CHAPTER II

1. Patterson [25] has described a method for estimating ρ, by the ratio of two linear functions of y's, for the curve $y = \alpha - \beta \rho^x$. Stevens [37] has described the least squares estimates for this exponential regression curve by providing table for n = 5,6, and 7 equally spaced ordinates. S.Lipton [20] has extended this tables upto n=12. Pimental Gomes [29] has shown that, with equally spaced ordinates, efficient estimates of ρ can be obtained by solving equations of the type

$$J_0 y_0 + J_1 y_1 + \ldots + J_{n-1} y_{n-1} = 0$$

where the J's are complicated polynomials in r.

In this chapter, we have extended the method as described by Patterson [25] to the curve $y = \alpha + \delta x + \beta \rho^x \ldots (A)$ by putting one more condition under fixed and increasing variance model, Shah [31]

2. Estimation of ρ under fixed model with five equally spaced ordinates:

Full information on ρ can be obtained, for four equally spaced ordinate, by the estimate

$$r = (y_3 - 2y_2 + y_1)/(y_2 - 2y_1 + y_0)$$

i.e. $y_3 - y_2(r+2) + y_1(2r+1) - ry_0 = 0$ ... (2.1)

We can write down a similar expression for five equally spaced ordinates as

$$y_4-y_3(r+2)+y_2(2r+1)-ry_1 = 0 \qquad \ldots \qquad (2.2)$$

Thus the sum of equations (2.1) and (2.2) leads to the estimate

$$r = (y_4-y_3-y_2+y_1)/(y_3-y_2-y_1+y_0) \qquad \ldots \qquad (2.3)$$

Note that if w_i are the coefficient of y_{n-i}, then

$$\sum w_i = 0 \text{ and } \sum iw_i = 0 .$$

Now the equations (2.1) and (2.2) can be combined with the relative weights $\mu:1$, where μ is to be determined.

$$y_4+y_3(\mu-r-2)+y_2(2r+1-\mu r-2\mu)+y_1(-r+2r\mu+\mu)$$

$$-r\mu y_0 = 0 \qquad \ldots \qquad (2.4)$$

The estimate of r given by this equation is

$$r=\left\{y_4+y_3(\mu-2)+y_2(1-2\mu)+\mu y_1\right\} / \left\{y_3+y_2(\mu-2)+y_1(1-2\mu)+\mu y_0\right\} \qquad \ldots \qquad (2.5)$$

In order to obtain full information provided by equations (2.1) and (2.2), μ should be so chosen so that the variance of r is minimum. For n = 5,

$$V(r)=\frac{\phi^2}{\rho^2}\left[\frac{1^2+(\mu-2-\rho)^2+(1-2\mu-\rho\mu+2\rho)^2+(\mu-\rho+2\rho\mu)^2+\mu^2\rho^2}{\{\rho^3+(\mu-2)\rho^2+(1-2\mu)\rho+\mu\}^2}\right]$$

where $\phi^2=\sigma^2/\beta^2$. (See Appendix)

Differentiating this with respect to μ and equating this to zero we get

$$\mu = \frac{4\rho^3+13\rho^2+12\rho+6}{6\rho^3+12\rho^2+13\rho+4} .$$

It is interesting to know that with these values of μ the estimate (2.5) are fully efficient. The value of μ range from $\mu=1.5$ when $\rho=0$ to $\mu=1$ when $\rho=1$. When $\mu=1$ equation (2.5) is same as equation (2.3). The estimate r of ρ given by equation (2.3) is therefore suitable for high values of ρ. Similarly $(2y_4-y_3-4y_2+3y_1)/(2y_3-y_2-4y_1+3y_0)$ will be most efficient for small values of ρ. Therefore it is necessary to choose the value of μ such that over all efficiency is 95%. It is found that $\mu=1.2$ is suitable for high efficiency. Thus the estimate

$$r=(5y_4-4y_3-7y_2+6y_1)/(5y_3-4y_2-7y_1+6y_0), \quad \ldots \quad (2.6)$$

is suitable for n = 5 equally spaced ordinates.

3. Estimation of ρ with six, seven and eight equally spaced ordinates:

Additional equation such as

$$y_5 - y_4 (r+2) + y_3(2r+1) - ry_2 = 0 \quad \ldots \quad (2.7)$$

are combined with equations (2.1) (2.2) using weights $\mu_2 : \mu_1 : 1$ etc.

