QUADRATIC FORMS OVER THE FIELD OF RATIONAL FUNCTIONS IN ONE VARIABLE OVER A FINITE FIELD.

INTRODUCTION

1. Dedekind [1,2] laid down the foundations of the arithmetic of algebraic function fields based on the axiomatic characterization of fields, as generalizations of the arithmetic of algebraic number fields.

The starting point of Dedekind's theory was ideals. The following definition of an ideal in K is true for any field which is the quotient field of an integral domain with a unit element.

Ideals in K : Let k be a prime field of characteristic $p \neq 2$ and $k(x)$ the field of rational functions in x over k. An ideal in K is defined, relative to the ring $k[x]$ of polynomials in x over k, as follows.

An ideal $\mathfrak{a}$ in K is the set of elements in K with the properties:

1. It is an additive subgroup of K.
2. If $a \in \mathfrak{a}, \lambda a \in \mathfrak{a}$ for any $\lambda \in k[x]$ and i.e. $\lambda a \in \mathfrak{a}, \lambda \in k[x]$ and for some $\lambda \in K$, $\lambda a \in \mathfrak{a}$, $\lambda \notin k[x]$

$\mathfrak{a}$ is called integral if there exists $\lambda \in k[x]$ such that $\lambda a \in \mathfrak{a}$ for every $a \in \mathfrak{a}$. Otherwise it is fractional, representable as a quotient of two integral ideals, that is, every element is representable as a/b, a belonging to one and b to another integral ideal.

The nonzero ideals in K form a group under this representation and the number of residue classes of $k[x]$ modulo $\mathfrak{a}$, the norm of $\mathfrak{a}$, denoted $N\mathfrak{a}$ is finite. These are the two properties on which the arithmetic of algebraic number fields and function fields is developed.

More important for further developments in arithmetic than ideals are the divisors defined as below. These give rise to the valuations in K defined a little later here.

Definition of a divisor in K and fields containing K

(Chevalley: Algebraic functions of one variable).

Let R be a field and K a subfield of R. By a V-ring in R (over K) is meant a subring $\mathfrak{v}$ in R which satisfies the following conditions:

1. $\mathfrak{v}$ contains K.
2. $\mathfrak{v}$ is not identical with R
3. If x is an element of R not in $\mathfrak{v}$ then x^{-1} is in $\mathfrak{v}$.

Let $\mathfrak{v}$ be a V-ring. Those elements x in $\mathfrak{v}$ for which x^{-1} are not in $\mathfrak{v}$ (we call them nonunits) form an ideal $\mathfrak{y}$ in $\mathfrak{v}$.

The ring $\mathfrak{v}$ is integrally closed in R (Chevalley)
The ring $\mathfrak{v}$ (of $\mathfrak{y}$) contains an element 't' such that $\mathfrak{y} = t\mathfrak{v}$ and $\bigcap_n t^n \mathfrak{v} = \{0\}$

If $x \in \mathfrak{v}$ there is, (by assumptions in Chevalley) a largest number n such that $x = t^n \theta$, $\theta \in \mathfrak{O}$; denote by $v_{\mathfrak{y}}(x)$ this integer. (θ is a unit)

If x and y are elements $\neq 0$ in $\mathcal{O}$ then

$$\nu_{\mathfrak{y}}(x) + \nu_{\mathfrak{y}}(y) = \nu_{\mathfrak{y}}(xy)$$

and, if $x + y \neq 0$

$$\nu_{\mathfrak{y}}(x+y) \geq \min\left(\nu_{\mathfrak{y}}(x), \nu_{\mathfrak{y}}(y)\right)$$

As consequences we have

$$\nu_{\mathfrak{y}}(xy) = \nu_{\mathfrak{y}}(x) + \nu_{\mathfrak{y}}(y) \qquad x, y \in K$$

$$\nu_{\mathfrak{y}}(x^{-1}) = -\nu_{\mathfrak{y}}(x)$$

$$0 = \nu_{\mathfrak{y}}(1)$$

$\nu_{\mathfrak{y}}(0) = \infty$ where ∞ is defined by $\infty > n$ for every n an integer $\infty \geq \infty$, $\infty + n = \infty$ for every n, an integer, $\infty + \infty = \infty$; further whenever

$$\nu_{\mathfrak{y}}(x) \neq \nu_{\mathfrak{y}}(y)$$
$$\nu_{\mathfrak{y}}(x+y) = \min\left(\nu_{\mathfrak{y}}(x), \nu_{\mathfrak{y}}(y)\right)$$

