CHAPTER V

We have seen Chapter IV that the efficiency is increased than that of linear or quadratic estimator. In this chapter, we will generalise the method of chapter IV. We will see that the over all efficiency is about 99% even for n=14. This indicate that this method is suitable for all practical purposes.

We have seen in previous chapter that r, the least-squares estimate of ρ for fitting the curve:

$$E(y_x) = \alpha + \delta x + \beta \rho^x \quad \ldots \quad (5.1)$$

is obtained by solving the polynomial equation

$$\hat{r} = \left(\sum w_x(\hat{r}) y_x \right) / \left(\sum w_x(\hat{r}) y_{x-1} \right), \quad \ldots \quad (5.2)$$

where $w_x(r)$ are polynomials of degree 3n-12. In previous chapter we have replaced $w_x(r)$ by the function $u_x + r v_x$. Here we will replace $w_x(r)$ by the function $u_x + r v_x + r^2 z_x$. This method is described by Khatri [15] in fitting the exponential regression curve.

1. Modified method for estimating ρ.

The function $u_x + r v_x + r^2 z_x$ is chosen to give as good approximation as possible to the polynomials $w_x(r)$ in (5.2). The resulting estimate which will be denoted simply by r, is then one of the roots of the cubic equation from

$$r = \frac{\sum u_x y_x + r \sum v_x y_x + r^2 \sum z_x y_x}{\sum u_x y_{x-1} + r \sum v_x y_{x-1} + r^2 \sum z_x y_{x-1}} \qquad \ldots (5.3)$$

$(\sum u_x = \sum x u_x = \sum v_x = \sum x v_x = \sum z_x = \sum x z_x = 0)$.

By the method of iteration, we can solve the above equation (5.3), taking the first estimate of ρ as one. The arithmetic is simple and the numerical computations are not increased for the solution of r .

A measure of the degree of approximation involved in estimating ρ by (5.3) is provided by the asymptotic efficiency of r when y_x are independent and have equal variance σ^2. Under these conditions the variance of r tends to

$$\frac{\sigma^2}{\beta^2} \frac{(1+\rho^2)\sum(u_x+\rho v_x+\rho^2 z_x)^2 - 2\rho \sum(u_x+\rho v_x+\rho^2 z_x)(u_{x-1}+\rho v_{x-1}+\rho^2 z_{x-1})}{\left[\sum(u_x+\rho v_x+\rho^2 z_x)\rho^{x-1}\right]^2} \qquad \ldots (5.4)$$

as σ^2/β^2 tends to zero, the corresponding variance of r is obtained by substituting $w_x(\rho)$ for $u_x + \rho v_x + \rho^2 z_x$ in (5.4), or more conveniently from

$$F_{rr}\, \sigma^2/\beta^2$$

where F_{rr} is the quantity defined in chapter I.

(a) When $n = 5$, the u_x, v_x and z_x can be chosen so that

$$u_x + r_i v_x + r_i^2 z_x \propto w_x(r_i) \quad \ldots \qquad (5.5)$$

for five different values of $r_i = r_1, r_2, r_3, r_4$ and r_5. When r_i are taken to be 0, 0.25, 0.50, 0.75 and 1, the values of u_x, v_x and z_x can be calculated as follows:
At ρ=0,1,1/4 and 3/4, we have the following four equations:

$$\begin{aligned} u_x &= p_1 w_x(0) \\ u_x + v_x + z_x &= p_2 w_x(1) \\ u_x + v_x/4 + z_x/16 &= p_3 w_x(1/4) \\ u_x + 3v_x/4 + 9z_x/16 &= p_4 w_x(3/4) \end{aligned} \quad \ldots \qquad (5.6)$$

The weights $w(\rho)$ at ρ=0, 1/4, 1/2, 3/4, 1 can be calculated as described in chapter III. They are for n=5:

x	w(0)	w(1/4)	w(1/2)	w(3/4)	w(1)
1	3	316	21	768	1
2	-4	-373	-23	-799	-1
3	-1	-202	-17	-706	-1
4	2	259	19	737	1

...(5.7)

From equations (5.6)and (5.7), (solving for $v_x + z_x$ from first two and last two equations of 5.6), we have,

$$\begin{aligned} p_1 &= 57p_3 - 31p_4 \, ; \\ p_2 &= -145p_3 + 675p_4 \end{aligned} \quad \ldots (5.8)$$

Now solving equation (5.6), using equations (5.7) and (5.8), we have the following vectors in p_3 and p_4:

u_x	v_x	z_x
$171p_3-93p_4$	$(2636p_3)/3+240p_4$	$-(3584p_3)/3+528p_4$
$-228p_3+124p_4$	$-(2693p_3)/3-395p_4$	$(3812p_3)/3-404p_4$
$-57p_3+31p_4$	$-(2522p_3)/3+70p_4$	$(3128p_3)/3-776p_4$
$114p_3-62p_4$	$-(841p_3)/3+85p_4$	$(3356p_3)/3+652p_4$.

