Synopsis of the Thesis entitled

A Study of Circular and Elliptical Restricted Three Body Problems with Perturbations

submitted by

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1 Introduction

In 17^{th} century, Kepler proposed three laws of planetary motion. Using these laws, Newton developed the formula for gravitational force between any two point masses. With the help of Newton's law of gravitation and three laws of Kepler, it is possible to find a closed mathematical solution for the two body problem. Later, Newton tried to get a closed form solution for three body problem but could not attain much success. Euler proposed a simplified form of three body problem in which the mass of one body is considered to be negligible compared to the masses of other two bodies. This is called Restricted Three Body Problem (RTBP). This phenomenon is very useful for studying the motion of celestial objects and has applications in astrophysics and astrodynamics. In the study of RTBP, two more massive bodies are called the primaries and the body with the infinitesimal mass is called the secondary. In a RTBP, if the primaries move in circular orbits around their barycenter, then it is called a Circular Restricted Three Body Problem (CRTBP) and if the primaries move in elliptic orbits, then it is called an Elliptic Restricted Three Body Problem (ERTBP). These are two widely studied particular cases of RTBP.

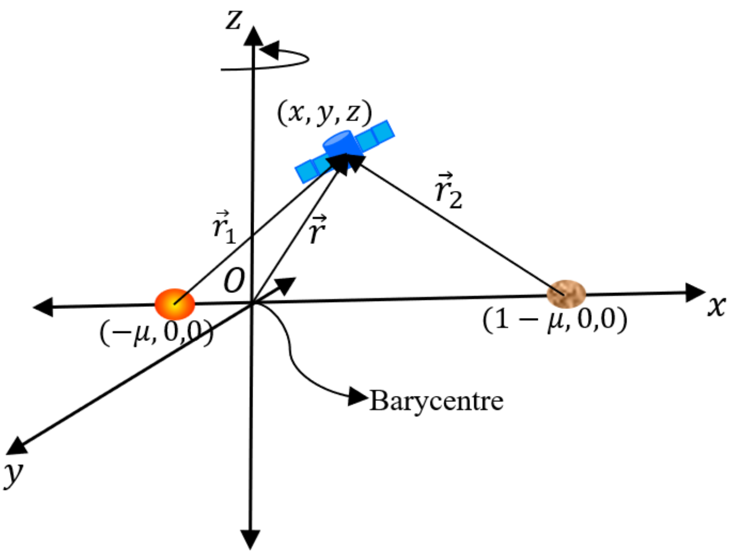

Figure 1: Schematic diagram of RTBP in a dimensionless synodic coordinate system

Infinitesimal body experiences forces other than the gravitational force such as radiation pressure force, force due to oblateness of the primary, atmospheric drag, etc. These forces are called perturbing forces. In this study, the perturbing forces due to solar radiation pressure and oblateness of primary are considered.

There are several locations in the space where the infinitesimal body experiences the balance between the gravitational force and all other perturbing forces. Such locations are called equilibrium points or libration points or Lagrangian points. Every RTBP has five planar Lagrangian points out of which three Lagrangian points lie on the line joining the primaries. These three Lagrangian points are called collinear Lagrangian points. Perturbing forces affect the location as well as stability of Lagrangian points.

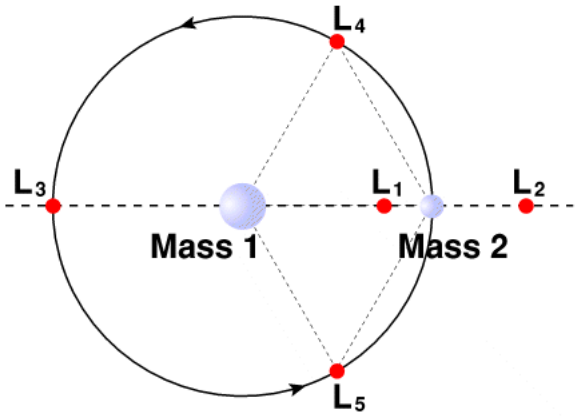

Figure 2: Lagrangian points

Halo family is a family of three dimensional periodic orbits around collinear Lagrangian points which arise as a bifurcation from the planar Lyapunov family. The concept of halo orbit was introduced by Fqrquhar in 1966 [6]. He discovered trajectories around Earth-Moon L_2 in which a communication satellite could be placed and that would allow a continuous link between the Earth and the far side of the Moon. He named this orbit as halo orbit as it appeared like a halo encircling the Moon. A satellite placed in a halo orbit serves many scientific purposes. International Sun-Earth Explorer 3 (ISEE-3) was the first halo orbits mission satellite. It was launched into a halo orbit around Sun-Earth L_1 for collecting the data on solar wind conditions upstream from Earth. ISRO is planning for Aditya L_1 mission which will orbit the L_1 point of the Sun-Earth system.