Actually $\nu_{\mathfrak{y}}$ satisfies the more general properties of a valuation and $\mathfrak{y}$ is called a place. A formal multiplication is defined between the places and it gives rise to the divisors. The places themselves are called the prime divisors under this multiplication. Because $K = k(x) = k(\frac{1}{x})$, $\frac{1}{x}$ is also a place.

K admits infinitely many places. The prime ideals in K can be identified with a subset of these places. Because the nonzero ideals in K form a group, these can be imbedded isomorphically in the group of all divisors.

<u>Definition of a valuation</u>: A valuation of the field K is a mapping of K^* of the nonzero elements of K on to an ordered multiplicative group W (Generally a subgroup of the real numbers) satisfying the following conditions:

(1) For $a, b \in K^*$, $v(ab) = v(a)\, v(b)$

(2) For $a, b \in K^*$, $a+b \in K^*$

$$v(a+b) \leq \max\,(v(a), v(b))$$

$$\text{or}\quad v(a+b) \geq \min\,(v(a), v(b))$$

(3) v is nontrivial; that is, there exists an $a \in K^*$ with $v(a) \neq 0$

$v_{\mathcal{X}}$ satisfies the properties of a valuation.

If all the places in $K = k(x)$ (defined by the prime polynomials f in increasing order of the degree of f and $1/x$) are taken suitably ordered the set of elements

$\{v_f(a)\}$ and $v_{\frac{1}{x}}(a)$ when $a \in K^*$ define an Idèle of the elements.

Starting with a set of values for all the f and $1/x$ arranged as above we can make these correspond to a ' p -adic' number in the rational number field.

2. In 1924, Artin, in his thesis, generalized the arithmetic of quadratic extensions of the rational number field to the quadratic extensions of the rational function field in one variable over a finite field, (taken as a prime field) hereafter referred to as the function field. Artin axiomatised the theory further by using the valuations; more generalizations were carried out

by students of Artin leading to important developments in Algebra and Algebraic geometry. Witt, in a paper published in 1937, generalized the arithmetic of quadratic forms to arbitrary fields.

In Artin's thesis $K_{1/x}$ is represented by power series of the form

$$a_n x^n + a_{n-1} x^{n-1} + \text{---} + a_0 + a_{-1} x^{-1} + \text{---}$$

a_i belong to the prime field k .
These can be made to correspond by means of the Idéales to power series of the form.

$$a_n p^n + a_{n-1} p^{n-1} + \text{---} + a_0 + a_{-1} p^{-1} + \text{---}$$

in the Euclidean space. Artin called

$$a_n x^n + \text{----} + a_0$$

the integral part and $a_{-1}x^{-1} + \text{-- ---}$ the fractional parts of the elements.
The number

$$a_n p^n + a_{n-1} p^{n-1} + \text{----}$$

can be imbedded in the Euclidean space to get the analogue of Dirichlet's lattice point principle for the field $K_{1/x}$. This is explained in paragraph 4 Chapter I.

Chapter I is devoted to a discussion of the reduction theory of quadratic forms over the field of rational functions in one variable over a finite field. Some of the known theorems are quoted here or given in alternative references because the original papers are not readily available to the Indian students and the author in particular.

The theory is based on a theorem of Tsen that a form $\mathfrak{S}[x]$ in $K_{1/x}$ in five variables represents zero nontrivially where $\mathfrak{S}[x]$ is the quadratic form of a symmetric matrix $\mathfrak{S}$ and $K_{1/x}$ is the completion at the valuation in $1/x$ of the field K, of rational functions in one variable over ~~over~~ the finite prime field k. A form $\mathfrak{S}[x]$ in K or $K_{1/x}$ is said to be definite if it does not represent zero nontrivially in $K_{1/x}$. Otherwise it is said to be indefinite.