... (5.9)

Now let us take $u_x+\rho v_x+\rho^2 z_x \propto w_x(\rho)$ at the fifth value of $\rho=1/2$. We have

$$u_x+\tfrac{1}{2}v_x+\tfrac{1}{4}z_x=p_5\,w_x(\tfrac{1}{2}). \qquad \ldots (5.10)$$

Taking first two values of u_x, v_x and z_y from (5.9) and substituting in equation (5.10) we will have two equations in terms of p_3, p_4 and p_5. Thus we have the following relations :

$$p_3=-(135p_5)/48990 \text{ and } p_4=(6735p_5)/48990 .$$

Hence, $p_3=-9$, $p_4=449$. ... (5.11)

Substituting these values of p_3 and p_4 in (5.9) we have the following vectors (with arbitrary factor) upto two decimal places:

x	u_x	v_x	z_x	
1	-3	6.92	17.17	
2	4	-11.73	-13.36	...(5.12)
3	1	2.70	-24.79	
4	-2	2.11	20.98	

The efficiency of the estimate of ρ using the values of (5.12) is very high (over 99.99%) for any value of ρ .

(b) When n=6, the u_x, v_x and z_x can be chosen so that

$$u_x + r_i v_x + r_i^2 z_x \propto w_x (r_i)$$

for four different values of $r_i = r_1, r_2, r_3$ and r_4. The weights at ρ=0.2, 0.5, 0.8 and 0.0 are :

x	w(0.2)	w(0.5)	w(0.8)	w(0)	
1	27.583	31.29	0.684	2	
2	-21.447	-17.81	-0.309	-2	
3	-22.102	-32.64	-0.792	-1	...(5.13)
4	-1.787	-6.45	-0.225	0	
5	17.753	25.61	0.642	1	

Herew we will outlined the method of calculating the vectors u_x, v_x and z_x as described below:

$$u_x = p_1 \underline{w}(0)$$

$$u_x+0.2v_x+.04z_x = p_2 \underline{w}(0.2)$$

$$u_x+0.5v_x+.25z_x = p_3 \underline{w}(0.5) \quad \ldots(5.14)$$

$$u_x+0.8v_x+.64z_x = p_4 \underline{w}(0.8)$$

Thus solving for u_x, v_x and z_x we have

$$u_x=p_1 \underline{w}(0)$$

$$v_x= -5p_4 \underline{w}(0.8)/3 \;\; +5p_2 \underline{w}(0.2)/3 \;\; +4p_3 \underline{w}(0.5)-4p_1 \underline{w}(0) \quad \ldots(5.15)$$

$$z_x= 10p_4 \underline{w}(0.8)/3 \;\; -10p_2 \underline{w}(0.2)/3 \;\; -4p_3 \underline{w}(0.5)+4p_1\underline{w}(0)$$

Then working as usual as described above we have the follow -ing equation to be solved for p_1, p_2, p_3 and p_4 i.e.

$$9p_1\underline{w}(0)-20p_2\underline{w}(0.2) + 16p_2\underline{w}(0.5)-5p_4\underline{w}(0.8)=0 \quad \ldots (5.16)$$

Thus we have, $p_3=1.4458p_2$, $p_4=100.8592p_2$ and

$$p_2=0.1042p_1 .$$

Thus by taking $p_1=1$, we have the required solution for p_1, p_2, p_3 and p_4 as

p_1	p_2	p_3	p_4
1	0.1042	0.15065	10.5095 .

Substituting these values in (5.15), we have the required

u_x, v_x and z_x as follows:

x	u_x	v_x	z_x	
1	2	3.67	3.53	
2	-2	-1.05	-0.65	
3	-1	-5.64	-4.40	...(5.17)
4	0	-0.25	-3.37	
5	1	3.27	4.89 .	

The efficiency of the estimate of ρ , using the values of (5.17) is very high (over 99.99%) for any value of ρ .

(c) When $n \geq 7$, the method to determine u_x, v_x and z_x is similar to as described in chapter IV. We mention here the outlines of the method.

From the values of $w_x(r_1)$, $w_x(r_2)$ and $w_x(r_3)$, the u_x, v_x and z_x (with arbitrary factor) can be written as

$$u_x = r_2 r_3 w_x(r_1) + r_1 r_3 p w_x(r_2) + r_1 r_2 q\, w_x(r_3) ,$$

$$v_x = -(r_2+r_3) w_x(r_1) - (r_1+r_3) p w_x(r_2) - (r_1+r_2) q\, w_x(r_3), \text{ and}$$

$$z_x = w_x(r_1) + p w_x(r_2) + q\, w_x(r_3) ,$$

Where p and q are unknown constants to be determined from the fourth value of ρ (say, r_4) such that the efficiency at that point is maximum.