The phenomenon of resonance is useful for studying the dynamics of solar system and celestial bodies. A resonance arises when there is a numerical relation between the frequencies or period of celestial bodies. In solar system, mainly two types of resonances are observed: spin-orbit resonance and orbit-orbit resonance. Spin-orbit resonance is generated by the relation between the rotational and orbital period of a single body. Most of the natural satellites are in 1 : 1 spin-orbit coupling. Due to this, only one side of Moon is visible form the Earth. Mercury is in a 3 : 2 spin-orbit resonance. Orbit-orbit resonance is generated due to relation between orbital periods of two or more bodies. The satellites Io and Europa of Jupiter are in 2 : 1 orbit-orbit resonance. Also, the structure of ring around the Jupiter is generated due to the resonance between the dust particles in Jupiter's gravitational field and the rotation of the magnetic field of the planet [5]. Many researchers have studied resonant periodic orbits in the CRTBP framework with various perturbations [15, 12, 14, 9, 10, 11, 8]. Orbit-orbit resonances are mean motion resonances as they are generated due to numerical relation between mean motions of bodies. Consider two bodies A and B of arbitrary masses orbiting around a central mass with mean motions n_A and n_B, respectively. If there exist integers p and q such that the ratio of mean motions $n_A : n_B$ is of the form $p : q$, then the mean motion resonance exists between the two bodies. Here, p corresponds to mean motion of A and q corresponds to mean motion of B. In general, orbit-orbit resonances are broadly classified into two types [20]:

- **Exterior resonances:**
  In this case, an integer ratio $p : q$ is such that $p < q$. Here, spacecraft spends majority of the time outside the vicinity of the second primary and has larger orbital period or semi-major axis compared to second primary.

- **Interior resonances:**
  In this case, an integer ration $p : q$ is such that $p > q$. The spacecraft spends majority of the time inside the orbit of the second primary as it has a smaller orbital period or semi-major axis compared to the second primary.

2 Equations of motion

The motion of the infinitesimal body is affected by the perturbing forces due to radiation pressure and oblateness of the primaries. The radiation pressure force changes with the distance in a similar way as the gravitational force but acts in the opposite direction to it. This reduces the effective mass of the radiating body. The mass reduction factor is defined as $q = 1 - (F_p/F_g)$, where F_p and F_g, respectively, represent the forces due to radiation pressure and gravitational attraction. To analyze the effect of radiation pressure on the motion of infinitesimal body, the mass reduction factor q is introduced into the equations of motion [16, 1, 12, 14]. Oblateness coefficient A, defined as $A = (R_e^2 - R_p^2)/(5R^2)$, is used in equations of motion for considering the perturbation due to oblateness of the primary [2, 3, 4]. Here, R_e and R_p, respectively, denote the equatorial and polar radius of the oblate primary and R is the distance between the primaries.

Then the motion of the infinitesimal body in a dimensionless synodic coordinate system with origin at the barycentre of the primaries in CRTBP framework is governed by the equations [1]

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{1}$$

where

$$\Omega = \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)q_1 A_1}{2r_1^3} + \frac{\mu q_2 A_2}{2r_2^3}, \tag{2}$$

and

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2), \tag{3}$$

$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2}, \\ r_2 &= \sqrt{(x+\mu-1)^2 + y^2 + z^2}. \end{aligned} \tag{4}$$

Here, an overhead dot denotes a differentiation with respect to time t. The quantities q_i and A_i, respectively, denote the mass reduction factor and oblateness coefficient of the i^{th} primary, $i = 1, 2$ and n is called the mean motion.