The general expression of the estimates obtained in this way is

$$r = \frac{y_{n-1} + (\mu_1 - 2)y_{n-2} + (\mu_2 - 2\mu_1 + 1)y_{n-3} \ldots (-2\mu_{n-4} + \mu_{n-5})y_2 + \mu_{n-4}y_1}{y_{n-2} + (\mu_1 - 2)y_{n-3} + (\mu_2 - 2\mu_1 + 1)y_{n-4} \ldots (-2\mu_{n-4} + \mu_{n-5})y_1 + \mu_{n-4}y_0},$$

with the condition $\sum w_i = 0$, and $\sum iw_i = 0$.

As usual $\mu_1, \mu_2, \ldots$ are so chosen that the efficiencies of the estimates are high over the whole range of ρ. The following values have been found to lead to estimates of high efficiency (over 90%):

$n = 6$, $\mu_1 = 1.9$, $\mu_2 = 1.5$,

$n = 7$, $\mu_1 = 2.1$, $\mu_2 = 2.6$, $\mu_3 = 1.7$,

$n = 8$, $\mu_1 = 2.5$, $\mu_2 = 3.5$, $\mu_3 = 3.3$, $\mu_4 = 2.0$.

The actual estimates are :

$$\frac{10y_5 - y_4 - 13y_3 - 11y_2 + 15y_1}{10y_4 - y_3 - 13y_2 - 11y_1 + 15y_0} \quad \text{for } n=6 \quad \ldots \quad (2.8)$$

$$\frac{10y_6+y_5-6y_4-14y_3-8y_2+17y_1}{10y_5+y_4-6y_3-14y_2-8y_1+17y_0} \quad \text{for } n=7 \quad \ldots \quad (2.9)$$

$$\frac{10y_7+5y_6-5y_5-12y_4-11y_3-7y_2+20y_1}{10y_6+5y_5-5y_4-12y_3-11y_2-7y_1+20y_0} \quad \text{for } n=8 \quad \ldots \quad (2.10)$$

The above proposed estimates (Shah [31]) can also be used to advantage as preliminary estimates of ρ as I have discussed in chapter I. When the computation is to be done on desk calculators then in such a case only single iteration is required. The biases in the estimates can be calculated by (2.27)

TABLE 2.1

Percentage efficiencies of proposed estimate of ρ.

Equ. / ρ	2.6	2.8	2.9	2.10
0.0	95.2	91.3	88.4	86.4
0.1	98.4	97.3	96.0	94.4
0.2	99.7	99.7	99.0	97.8
0.3	99.9	99.7	98.4	96.9
0.4	99.6	98.4	95.5	93.0
0.5	99.0	96.1	91.4	87.5
0.6	98.1	93.7	87.1	82.6
0.7	97.3	91.4	83.0	76.1
0.8	96.3	89.4	79.1	71.8
0.9	95.4	86.7	75.9	68.2
1.0	94.5	84.7	78.4	64.6

TABLE 2.2

Biases in equations (2.6),(2.8),(2.9) & (2.10)

Equ. / ρ	2.6	2.8	2.9	2.10
0.0	0.270	0.091	-0.093	-0.178
0.1	0.348	0.129	0.007	-0.080
0.2	0.409	0.219	0.106	0.023
0.3	0.455	0.298	0.200	0.124
0.4	0.487	0.364	0.283	0.218
0.5	0.507	0.414	0.352	0.300
0.6	0.517	0.452	0.406	0.401
0.7	0.519	0.477	0.446	0.421
0.8	0.516	0.492	0.474	0.459
0.9	0.510	0.499	0.491	0.484
1.0	0.500	0.500	0.500	0.500

4. Increasing variance Model:

When equation (A) represents a biological growth curve, x is a measure of time, and it may be appropriate to incorporate into the model a variance that changes with time. One way of doing this is by means of a continuous autoregressive scheme in which the expected rate of growth at any time is given by a differential equation corresponding to(A), but in each element of time individuals are subject to an error distribution. This will produce a correlation between successive observations in respect of their deviations from the average curve as well as a steadily increasing total variance. The theory that follows in this section has analogies with the theory of Brownian movement, based upon Langevin's equation (Chandrasekar [2], Finney [7]).

Let us write

$$\log \rho = -\gamma \quad \ldots \quad (2.11)$$

and

$$\rho^x = u. \quad \ldots \quad (2.12)$$

Then, from equation (A), $\frac{\partial y}{\partial x} = \delta + \gamma(\alpha + \delta x - y)$ ... (2.13)

Define $K_y(\theta, x)$ as the cumulant generating function of y for a particular value of the independent variate, x. Define also $L(\theta, x)\,dx$ as the cumulant generating function for the distribution of the additional 'error' acquired by an individual in the time interval $(x, x + dx)$.