Table 5.1 gives the values of the u_x, v_x and z_x for n=7 to 14, derived by using the values for $w_x(r_i)$ with r_i=0.2, 0.5 and 0.8 and maximising at r_4=0. Table 5.2 gives the efficiency of the estimate obtained from equation (5.3) by using the values of Table 5.1 for n=14 and Table 5.3 gives the efficiencies at ρ =0 and ρ =1 for various values of n. This shows that it is possible to keep the efficiency of the estimate more than 94% for any value of ρ even for n=20.

Example:

The following data is taken from Stevens [37].

x :	0	1	2	3	4	5
y :	50.0	90.0	111.0	125.7	136.0	143.2 .

The estimate of ρ is obtained by solving

$$r = \frac{24.5 + 60.934r + 65.464r^2}{55.0 + 123.745r + 128.672r^2} .$$

Taking r=1, the first approximate value is found to be 0.49. Then taking r=0.49, we found the value of the estimate to be 0.4782, then next value is 0.47778. While the least square solution upto three decimal places only, given in chapter I, is r=0.478. Thus the value of the estimate agree very closely to the least-square solution.

Table 5.1

Values of u_x, v_x and z_x for n=7 to 14.

n=7

u_x :	20.51	-16.23	-10.72	-3.45	1.43	8.46
v_x :	20.02	13.65	-39.98	-24.08	13.40	16.99
z_x :	30.92	2.35	-20.68	-43.86	-14.24	45.51

n=8

u_x :	26.75	-17.24	-13.79	-5.89	-2.03	2.59	9.61
v_x :	12.29	30.52	-32.90	-36.52	-2.43	19.68	9.36
z_x :	35.99	7.92	-6.69	-43.76	-48.81	0.12	55.23

n=9

u_x :	33.51	-18.01	-16.65	-8.06	-4.11	-1.33	3.97
v_x :	-1.50	48.54	-20.94	-42.29	-16.59	12.16	18.57
z_x :	36.96	3.21	9.04	-23.75	-53.86	-39.76	12.92
u_x :	10.68						
v_x :	2.05						
z_x :	55.24						

n=10

u_x :	39.18	-17.78	-18.68	-9.75	-5.48	-3.56	-0.41
v_x :	-13.14	60.75	-8.93	-43.51	-26.77	2.49	18.23
z_x :	37.96	-1.25	18.53	-4.45	-45.38	-57.41	-27.23
u_x :	5.27	11.21					
v_x :	14.15	-3.27					
z_x :	24.35	54.88					

n=11

u_x :	44.80	-17.26	-20.54	-11.38	-6.46	-4.85	-3.06
v_x :	-27.24	71.27	4.75	-40.79	-34.52	-7.19	13.84
z_x :	39.32	-10.07	21.61	13.50	-26.68	-55.98	-50.48
u_x :	0.68	6.46	11.61				
v_x :	18.74	8.85	-7.71				
z_x :	-13.98	31.13	51.63				

n=12

u_x :	49.92	-16.42	-22.02	-12.81	-7.23	-5.60	-4.62
v_x :	-41.17	79.20	18.17	-35.78	-39.57	-16.09	7.52
z_x :	41.12	-19.69	19.68	26.62	-6.71	-44.01	-57.07
u_x :	-2.30	1.89	7.39	11.80			
v_x :	18.61	15.91	4.14	-10.94			
z_x :	-39.67	-2.32	34.47	47.58			

n=13

u_x :	54.60	-15.41	-23.15	-14.09	-7.93	-6.00	-5.51
v_x :	-54.63	85.29	30.31	-29.36	-41.89	-23.79	0.52
z_x :	43.54	-29.42	14.86	34.75	10.64	-27.69	-52.69
u_x :	-4.22	-1.24	3.06	8.11	11.78		
v_x :	16.08	18.50	11.83	0.01	-12.87		
z_x :	-51.97	-27.62	6.82	35.73	43.05		

n=14

u_x :	58.88	-14.23	-24.01	-15.27	-8.58	-6.28	-5.96
v_x :	-67.67	89.39	41.71	-21.86	-42.27	-29.80	-6.63
z_x :	46.73	-38.22	7.64	38.38	24.52	-11.04	-42.09
u_x :	-5.36	-3.44	-0.13	4.09	8.63	11.66	
v_x :	11.68	19.00	16.21	7.57	-3.24	-14.09	
z_x :	-53.66	-43.05	-16.80	13.49	35.42	38.68 .	

Table 5.2

% efficiencies of the estimates of ρ obtained from equation (5.3) using u_x, v_x and z_x of Table 5.1

For n = 14,

ρ	% Eff.
0.0	97.64
0.1	99.65
0.2	100.00
0.3	99.94
0.4	99.96
0.5	100.00
0.6	99.96
0.7	99.95
0.8	100.00
0.9	99.76
1.0	98.74

Table 5.3

% efficiencies of the estimates of ρ obtained from equation (5.3) using u_x, v_x and z_x of Table 5.1

ρ \ n	7	8	9	10	11	12	13	14
0	99.87	99.62	99.33	99.02	98.66	98.29	97.97	97.64
1	99.99	99.98	99.95	99.89	99.77	99.57	99.24	98.74