0.75		0.0084	0.0054	
0.9	1.0	0.0101	0.0020	0.0384
1.0		0.0074	0.0001	

Absolute Error for m=16

r	t	Haar Bases	Walsh Bases	Walsh Functions
0.75	0.0625	0.0135	0.0015	0.0027
0.9		0.0125	0.0009	
1.0		0.0119	0.001	
0.75	0.1250	0.0068	0.0013	0.0031
0.9		0.0064	0.0011	
1.0		0.0064	0.0018	
0.75	0.1875	0.007	0.002	0.0032
0.9		0.0068	0.0016	
1.0		0.0066	0.0025	
0.75	0.2500	0.0001	0.0032	0.0034
0.9		0.0005	0.0032	
1.0		0.0009	0.0035	
0.75	0.3125	0.0099	0.0031	0.0027
0.9		0.0099	0.0035	
1.0		0.0099	0.0042	
0.75	0.3750	0.0033	0.0017	0.0007
0.9		0.0036	0.0021	
1.0		0.0041	0.001	
0.75	0.4375	0.0024	0.0015	0.0001
0.9		0.0032	0.0021	
1.0		0.0038	0.0016	
0.75	0.5000	0.0041	0.0012	0.0002
0.9		0.0034	0.0018	
1.0		0.0027	0.0023	

r	t	Haar Bases	Walsh Bases	Walsh Functions
0.75	0.5625	0.0044	0.001	0.0006
0.9		0.006	0.002	
1.0		0.0066	0.0025	
0.75	0.6250	0.0064	0.0009	0.0009
0.9		0.0081	0.0019	
1.0		0.0087	0.003	
0.75	0.6875	0.0074	0.0006	0.0018
0.9		0.0087	0.0018	
1.0		0.0097	0.0027	
0.75	0.7500	0.0099	0.0002	0.0024
0.9		0.0111	0.0014	
1.0		0.0121	0.0025	
0.75	0.8125	0.0018	0.0005	0.0029
0.9		0.0037	0.0011	
1.0		0.0058	0.0022	
0.75	0.8750	0.0104	0.0012	0.0042
0.9		0.0037	0.0011	
1.0		0.0124	0.0017	
0.75	0.9375	0.0085	0.0019	0.0055
0.9		0.0096	0.0001	
1.0		0.0105	0.001	
0.75	1.0	0.0179	0.0035	0.0069
0.9		0.0184	0.0013	
1.0		0.0187	0	