Suppose the primaries are moving in elliptic orbit around their barycentre. Then the equations of motion of infinitesimal body in the synodic dimensionless coordinate system are given by [17]

$$\begin{aligned} x'' - 2y' &= \Omega_x, \\ y'' + 2x' &= \Omega_y, \\ z'' &= \Omega_z, \end{aligned} \tag{5}$$

where

$$\Omega = \frac{1}{\sqrt{1-e^2}}\left[\frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2}\right]. \tag{6}$$

Here, a prime denotes a derivative with respect to independent variable E, the eccentric anomaly, e is the eccentricity of the orbit of the primaries, q_i is mass reduction factor of $i^{th}, i = 1, 2$, primary and r_1 and r_2 are as defined in (4).

3 Methodology

In this section, different methods used for finding periodic orbits in two and three dimensions are described. Lindstedt-Poincaré method is an analytical method which is useful for finding the initial state vector for planar Lyapunov orbits and three dimensional halo orbits in CRTBP and ERBP framework. The initial condition obtained using the Lindstedt-Poincaré method is revised with help of numerical method of Differential Corrections (DC) for getting more accurate solution.

Poincaré Surface of Sections can be used for getting the initial conditions for various planar periodic orbits. For getting a PSS, system (5) is solved using the Runge-Kutta-Gill method with fixed step size and the point (x, x') corresponding to each solution for which $y = 0$ and $y' > 0$ is plotted. Periodic orbits give rise to fixed points that are the centre of islands of stability and islands correspond to the quasi–periodic orbits librating around the stable positions.

3.1 Lindstedt-Poincaré Method

Systems (1) and (5) contain non-linear terms which change the frequency of the linearized system and give rise to secular terms. The terms whose amplitude grow with time are called secular terms. Lindstedt-Poincaré method is an analytical method which uses the method of perturbations for removing the secular terms appearing in the solution. For this, a new independent variable $\tau = \omega t$, where t is current/existing independent variable and ω, called the frequency connection term, is considered. Then the systems (1) and (5) are expressed in terms of new independent variable τ and the solutions of these systems are assumed in the perturbation form as

$$X(\tau) = \epsilon X_1(\tau) + \epsilon^2 X_2(\tau) + \epsilon^3 X_3(\tau) + \dots, \tag{7}$$

$$Y(\tau) = \epsilon Y_1(\tau) + \epsilon^2 Y_2(\tau) + \epsilon^3 Y_3(\tau) + \dots, \tag{8}$$

$$Z(\tau) = \epsilon Z_1(\tau) + \epsilon^2 Z_2(\tau) + \epsilon^3 Z_3(\tau) + \dots, \tag{9}$$

and

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \dots . \tag{10}$$

Here, the aim is to select the values of $\omega_i,\ i = 1,\ 2,\ 3,\ \dots$ in such a manner that terms giving rise to secular terms are avoided from the equations of motion. To accomplish this aim, the solutions (7)-(10) are substituted in the systems (1) and (5) and the coefficients of n^{th} powers of ϵ are equated to get the n^{th} order approximate solution, $n = 1,\ 2,\ 3,\ \ldots$. Usually, the series in equations (7)-(10) are terminated after four or five terms giving the fourth or fifth order approximate solution.

3.2 Differential Corrections Method

The numerical method of differential corrections (DC) or multi-dimensional Newton-Raphson's method is useful for modifying the state vectors of trajectories having certain constraints. Halo orbits are three dimensional periodic orbits which are symmetric about xz plane and intersect this plane perpendicularly. This characteristic makes the computation of halo orbits similar to solving a two point boundary value problem. In DC method, design variables are modified in such a manner that all given constraints are satisfied simultaneously. Suppose

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

is a free variable vector with n independent design variables $X_1, X_2, \ldots, X_n$. In most of the cases, $\mathbf{X}$ contains the elements of state vector and integration time. The design variables can be modified subject to m scalar constraint equations

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} F_1(\mathbf{X}) \\ F_2(\mathbf{X}) \\ \vdots \\ F_m(\mathbf{X}) \end{bmatrix} = \mathbf{0}.$$