Express the condition that the cumulant generating function at time $(x + dx)$ is the sum of the functions at x for the variate $(y + dy)$ and for the error, or

$$K_y(\theta, x+dx) = K_{y+dy}(\theta, x) + L(\theta, x)dx \quad \ldots \quad (2.14)$$

where $dy = [\delta + \gamma(\alpha + \delta x - y)]\,dx.$

Then

$$K_y(\theta,x)+\frac{\partial K}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\partial x = K_{(\delta+\gamma\alpha+\gamma\delta x)dx+y(1-\gamma dx)}(\theta,x)+ L(\theta,x)dx.$$

$$= \frac{(\delta+\gamma\alpha+\gamma\delta x)dx}{1-\gamma dx}(i\theta)+ K_y(\theta,x) - \gamma(i\theta)dx\,\frac{\partial K}{\partial(i\theta)} + L(\theta,x)dx.$$

Hence, in the limit,

$$\gamma u(\partial K/\partial u)-(\delta+\gamma\alpha+\gamma\delta x)\;(i\theta)\;+\gamma(i\theta)(\partial K/\partial(i\theta))-L(\theta,x)=0,$$

or

$$u(\partial K/\partial u)-(i\theta)\;\partial K/\partial(i\theta)\;+(\delta\gamma^{-1}+\alpha+\delta x)(i\theta)\;+ \gamma^{-1}L(\theta,x)=0 \quad \ldots \qquad (2.15)$$

Moreover, if time is measured from the point at which the error begins to effect, this differential equation is subject to the end condition

$$K_y(\theta,0) = (\alpha+\beta)\;(i\theta) \qquad \ldots \qquad (2.16)$$

(i) Suppose now that the error increment is normally distributed and a constant independent of y, that is to say

$$L(\theta,x) = (1/2)\,\sigma^2\,(i\theta)^2 \qquad \ldots \qquad (2.17)$$

Then, the solution of (2.15) subject to (2.16) is

$$K_y(\theta,x)=(\alpha+\beta u+\delta x)(i\theta) + (\sigma^2/2\gamma)(1-u^2)(i\theta)^2/2! \quad \ldots(2.18)$$

Thus y is normally distributed, with expectation still given by $y=\alpha+\delta x+\beta u$, but with variance now expressed by

$$V(y) = \frac{\sigma^2}{2\gamma}(1-\rho^{2x}) \quad \ldots \quad (2.19)$$

and tending to the limit $\sigma^2/2\gamma$ as x becomes large.

(ii) A second possibility is to have a normally distributed error increment whose magnitude is proportional to the expectation of y. If $k_s(u)$ is written for the sth cumulant, then

$$L(\theta,x) = \tfrac{1}{2}\sigma^2 k_1(u)(i\theta)^2 \quad \ldots \quad (2.20)$$

equating coefficient of $(i\theta)$ in (2.15) then gives

$$uk_1'(u)-k_1(u) + (\alpha+\delta\gamma^{-1}+\delta x) = 0$$

$$uk_2'(u)-2k_2(u) + (\sigma^2/\gamma)\, k_1(u) = 0$$

$$uk_s'(u)-sk_s(u) = 0 \quad \text{for } s \geqslant 3,$$

whence, in virtue of the end condition (2.16),

$$K_y(\theta,x)=(\alpha+\delta x+\beta u)(i\theta)+(\sigma^2/2\gamma)\Big[(1-\rho^x)\Big\{\alpha-\tfrac{1}{2}(\delta/\gamma)+\rho^x(\alpha-\tfrac{1}{2}\tfrac{\delta}{\gamma}+2\beta)\Big\}+\delta x\Big](i\theta)^2/2!\ldots(2.21)$$

Thus, the expectation of y is the same, but now the variance is

$$V(y)=(\sigma^2/2\gamma)\left[(1-\rho^x)\left\{(\alpha-\tfrac{1}{2}\tfrac{\delta}{\gamma})+\rho^x(\alpha-\tfrac{1}{2}\tfrac{\delta}{\gamma}+2\beta)\right\}+\delta x\right] \quad \ldots \quad (2.22)$$

Illustration: Thus for n=5, the estimator can be written in the form (2.5), and the variance is

$$V(r_p)=\frac{\psi^2(1-\rho^2)}{\rho^{2X}} \quad \frac{6\mu^2-8\mu+6}{(1-\rho)^4(\rho+\mu)^2} \quad \ldots \quad (2.23)$$

Where X indicates time for the first observation, and $\psi^2 = \sigma^2/2\gamma\beta^2$.