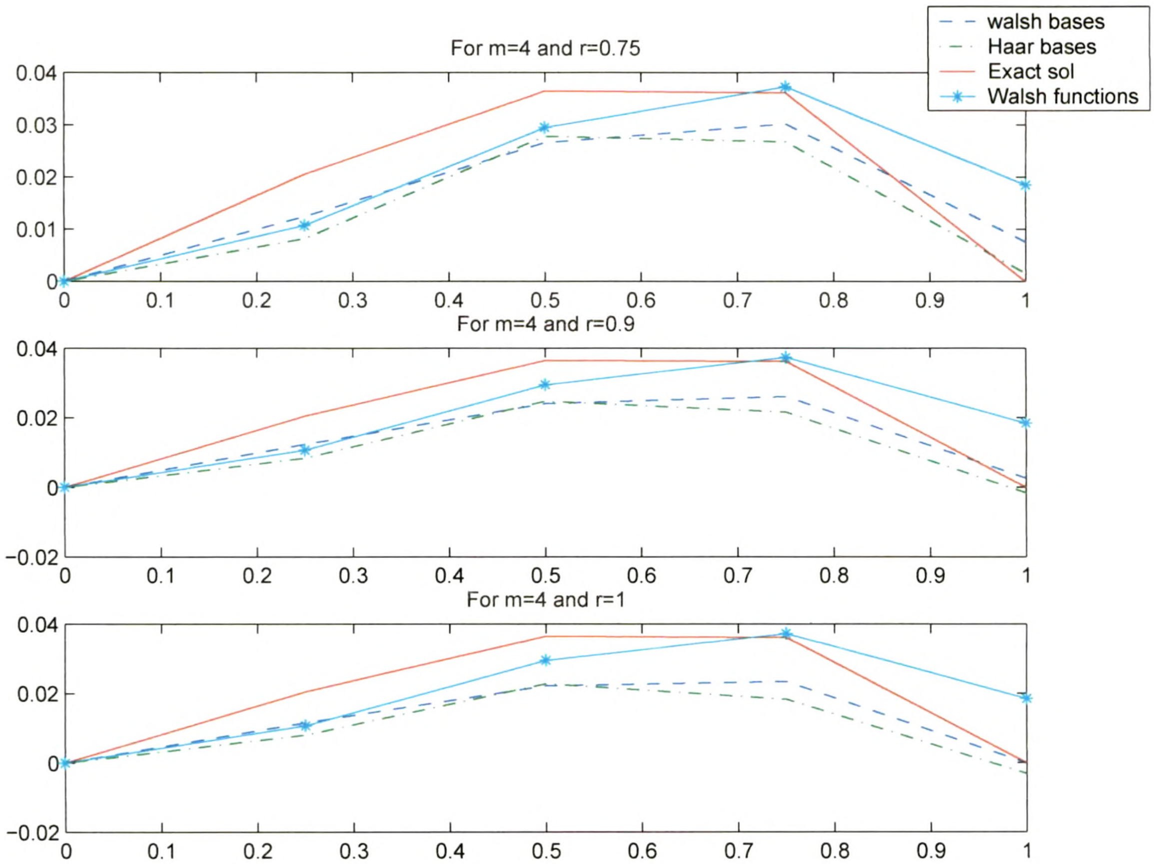

Figure 4.1: Solution Using Wavelet Packets for m=4

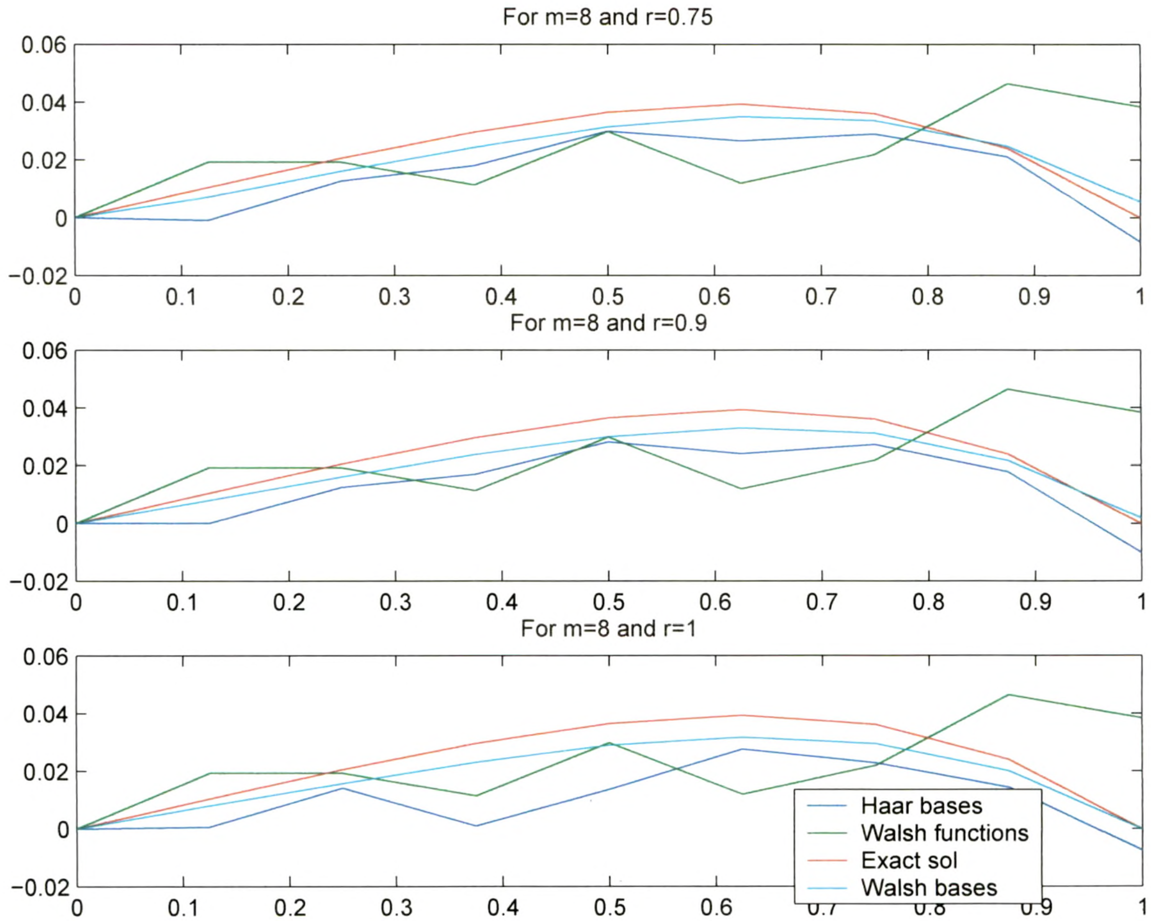

Figure 4.2: Solution Using Wavelet Packets for m=8

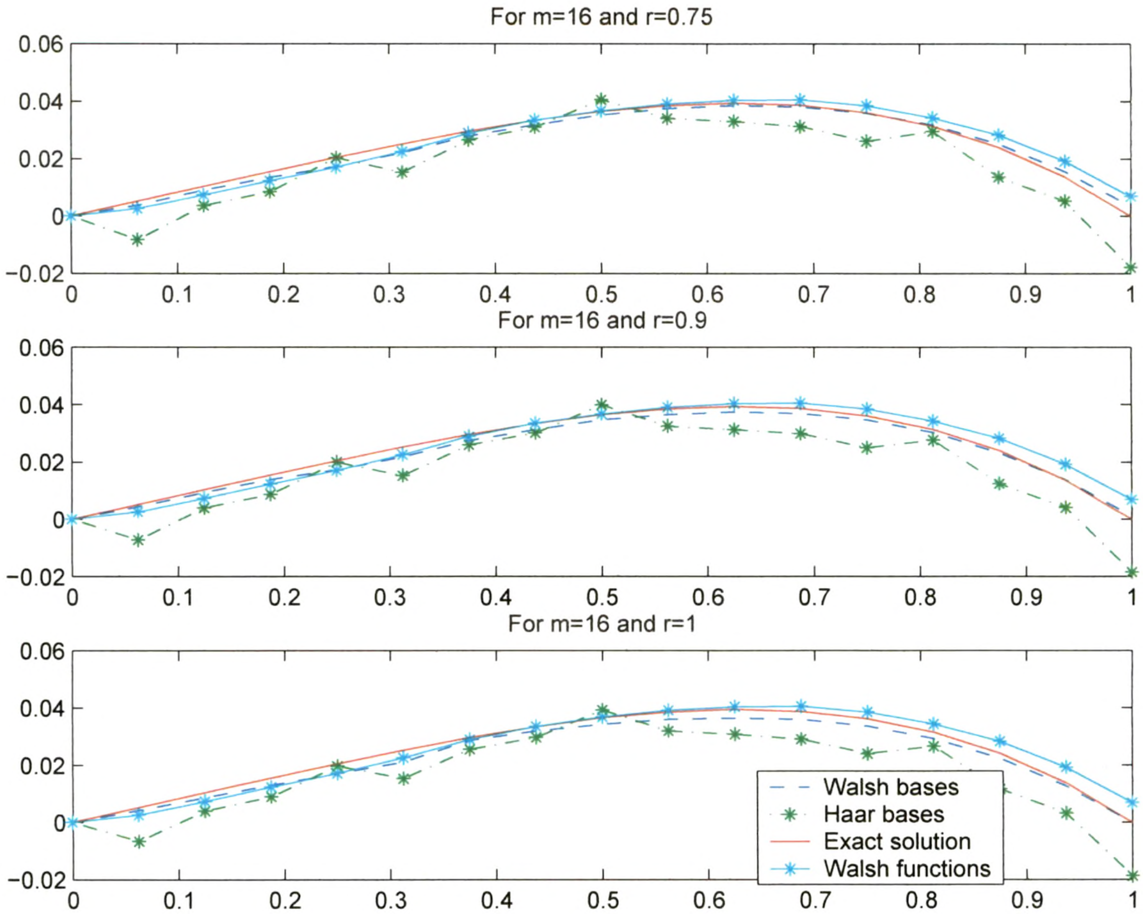

Figure 4.3: Solution Using Wavelet Packets for m=16

4.5 Conclusion

On the basis of the solution obtained for the problem, we can draw the following conclusion :

1. The Walsh wavelet packet gives better solution than that obtained using Walsh functions.
2. For larger values of m as well as r we can have solution which is more closer with the exact solution.