In most of the cases, constraints are position, time of flight and velocity. Consider an initial guess $\mathbf{X}^0$ for determining a free variable vector $\mathbf{X}^*$ such that $\mathbf{F}(\mathbf{X}^*) = 0$. Expanding the constraint vector in a Taylor series about initial guess $\mathbf{X}^0$,

$$\mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{X}^0) + \frac{\partial \mathbf{F}(\mathbf{X}^0)}{\partial \mathbf{X}^0}(\mathbf{X} - \mathbf{X}^0) + \ldots\ .$$

Now, denoting $\partial \mathbf{F}(\mathbf{X}^0)/\partial \mathbf{X}^0$, an $m \times n$ Jacobian matrix of partial derivatives of constraint vector as $D\mathbf{F}(\mathbf{X}^0)$ and truncating the Taylor series to first order gives

$$\mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{X}^0) + D\mathbf{F}(\mathbf{X}^0)(\mathbf{X} - \mathbf{X}^0). \tag{11}$$

Since for a solution $\mathbf{F}(\mathbf{X}) = 0$, equation (11) in an iterative update form can be written as

$$\mathbf{F}(\mathbf{X}^j) + D\mathbf{F}(\mathbf{X}^j)(\mathbf{X}^{j+1} - \mathbf{X}^j) = \mathbf{0} \tag{12}$$

where $\mathbf{X}^j$ is the current iteration of the free variable vector, $\mathbf{X}^{j+1}$ is the next iteration of the free variable vector, and $\mathbf{F}(\mathbf{X}^j)$ is the value of the current constraint vector as evaluated after propagating the equations of motion from the initial condition $\mathbf{X}^j$. The value of $D\mathbf{F}(\mathbf{X}^j)$ can be obtained with the help of $\mathbf{X}^j$ and $\mathbf{F}(\mathbf{X}^j)$. Equation (12) represented in the form

$$\mathbf{X}^{j+1} = \mathbf{X}^j - D\mathbf{F}(\mathbf{X}^{\mathbf{j}})^{-1}\mathbf{F}(\mathbf{X}^j) \tag{13}$$

is used as an update equation until $\left\|\mathbf{F}(\mathbf{X}^{j+1})\right\|_2 < 10^{-12}$. In most of the cases, desired accuracy is reached within 10 iterations.

The initial state vector of halo orbit obtained using the Lindstedt-Poincaré method is modified using the DC method. Since halo orbits are symmetric about xz plane, we must have $y = 0$ at half period and also, these orbits intersect xz plane perpendicularly so at half period, we must have $\dot{x} = \dot{z} = 0$. Then the free variable vector for revising the state vector of halo orbit is

$$\mathbf{X} = \begin{bmatrix} x \\ \dot{y} \\ T/2 \end{bmatrix}$$

and the constraint vector is

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} y \\ \dot{x} \\ \dot{z} \end{bmatrix} = \mathbf{0}$$

with the Jacobian matrix

$$D\mathbf{F}(\mathbf{X}) = \begin{bmatrix} O & I_3 \\ \mathcal{U} & K \end{bmatrix},$$

where

$$\mathcal{U} = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} & \Omega_{xz} \\ \Omega_{yx} & \Omega_{yy} & \Omega_{yz} \\ \Omega_{zx} & \Omega_{zy} & \Omega_{zz} \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

and the matrix O is 3×3 null matrix and I_3 is 3×3 identity matrix. Here, (x, y, z) is position vector, $(\dot{x}, \dot{y}, \dot{z})$ is velocity vector and T is period of halo orbit. In this case, the z coordinate of the position vector is not considered in the free variable vector so its value will remain unchanged throughout the correction scheme. Further, it is possible to keep x coordinate of position vector fixed by removing it from the free variable vector and inserting z coordinate instead.

3.3 Runge-Kutta-Gill Method

Runge-Kutta-Gill (RKG) method is a numerical method useful for solving first order Initial Value Problems (IVPs) numerically. The algorithm for solving an autonomous IVP using RKG method with fixed step size is given below:
Consider the Initial Value Problem

$$\frac{dy}{dx} = y' = f(y),\ y(x_0) = y_0.$$

1. Select the step size h.

2. Find the quantity: $k_1 = hf(y_i)$.

3. Update y_i as $y_{i,1} = y_i + 0.5k$.

4. Calculate k_2 as $k_2 = hf(y_{i,1})$.

5. Further update y_i as $y_{i,2} = y_{i,1} + 0.5k_1(-1 + \sqrt{2}) + k_2(-1 - 0.5\sqrt{2})$.

6. Compute the quantity: $k_3 = hf(y_{i,2})$.

7. Update y_i as $y_{i,3} = y_{i,2} - \left[\frac{k_2}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right)k_3\right]$.