The axiomatic characterization given by Dedekind enables us to consider some results from Hasse, Witt and Siegel as known results with the prime numbers in the rational number field replaced by the prime polynomials in K, the integers by the more general polynomials and the rational numbers by the rational functions in K. Accordingly $\mathfrak{p}$-adic numbers and real numbers have their generalizations.

3. The main purpose of this thesis is to establish the analogue of Siegel's famous identity on the representation theory of quadratic forms over the rational number field Γ to quadratic forms over K. The notions of equivalence are primary in the statement of the main theorem of Siegel. Equivalence of symmetric matrices gives rise to the reduction theory and semiequivalence to the results on the genera. Let $\mathfrak{S}[x]$, $\mathfrak{T}[y]$ be two quadratic forms in the variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ with coefficients in one of the fields mentioned above. A quadratic form $\mathfrak{S}[x] = \sum_{i,j=1}^{m} A_{ij} x_i x_j$ is said to be equivalent to a form $\mathfrak{T}[y] = \sum_{i,j=1}^{m} B_{ij} y_i y_j$ if the transformation $x_k = \sum_{l=1}^{m} h_{kl} y_l$, $k = 1, \ldots, m$ transforms

$\gamma[x]$ into $\eta[Y]$ and if a similar transformation takes $\eta[Y]$ into $\gamma[x]$ where $A_{ij} \in k[x]$ and $B_{ij} \in k[x]$. Let Um denote the group of unimodular matrices from $k[x]$ of order m (integral matrices with determinant a unit). Let γ be the given symmetric matrix of order m with elements in $k[x]$ and $U \in U_m$. For $U \in U_m$, $U'\gamma U$ is said to be equivalent to γ. It is an equivalence relation. Equivalent forms take the same values if x_i, y_i take values in $k[x]$. If U is such that $U'\gamma U = \gamma$, U is called a unit. Units form a group.

$\gamma[x]$ and $\eta[Y]$ are said to be semiequivalent if for every polynomial h there exists a linear transformation $X_k = \sum_{l=1}^{m} h_{kl} Y_l$, which takes $\gamma[x]$ to $\eta[Y]$ such that denominations of h_{kl} are prime to h and a similar transformation which takes $\eta[Y]$ to $\gamma[x]$ in the same way.

Given the value of the determinant in the set of symmetric matrices in $\mathcal{R}$, the reduced space, with this value for the determinant there are at least two equivalent classes. This fact is used in the construction of the fundamental space for the discontinuous group of mappings $R \to U'R\,U$ where U is the unit of the symmetric matrix γ, $R \in \mathcal{R}$ and the mapping is into a subspace of the space of symmetric matrices with a given determinant equal to that of γ in value with respect to $1/x$. This is done in Chapter III.

Chapter II consists mainly of the proof of the main theorem for definite forms over K. Results of Artin [3]* using quadratic extensions of K and their arithmetic are exhibited as special cases of these results. In fact results of Artin [3]* from his thesis are used initially in the induction part of the proof of the main theorem.

* pp 153-203 pp 227

These results cannot be considered as particular cases of results on algebraic function fields over k because the evaluations of certain quantities connected with the main theorem of Siegel are made strictly for K. Also it is not proper to consider the algebraic function fields, as far as this identity is concerned, as generalizations of K or the algebraic number fields. These can be dealt on their own right though one has to borrow the ideas from the techniques given here to give the more general results. Besides the $\mathfrak{f}$-adic densities must also be defined as measures of representation for algebraic function fields. This is not difficult because the reduction theory in Chapter I and the results on the units in Chapter III can be carried out for discretely valued and complete fields over a finite field using the power series representations. But the explicit evaluation of $A_0(\mathfrak{S}, \mathfrak{T})$ could be more involved though it might be simplified using the Dirichlet lattice point principle. The results for the indefinite forms are given in detail in Chapter - III.

4. The equivalence and semiequivalence of symmetric matrices have already been defined; it follows that if a matrix belongs to a genus the whole class of the matrix belongs to the genus so that there are only a finite number of genera of matrices with a given determinant and in each genus a finite number of classes. Let $\mathfrak{S}_1, \ldots, \mathfrak{S}_h$ be the representants of the classes in a genus; let $A(\mathfrak{S}_i, \mathfrak{T})$ be defined as the number of integral $\mathfrak{X}^{(m,n)}$ such that $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{T}$ when $\mathfrak{S}$ and $\mathfrak{T}$ are integral and definite.