The variance (2.23) is minimised by

$$\mu = (2\rho+3)/(3\rho+2) \quad \ldots \quad (2.24)$$

which leads to

$$V_{min}(r_p)= 10\,\psi^2(1+\rho)/\rho^{2X}(1-\rho)^3(3\rho^2+4\rho+3).$$

Thus μ=1.2 is a good approximation at moderate values of ρ, for large ρ, μ=1 is superior and for small ρ, μ=1.5 is superior. The variances are

$$V(r_{p,1.2}) = \frac{\psi^2(1+\rho)}{\rho^{2X}(1-\rho)^3} \quad \frac{5.04}{(\rho+1.2)^2},$$

$$V(r_{p,1}) = \frac{\psi^2(1+\rho)}{\rho^{2X}(1-\rho)^3} \quad \frac{4}{(1+\rho)^2},$$

$$V(r_{P,1.5}) = \frac{\psi^2(1+\rho)}{\rho^{2X}(1-\rho)^3} \quad \frac{30}{(2\rho+3)^2} \; .$$

TABLE 2.3

Multipliers $\psi^2/\{\rho^{2X}(1-\rho)\}$ in the asymptotic variance of various estimators for the increasing variance model

ρ	Estimator r	$r_{P,1.2}$	$r_{P,1}$	$r_{P,2}$
0.0	3.333	3.500	4.000	3.333
0.1	2.889	2.955	3.276	2.903
0.2	2.469	2.488	2.688	2.511
0.3	2.092	2.095	2.213	2.165
0.4	1.767	1.767	1.832	1.864
0.5	1.491	1.495	1.524	1.607
0.6	1.260	1.270	1.276	1.390
0.7	1.068	1.084	1.074	1.203
0.8	0.908	0.929	0.911	1.046
0.9	0.776	0.801	0.777	0.913
1.0	0.667	0.694	0.667	0.800

5. Appendix.

Let $r = A/B$ ... (2.25)

where A and B are functions of y's.

Let $E(A) = \xi$, $E(B) = \eta$.

Write $R = \xi/\eta$

and define the bivariate moments of A and B by

$$v_{st} = E\left\{(A-\xi)^s (B-\eta)^t\right\}. \qquad \ldots \qquad (2.26)$$

Then, by writing

$$r = R\left(1+\frac{A-\xi}{\xi}\right)\left(1+\frac{B-\eta}{\eta}\right)^{-1},$$

expanding in series and taking expectations, we have

$$E(r) = R-(v_{11}-Rv_{02})/\eta^2 + (v_{12}-Rv_{03})/\eta^3 - (v_{13}-Rv_{04})/\eta^4$$
$$+ \ldots \qquad \ldots \qquad (2.27)$$

Similarly we have an asymptotic expansion for the variance of r : $V(r) = E\left\{r- E(r)\right\}^2$

$$=(v_{20}-2Rv_{11}+R^2v_{02})/\eta^2 - 2(v_{21}-2Rv_{12}+R^2v_{03})/\eta^3$$
$$+\left\{(3v_{22}-v_{11}^2)-2R(3v_{13}-v_{11}v_{02})+R^2(3v_{04}-v_{02}^2)\right\}/\eta^4$$
$$- \ldots \qquad \ldots \qquad (2.28)$$

When A and B are normally distributed, all odd moments vanish; and we have

$$V(r) = (v_{20} - 2Rv_{11} + R^2 v_{02})/\eta^2 + (3v_{20}v_{02} + 5v_{11}^2$$

$$- 16Rv_{11}v_{02} + 8R^2 v_{02}^2)/\eta^4 + \dots \qquad (2.29)$$

These expressions agree with those obtained by Merrill [22].

In particular, the general linear estimator can be put in the form

$$r_p = \frac{\mu_1 y_n + \mu_2 y_{n-1} + \dots + \mu_{n-1} y_2}{\mu_1 y_{n-1} + \mu_2 y_{n-2} + \dots + \mu_{n-1} y_1} \quad \dots (2.30)$$

where $\sum \mu_i = 0$, and $\sum i \mu_i = 0$.

From equation (2.29) the variance is

$$V(r_p) = \frac{\phi^2}{\rho^2} \frac{\mu_1^2 + (\mu_2 - \rho\mu_1)^2 + (\mu_3 - \rho\mu_2)^2 + \dots \rho^2 \mu_{n-1}^2}{(\mu_1 \rho^{n-2} + \mu_2 \rho^{n-3} + \dots + \mu_{n-1})^2}$$