8. Evaluate the quantity: $k_4 = hf(y_{i,3})$.

9. Then, the new iterate of y_i, y_{i+1}, can be obtained as

$$y_{i+1} = y_{i,3} + \frac{1}{6}[k_1 + (2 - \sqrt{2})k_2 + (2 + \sqrt{2})k_3 + k_4], \quad i \geq 0.$$

This process is repeated till the desired accuracy is obtained. Since system (5) contains second order differential equations, it is converted into equivalent first order system and then RKG method is applied to a system of six first order equations with step size $h = 0.001$.

3.4 Poincaré Surface of Sections

The study of a complex dynamical system can be simplified by reducing its dimension. Poincaré map is one such useful tool which effectively reduces the dimension of the dynamical system and converts a continuous system into a corresponding discrete system. The technique of Poincaré map was introduced by Henri Poincaré in 1981 in which the crossing of a trajectory to a particular hyperplane is recorded. First, a particular value of Jacobi constant or energy constant is selected for the numerical propagation. RKG method with fixed step size is used commonly for numerical propagation of system (5). Jacobi constant is given by

$$C = \frac{1}{\sqrt{1-e^2}}\left[x^2 + y^2 + \frac{2(1-\mu)q}{r_1} + \frac{2\mu}{r_2}\right] - x'^2 - y'^2. \qquad (15)$$

Equation (15) shows selecting a particular value of C reduces the degree of freedom and hence the orbits will lie in three dimensional subspace $C(x, y, x', y') = C$ embedded in a four dimensional phase space. Further, by specifying a hyperplane, three dimensional subspace is projected onto a two dimensional (x, x') plane [7]. In most of the cases, the plane $y = 0$ is considered as hyperplane and it is further assumed that at initial time, the infinitesimal body lies on the x axis and there is no velocity in the x direction. Then the velocity in the y direction can be obtained from

$$y' = \sqrt{\frac{1}{\sqrt{1-e^2}}\left(x^2 + y^2 + \frac{2(1-\mu)q}{r_1} + \frac{2\mu}{r_2}\right) - C - x'^2}. \qquad (16)$$

The equations (15) and (16) corresponding to CRTBP can be obtained by putting $e = 0$.

4 Layout of Thesis

This Thesis is divided into eight chapters. **Chapter 1** is introduction in which the motivation for the study and different mathematical tools and techniques used in the study of CRTBP and ERTBP are presented. At the end, summary of subsequent chapters is given.

In Chapter 2, computation of halo orbits around L_1, L_2 and L_3 using the analytic and numerical method in CRTBP framework is given. By considering the perturbation due to radiation pressure and oblateness of both the primaries, analytic solution for computing halo orbits upto fifth order approximation using Lindstedt-Poincaré technique is obtained. Using this analytic solution as a first guess in DC scheme, halo orbits around L_1 and L_2 of the Sun-Earth system are computed numerically for different solar radiation pressure and oblateness of Earth. Also, the third and fourth order analytical solutions were used for finding halo orbits for analyzing the accuracy of the solutions. It was observed that the separation between halo orbit decreases as the order of solution increases. So, fifth order initial solution provides more precise initial guess than third or fourth order solution.

Further, the effects of perturbing forces due to radiation pressure and oblateness on location, size, period, frequency correction term and other parameters of halo orbits around L_1, L_2 and L_3 were studied. [19] computed fourth order analytic solution for halo orbits in the photogravitational Sun-Earth CRTBP with oblateness. Variation in parameters of halo orbits around L_1 and L_2 due to variation in q_1 and A_2 was similar to observations of [19]. Due to increase in solar radiation pressure, halo orbits around L_3 shrink and move towards the more massive primary. Further, period of these orbits decrease. Oblateness of second primary shifts orbits around L_3 towards the more massive primary and decreases period. To study the effect of oblateness of more massive primary on halo orbits around L_1 and L_2, the Earth-Moon system with actual oblateness of the Earth was considered. It was observed that due to increase in A_1, halo orbits around L_1 and L_2 both elongate and move towards the second primary, and period decreases. Radiation pressure of second primary shrinks halo orbits around L_1 and enlarges halo orbits around L_2. A decrease in q_2 shifts orbits around L_1 towards m_2 and orbits around L_2 towards m_1, and period of orbits around L_1 increase while around L_2 decreases.