Put $$\sum \frac{A(\mathfrak{S}_i, \mathfrak{T})}{E(\mathfrak{S}_i)} \Big/ \sum \frac{1}{E(\mathfrak{S}_i)} = \bar{A}(\mathfrak{S}, \mathfrak{T}).$$

Notice $A(\gamma_i, \mathfrak{z})$ is the same as $E(\gamma_i)$ when γ_i is equivalent to $\mathfrak{z}$; otherwise zero.* Let $A_{f^r}(\gamma, \mathfrak{z})$ be the number of solutions of $\mathfrak{X}'\gamma\mathfrak{X} \equiv \mathfrak{z} \pmod{f^r}$ for f irreducable and r an integer from the rational number field.

$$\lim_{r \to \infty} \frac{A_{f^r}(\gamma, \mathfrak{z})}{|f|^{r\left(mn - \frac{n(n+1)}{2}\right)}}$$

exists and is denoted by $\alpha_f(\gamma, \mathfrak{z})$

Then following the statement

$$\frac{\bar{A}(\gamma, \mathfrak{z})}{A_0(\gamma, \mathfrak{z})} = \rho(\gamma)\ \epsilon_{mn} \prod_f \alpha_f(\gamma, \mathfrak{z}) \qquad (1)$$

it is proved that the right hand side converges. The proof is the same as it was done by Siegel [5] for the rational number field. It is included here for the sake of completeness.

Also the arithmetical part of the results can be obtained as a generalization of the results of Siegel for the indefinite forms.

$A_0(\gamma, \mathfrak{z})$ is a quantity depending on the values of the determinants of γ and $\mathfrak{z}$ and the orders of γ and $\mathfrak{z}$. Siegel defined it for the rational number

* If $\gamma_i \sim \mathfrak{z}$ i.e. γ_i and $\mathfrak{z}$ are in the same genus.

field in the following fashion. $\mathfrak{S}$ is represented as a point in the space $\Gamma^{\frac{m(m+1)}{2}}$, $m(m+1)/2$ dimensional product space of Γ and $\mathfrak{T}$ as a point in the space $\Gamma^{\frac{m(m+1)}{2}}$. For a certain neighbourhood of $\mathfrak{T}$ with the ordinary distance metric the point $\mathfrak{X}$, satisfying the equation $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{T}$ is represented as a point in the mn dimensional space and $\mathfrak{T}_1$ is taken as a point in the neighbourhood of $\mathfrak{T}$. The volume of the $\mathfrak{X}$ space traced when $\mathfrak{T}_1$, traces the neighbourhood of $\mathfrak{T}_1$ divided by the volume of the $\mathfrak{T}$ neighbourhood tends to a finite limit when the '$\mathfrak{T}$' neighbourhood shrinks to $\mathfrak{T}$. As it would be expected Siegel used integration as the available tool in the rational number field.

<u>Dirichlet's lattice point principle with special reference to Siegel's theory</u>.

In Chapter II of this thesis a set of Lemmas due to Siegel are generalized to $k(x)$, the field of rational functions over the prime field k.

Let $\mathfrak{S}$ and $\mathfrak{T}$ both lie in K, K_f or $K_{1/x}$. For n=m the equation $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{T}$ defines an irreducible manifold of dimension $mn - \frac{n(n+1)}{2} = \nu$. For n = m by the adjunction of $(|\mathfrak{T}|/|\mathfrak{S}|)^{1/2}$ to the corresponding field an extension field, is obtained, on which there exist exactly two different algebraic manifolds of dimension ν on one of which $|\mathfrak{x}| = \rho$ and on the other $|\mathfrak{x}| = -\rho$, if -1 is a quadratic residue modulo p these two manifolds can be identified.

In virtue of this lemma for the solutions of the equation $\mathfrak{x}'\mathfrak{S}\mathfrak{x} = \mathfrak{T}$ Dirichlet's lattice point principle can be applied which justifies the following evaluation of $A_0(\mathfrak{S},\mathfrak{T})$ for function fields.

Dirichlet's lattice point principle for the rational number field states that the number of lattice points or points with integral coordinates in a convex bounded domain of the Euclidean space tends to the volume of the domain as its boundary tends to infinity.

In the case of the function fields $A_0(\mathfrak{S},\mathfrak{T})$ is defined thus: the above process applied to $A(\mathfrak{S},\mathfrak{T})$ is applied to the average $\Sigma A(\mathfrak{S}_i,\mathfrak{T})/h$. Finally it is proved that it is independent of the genus; with the result the definition can be taken to be the same as for the rational number field. Besides one makes note of the fact that the ratio of the two volumes, defined above, in the limit when the neighbourhood expands is the ratio of the number of lattice points in one to that in the other (Dirichlet's Principle) After making note of all these observations $A_0(\mathfrak{S},\mathfrak{T})$ is defined thus. Take the average $\Sigma A(\mathfrak{S}_i,\mathfrak{T})/h$

in a neighbourhood of $\mathfrak{F}$ in the valuation with respect to π .
Define $P(\mathfrak{S},\mathfrak{F}) = \Sigma A(\mathfrak{S}_i,\mathfrak{F})/h$
and take all the elements (when n = 1) such that

$$p^{-M} \leq |\mathfrak{F}'-\mathfrak{F}| \leq p^{-N} \quad \text{for } N$$

and M sufficiently large . $\Sigma P(\mathfrak{S},\mathfrak{F})$ / number of $\mathfrak{F}'$
as $N \to \infty$ is the definition of $A_0(\mathfrak{S},\mathfrak{F})$
When $n > 1$ the inequalities are taken for each of the elements of $\mathfrak{F}$ with the corresponding elements of $\mathfrak{F}'$.

$\mathfrak{F}^{(n)}$ is represented as a point in the $\frac{n(n+1)}{2}$ dimensional space over the completion of K at π .
Consider the equation $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{F}$. For points in the neighbourhood of $\mathfrak{F}$, denoted by $\mathfrak{D}$, $\mathfrak{X}$ is one of a set of points $\mathfrak{D}'$. Instead of taking all points in the neighbourhood of $\mathfrak{F}$ we take $\mathfrak{G}$ with elements g_{ij} which are polynomials satisfying the conditions,

$$p^{-M} \leq |g_{ij} - f_{ij}| \leq p^{-N}$$

The remaining steps of the analytical part are once again the same as those in Siegel [5,6] and a few more observations are made in the introduction of Chapter - III.

Quite apart from the algebraic part of the proof, in order to formulate the theorem for the indefinite forms one needs a measure for the unit group because the units are no more finite in the case of the indefinite forms. While securing the measure one has the result that the units of an indefinite symmetric matrix over K are finitely generated. Also the notion of measure of representation is introduced. This is a hurdle of specific importance in the whole work though the results that one obtains thereafter are not apparently different from those of Siegel [6] if one makes use of the preparation in Chapter I and II. Chapter II can be taken, as far as these results are concerned, to be a particular case of Chapter - III. This is exhibited explicitly in Chapter III.

The equation (1) with the nature of $\rho(\mathfrak{S})$ determined (proved equal to one in the rational number field) is the main theorem of Siegel on the representation theory of quadratic forms. The $\alpha_p(\mathfrak{S},\mathfrak{T})$ are the p-adic densities of the representations of $\mathfrak{T}$ by $\mathfrak{S}$.

Preceding the proof of the convergence of the right hand side of (1) certain lemmas proved by Siegel are summarized here with an introduction to the methods of Siegel.

5. Siegel [5] contains a collection of lemmas (apart from the other preliminaries to his papers) where actually all the important notions, as far as the arithmetic is concerned, are included. These lemmas can be classified into three sets - one leading to the proof of the convergence of the p-adic densities, the second leading to the formulae of Gauss and Eisenstein and the third leading to the induction part of the proof of the main theorem with the more intricate methods for the estimation of $\rho(\mathfrak{S})$. A short discussion of the lemmas leading to the convergence of the product of the p-adic densities is given in Chapter - II. The set of important lemmas leading to the proof of the formula of Gauss and Eisenstein has not been dealt with