Chapter 3 analyzes the effects of mass ratio $\mu = m_2/(m_1+m_2)$ on parameters of halo orbits around L_1, L_2 and L_3 in CRTBP framework. Different random values of μ in the interval $[10^{-8}, 0.5]$ were considered. Starting with 10^{-8}, value of μ was increased with a fixed step size of 10^{-6} until $\mu = 0.5$. It was observed that as the value of μ increases, Lagrangian point L_1 and halo orbits around it shift towards the more massive primary while Lagrangian point L_2 and corresponding halo orbits recede from second primary till $\mu_0 = 0.17894$ and for $\mu > \mu_0$, orbits move towards the second primary. For verifying these results, Sun-Mars, Sun-Earth, Sun-Earth+Moon, Sun-Saturn and Sun-Jupiter systems were considered and halo orbits around all three collinear Lagrangian points were computed. It was observed that as the value of μ increases, Lagrangian point L_3 and corresponding orbits move towards the more massive primary. Suppose A_x, A_y and A_z represent the amplitudes of halo orbits in the x, y and z

direction, respectively. For a halo orbit, A_y is a multiple of A_x, and A_x and A_z are related by amplitude constraint, it is enough to study the variation in either A_x or A_z. For $A_z = 3.25 \times 10^{-4}$, corresponding value of A_x was obtained using amplitude constraint relation for $\mu \in [10^{-8}, 0.5]$. It was found that μ and A_x are inversely proportional for orbits around L_1 while they are directly proportional for orbits around L_2. For analyzing the variation in amplitude of halo orbits around L_3, $A_x = 0.045$ was considered for finding corresponding A_z value. Study shows that A_z increases with the increase in μ. Period of halo orbits around L_1 and L_3 decreases while it increases for orbits around L_2 with the increase in μ. The size, initial distance from origin and initial velocity of orbits are also affected by the value of μ. The analysis shows orbits around L_1 and L_2 both elongate as μ increases. Further, with the increase in the value of μ, halo orbits around L_1 come close to the origin and orbits around L_2 move away from the origin. The initial velocity of spacecraft in orbits around L_1 as well as L_2 increase with the increase in mass ratio.

In Chapter 4, computation of halo orbits around L_1, L_2 and L_3 in the photogravitational Sun-Mars ERTBP is given. [18] has discussed the motion of an infinitesimal body in a dimensionless synodic pulsating coordinate system in a ERTBP framework which is a non-autonomous system with true anomaly as independent variable. This non-autonomous system has been converted into an autonomous system by averaging the system with respect to new independent variable as the eccentric anomaly E of the second primary. Computation of locations of collinear equilibrium points in this system shows due to solar radiation pressure, location of equilibrium points vary. The location of Lagrangian points do not change with the change in the eccentricity of the orbit of the primaries. Computation of the third order approximate solution using Lindstedt-Poincaré method is described and the procedure of finding halo orbits using differential correction method is given. Monodromy matrix is the State Transition Matrix (STM) evaluated at one period of halo orbit. The eigenvalues of monodromy matrix are used for analyzing the stability of periodic orbits and finding bifurcations. If λ_i $(i = 1, 2, \ldots, 6)$ are eigenvalues of monodromy matrix, then the stability index is defined as $\nu_i = (\lambda_i + 1/\lambda_i)/2$. Since eigenvalues of monodromy matrix are always in reciprocal pairs, there are three stability indices corresponding to a periodic orbit. Further, two eigen values of monodromy matrix are always unity and hence the stability index, say ν_2, corresponding to this pair is always unity [21]. Halo orbits are obtained as tangent bifurcation from planar Lyapunov orbits when the out-of-plane stability index ν_3 crosses the line $\nu_3 = 1$. A periodic oribt is stable if all stability indices have value between -1 and 1 [20]. Due to solar radiation pressure of the Sun, the separation between the halo and axial bifurcation increases. This holds true for orbits around L_1 and L_2 both. Halo orbits around L_1 shrink, move towards the Sun and period of orbits increases due to increase in solar radiation pressure. But orbits around L_2 enlarge, move towards the Sun and period decreases due to increases in solar radiation pressure. The effect of solar radiation pressure on halo orbits around L_1 and L_3 were found to be similar. A graphical comparison of size of halo orbits show that due to non-zero eccentricity of the orbit of the primaries, halo orbits shrink.