in detail. After writing the equation

$$\mathfrak{w}'\mathfrak{S}\mathfrak{w} \equiv |\mathfrak{T}|^{-1}\mathfrak{z} + \mathfrak{y}'\mathfrak{T}^{-1}\mathfrak{y} \pmod{\tfrac{f}{|\mathfrak{T}|}}$$

by lemma 24, Siegel [5]; the existence of an $\mathfrak{S}_1$ semiequivalent to $\mathfrak{S}$ with $\mathfrak{S}_1 \equiv \mathfrak{S} \pmod{f/|\mathfrak{T}|}$ and of an integral $\mathfrak{C}$ such that $\mathfrak{C}_1 \equiv \mathfrak{C} \pmod{f/|\mathfrak{T}|}$ so that $\mathfrak{C}_1'\mathfrak{S}\mathfrak{C}_1 = \mathfrak{T}$

have to be assumed to proceed with the rest of the proof (Siegel [5] equations, 50, 51, 52). The construction of the reduced $\mathfrak{z}$ and $\mathfrak{y}$ and also of $\mathfrak{w}$ is the important step that precedes the above argument (art 8, equations 45, 46 Siegel [5]). The equations 47, 48 and 49 Siegel [5] and subsequently upto 52 make use of the construction of the reduced $\mathfrak{z}$ and $\mathfrak{y}$. (Lemmas 20,21 Siegel [5]) are applied repeatedly. Lemma 22 Siegel [5] is the Hasse - Witt theorem for function fields.

One can see from the proofs that each of these lemmas 20,21 and 24 Siegel [5] is an improvement of the previous. Lemmas 16 and 19, Siegel [5] arent needed here. We can have now a systematic discussion of (1) the seventeen lemmas generalized from Siegel [5] , (2) the three lemmas mentioned above and (3), the construction of the reduced $\mathfrak{z}$ and $\mathfrak{y}$ which has to be dealt with in detail once again in view of the modifications in Chapter III for indefinite forms. The improvements on (3) for indefinite forms can be carried out in two ways. I call the one the geometrical and the other the arithmetical approach. Either of these is needed for further improvements in the work. The geometrical approach is direct generalization of Siegel [6] and the arithmetical is just the procedure in Siegel [5] with a suitable restriction on the degrees of the elements in the matrices $\mathfrak{C}, \mathfrak{y}, \mathfrak{b}, \mathfrak{z}$ and $\mathfrak{w}$. The two methods are possible once again because of the discreteness of the valuation.

In detail

(3) <u>Construction of the reduced $\mathfrak{z}$ and $\mathfrak{y}$ for definite and indefinite forms</u>.

This construction is carried out on page following Siegel [5]. Let $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ be a particular primitive representation in $k[x]$. If $\mathfrak{w}_0$ is a complement of $\mathfrak{C}$, $(\mathfrak{C}\ \mathfrak{w}_0) = \mathfrak{b}_0$

$$\mathfrak{y}_0 = \mathfrak{C}'\mathfrak{S}\mathfrak{w}_0 \qquad \mathfrak{h}_0 = \begin{pmatrix} \mathfrak{T} & \mathfrak{y}_0 \\ \mathfrak{n} & \mathfrak{E} \end{pmatrix}$$

Then $|\mathfrak{z}_0| = |\mathfrak{S}||\mathfrak{T}|^{m-n-1}$ and

$$\mathfrak{b}_0'\mathfrak{S}\mathfrak{b}_0 = \mathfrak{h}_0' \begin{pmatrix} \mathfrak{T}^{-1} & \mathfrak{n}' \\ \mathfrak{n} & |\mathfrak{T}|^{-1}\mathfrak{z}_0 \end{pmatrix} \mathfrak{h}_0$$

$$= \begin{pmatrix} \mathfrak{T} & \mathfrak{y}_0 \\ \mathfrak{y}_0' & |\mathfrak{T}|^{-1}\mathfrak{z}_0 + \mathfrak{y}_0'\mathfrak{T}^{-1}\mathfrak{y}_0 \end{pmatrix}$$

For any general complement $\mathfrak{w} = \mathfrak{C}\,\mathfrak{F} + \mathfrak{w}_0\mathfrak{W}$

with integral $\mathfrak{F}$ and unimodular $\mathfrak{W}$ the following equations are true.

$$\mathfrak{b} = \mathfrak{b}_0 \begin{pmatrix} \mathfrak{E} & \mathfrak{F} \\ \mathfrak{n} & \mathfrak{W} \end{pmatrix} \qquad \mathfrak{y} = \mathfrak{T}\mathfrak{F} + \mathfrak{y}_0\mathfrak{W}$$

$$\mathfrak{z} = \mathfrak{W}'\mathfrak{z}_0\mathfrak{W}$$

Given $\mathfrak{C}$ and $\mathfrak{A}_0$, $\mathfrak{h}_0$ is fixed uniquely and $\mathfrak{h}$ is in the same class as $\mathfrak{h}_0$. That is, the class of $\mathfrak{h}$ is uniquely fixed and $\mathfrak{W}$ is determined in E ($\mathfrak{h}$) ways. in the case of indefinite forms instead of E ($\mathfrak{h}$) we have to use the measure of the unit group, Also for the number of primitive representations $\mathfrak{C}$ the notion of measure of representation has to be used. These have been introduced in Chapter - III. At this stage the method in Siegel [6] can be compared with that in [5] to establish the formula of Gauss and Eisenstein in the large. Let $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ be a representation of $\mathfrak{T}$ by $\mathfrak{S}$. To a representation $\mathfrak{C}'\mathfrak{S}\mathfrak{C} = \mathfrak{T}$ of $\mathfrak{T}$ by $\mathfrak{S}$ let $\mathfrak{U}$ be a unit of $\mathfrak{S}$ such that $\mathfrak{U}\mathfrak{C} = \mathfrak{C}$. Let $\mathfrak{X}_0$ m (m-n) be a matrix in $K_{1/x}$ such that ($\mathfrak{C}\,\mathfrak{X}_0$) has a determinant different from zero. Then put

$$(\mathfrak{C}\,\mathfrak{X}_0)'\,\mathfrak{S}\,(\mathfrak{C}\,\mathfrak{X}_0) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Y}_0 \\ \mathfrak{Y}_0' & \mathfrak{R}_0 \end{pmatrix}$$ Lemma 10, Siegel [6]

so that $\mathfrak{C}'\,\mathfrak{S}\,\mathfrak{X}_0 = \mathfrak{Y}_0$, $\mathfrak{X}_0'\,\mathfrak{S}\,\mathfrak{X}_0 = \mathfrak{R}_0$ (2)

Then it shall be shown that

$$(\mathfrak{C}\,\mathfrak{X})'\,\mathfrak{S}\,(\mathfrak{C}\,\mathfrak{X}) = \begin{pmatrix} \mathfrak{T} & \mathfrak{Y} \\ \mathfrak{Y}' & \mathfrak{R} \end{pmatrix}$$ Eqn (48) page 246 Siegel [6]

possesses a solution $\mathfrak{X}$ in $K_{1/p}$ and $\mathfrak{R} = \mathfrak{R}'$ lie sufficiently neat to $\mathfrak{Y}_0$ and $\mathfrak{R}_0$. Here lemma 2, Chapter - II is applied.

In order to solve the equations $\mathfrak{C}'\mathfrak{S}\mathfrak{X} = \mathfrak{Y}$ $\mathfrak{X}'\mathfrak{S}\mathfrak{X} = \mathfrak{R}$, put $\mathfrak{X} = \mathfrak{C}\mathfrak{F} + \mathfrak{X}_0\mathfrak{W}$ with unknown $\mathfrak{F}^{(n,m-n)}$ and $\mathfrak{W}^{(m-n)}$. with the abbreviations

$$\mathfrak{R} - \mathfrak{Y}_0'\mathfrak{T}^{-1}\mathfrak{Y}_0 = \mathfrak{H}_0 \quad \text{and} \qquad (3)$$
$$\mathfrak{R} - \mathfrak{Y}'\mathfrak{T}^{-1}\mathfrak{Y} = \mathfrak{H}$$