Chapter 5 contains evolution of f-family orbits in the photogravitational Sun-Saturn ERTBP framework. The technique of PSS is extended from CRTBP to ERTBP for exploring periodic orbits. Variations in parameters of f-family orbits due to variation in eccentricity of the orbit of the primaries, solar radiation pressure and Jacobi constant are observed. The existence of energy integral puts a constraint on the value of Jacobi constant. So, it is necessary to find the maximum value of C, say C_M, corresponding to each pair (q, e) such that for $C \leq C_M$, $v^2 \geq 0$, where v is the velocity of the infinitesimal body. For $e \in [0, 0.1]$ and $q = 0.98, 0.99$ and 1 computation of C_M shows that a quadratic polynomial in e provides the curve of best fit for approximating C_M for $q = 0.98, 0.99$ and 1. Further, it has been observed that the excluded region shifts towards the second primary due to increase in eccentricity of orbit of the primaries. Analysis shows f-family orbits shift towards the more massive primary and their diameter and period increases with increase in the value of e. An increase in solar radiation pressure decreases the value of C_M and expands the excluded region of motion for a satellite. Regression analysis shows that the functional relation between the length of excluded region and e depends on solar radiation pressure of the Sun as well. Since solar radiation pressure is a repulsive force, orbits move towards the second primary and their diameter decreases. Due to perturbing force of solar radiation pressure, the value of C and the difference of energy levels at separatrices decreases and variation in size and shape of islands and f-family orbits is also observed. By considering different values of Jacobi constant C in the interval $[2.77, 3.017]$, variations in parameters of f-family orbits were analyzed and the results agree with [13] for CRTBP framework.

Chapter 6 is devoted to the study of first order exterior resonant periodic orbits in the photogravitational Sun-Saturn ERTBP framework. Using the numerical technique of PSS, 1:2, 2:3, 3:4, 4:5 and 5:6 resonant periodic orbits were obtained and the effects of eccentricity of the orbit of the primaries (e_p), solar radiation pressure (q) and Jacobi constant (C) on location, period, eccentricity (e_s) and semi-major axis (a_s) of these periodic orbits were studied. For an exterior resonance, in the ratio $p : p + q$, p denotes number of loops in the orbit of a spacecraft and q denotes the order of resonance. It was observed that the first order exterior resonant orbits lie on the right side of f-family orbits. For observing the effects of variation in e_p on parameters of resonant periodic orbits, e_p was varied in the interval $[0, 0.1]$. The observations show that the orbits move towards the Sun due to increase in the value of e_p. Further, an increase in period and a decrease in semi-major axis of orbits is observed due to non-zero value of e_p. The variation in e_s is not similar for all orbits. The eccentricity e_s of $1 : 2$ resonant orbits decreases with the increase in e_p while e_s increases with the increase in e_p for $p : p + 1,\ p \in \{2, 3, 4, 5\}$ resonant orbits. Effects of solar radiation pressure and Jacobi constant are similar in CRTBP and ERTBP framework.

In Chapter 7, analysis of first order interior resonant orbits is performed. These orbits lie on the left side of f-family orbits. In this case, the resonance ratio is of the form $p+q : p$ in which q denotes the order of resonance and $p+q$ denotes the number of loops in the orbit of spacecraft. The number of islands corresponding to a $p+q : p$ resonant orbit denotes the order of resonance. For distinct values of $e_p \in [0, 0.09]$, 2:1, 3:2, 4:3 and 5:4 resonant periodic orbits were computed. The study shows that these orbits recede from the Saturn and advance towards the Sun. Further, with the increase in e_p, the period, semi-major axis (a_s) and eccentricity (e_s) of these orbits increase. The analysis of size and shape of these orbits revels that orbits shrink while their loops enlarge due to increment in the value of e_p. Due to solar radiation pressure, orbits advance towards the Saturn and period, semi-major axis (a_s) and eccentricity (e_s) decrease. Further, orbits enlarge while the loops of these orbits shrink due to solar radiation pressure. By considering five different values of Jacobi constant in the interval $[2.88, 2.92]$, the effects of Jacobi constant C on parameters of resonant orbits are analyzed. Using the non linear multiple regression analysis, an estimator function for computing approximate locations of resonant periodic orbits is obtained.

Chapter 8 contains conclusions and a brief overview of future scopes for research in this field and is followed by list of publications and bibliography.

5 List of Publications/ Communications

1. Dhwani Sheth, V. O. Thomas, Elbaz I. Abouelmagd and Vineet K. Srivastava : Fifth order solution of halo orbits via Lindstedt–Poincaré technique and differential correction method. *New Astronomy* **87** (2021): 101585. doi: `https://doi.org/10.1016/j.newast.2021.101585` (Indexed in SCOPUS, Web of Science, Scimago, SCI)

2. Dhwani Sheth and V. O. Thomas : Effects of mass ratio on halo orbits about L_1 and L_2. *AIP Conference Proceedings* **2451** 1 (2022): 020040. doi: `https://doi.org/10.1063/5.0095250` (Indexed in SCOPUS, Scimago, SCI)

3. Dhwani Sheth and V. O. Thomas : Halo orbits around L_1, L_2, and L_3 in the photogravitational Sun–Mars elliptical restricted three-body problem. *Astrophysics and Space Science* **367** 10 (2022): 1-20. doi: `https://doi.org/10.1007/s10509-022-04130-w` (Indexed in SCOPUS, Web of Science, Scimago, SCI)

4. Dhwani Sheth, Niraj M. Pathak, V. O. Thomas and Elbaz I. Abouelmagd : Periodic orbits analysis of elliptical Sun-Saturn system, is communicated for publication.

5. Dhwani Sheth, Niraj M. Pathak, V. O. Thomas and Elbaz I. Abouelmagd : Analysis of Exterior Resonant Periodic orbits in the Photogravitational ERTBP, is communicated for publication.

Papers Presented

1. Presented a paper as a poster entitled ***Halo Orbits around*** L_1 ***in the Photogravitational Sun-Earth System with Oblateness*** in the WOMEN'S SCIENCE CONGRESS – 2020 held during January 5-6, 2020 during 107^{th} Indian Science Congress at university of Agricultural Sciences, GKVK, Bengaluru, Karnatak. (*Received First prize*)

2. Presented an article titled ***Effects of Mass ratio on Halo orbits around*** $\mathbf{L_1}$ ***and*** $\mathbf{L_2}$ at International Conference on Advances in Multi-Disciplinary Sciences and Engineering Research organized by Chitkara University on July 2-3, 2021. (Virtual Mode)

3. Presented a paper in 4^{th} International Conference on Recent Advances in Mathematical Sciences and Applications (RAMSA-2021) titled ***Effect of radiation and oblateness of the primaries on the frequencies of the short and long periodic orbits around $\mathbf{L_4}$ and $\mathbf{L_5}$ in ERTBP*** organized by Gayatri Vidya Parishad college of Engineering during December 21-24, 2021. (Virtual Mode)

4. Presented an article entitled ***Effects of eccentricity of orbit of primaries on separatrices of Sun-Saturn ERTBP using PSS*** at 108^{th} Indian Science Congress held at R. T. M. Nagpur University, Nagpur, Maharashtra during January 3-7, 2023. (*Received Young Scientist Award*)

5. Presented a paper entitled ***Halo orbits around collinear Lagrangian points in ERTBP*** at the International Workshop on Celestial Mechanics and Dynamical Astronomy organized by Central University of Rajasthan, Ajmer, Rajasthan and Inter University Centre for Astronomy and Astrophysics (IUCAA), Pune, Maharashtra at Central University of Rajasthan, Ajmer, Rajasthan during January 6-8, 2023.

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