$$\begin{pmatrix} \mathfrak{T} & \mathfrak{Y}_0 \\ \mathfrak{Y}_0' & \mathfrak{R}_0 \end{pmatrix} = \begin{pmatrix} \mathfrak{T} & \mathfrak{N} \\ \mathfrak{Y}_0' & \mathfrak{E} \end{pmatrix} \begin{pmatrix} \mathfrak{T}^{-1} & \mathfrak{N} \\ \mathfrak{N} & \mathfrak{H}_0 \end{pmatrix} \begin{pmatrix} \mathfrak{T} & \mathfrak{Y}_0 \\ \mathfrak{N} & \mathfrak{E} \end{pmatrix} \quad \text{Eqn 53, Siegel [6]}$$

then $|\mathfrak{H}_0| \neq 0$ and we have further the equations

$\mathfrak{T}\mathfrak{F} + \mathfrak{Y}_0\mathfrak{W} = \mathfrak{Y}$ and $\mathfrak{W}'\mathfrak{H}_0\mathfrak{W} = \mathfrak{H}$

In this situation $\mathfrak{H}$ must be sufficiently near to $\mathfrak{H}_0$. In the (m-n) (m-n+1)/2 dimensional space of pairs that set of points, for which(4) is soluble is chosen. By means of (4) this space B is mapped to B' (the m (m-n) dimensional) of the $\mathfrak{X}$ space. Any two points $\mathfrak{X}_1, \mathfrak{X}_2$ of the $\mathfrak{X}$ space are called associated if for a certain unit U of the equation $\mathfrak{X}_2 = U\mathfrak{X}_1$ is true. If B is the reduced space of $\mathfrak{X}$ in B' for this equivalence relation volume of B exists and is different from zero. Also for a certain neighbourhood B of $\mathfrak{Y}, \mathfrak{R}$ (Lemma 13, Siegel [6]) $\dfrac{\text{Volume of B}'}{\text{Volume of B}}$ in the limit when B tends to $\mathfrak{Y}, \mathfrak{R}$ is the same as $\bar{f}(\mathfrak{C}, \mathfrak{S})$ if $\mathfrak{C}$ is a primitive representation and ($\mathfrak{C}\ \mathfrak{X}_0$) is unimodular.

The construction of the measure $\bar{\rho}(\mathfrak{L},\mathfrak{T})$, its existence and the interrelation with the reduced $\mathfrak{z}$ and $\mathfrak{y}$ are given by lemmas 11 and 12, Siegel [6]. These reduced $\mathfrak{z}$ and $\mathfrak{y}$ are defined just as for definite forms. Refer back to the equations (2) and (3). $\mathfrak{z}$ is called reduced once its class ($\mathfrak{z}$) is fixed. Of the possibilities for $\mathfrak{W}$ (which can be measured by $1/\mu^{-1}(\mathfrak{z})$) one is chosen and to fix $\mathfrak{F}$ in $\mathfrak{x} = \mathfrak{L}\mathfrak{F} + \mathfrak{x}_0\mathfrak{W}$, $\mathfrak{y}$ is determined uniquely so that

$$\mathfrak{y} = \mathfrak{T}\mathfrak{F} + \mathfrak{y}_0\mathfrak{W}$$

is a given representant ($\mathfrak{y}$) of its left residue class modulo $\mathfrak{T}$ Here $\mathfrak{y}$ is reduced. If $\mathfrak{z}$ and $\mathfrak{y}$ are both reduced $\mathfrak{x}$ is also called reduced. To call $\mathfrak{z}$ actually reduced for indefinite forms it must be chosen from a certain reduced space. There spaces are dealt with in Chapter - III. The quantities $B(\mathfrak{z}), B(\mathfrak{T},\mathfrak{T}), C(\mathfrak{z},\mathfrak{T})$ and $E(\mathfrak{z})$ have their corresponding generalizations,

$$\mu_B(\mathfrak{z}) \quad , \mu_B(\mathfrak{T},\mathfrak{T}) \; , \mu_C(\mathfrak{z},\mathfrak{T}) \text{ and } 1/\mu^{-1}(\mathfrak{z})$$

formula of Gauss and Eisenstein, with all this preparation, is still not immediate for indefinite forms. The rest of the explanation is to be found in Chapter - III. It is here the two methods of explanations are used, which we call the geometrical and algebraic approaches. In the algebraic approach a restriction is placed on the degrees of the terms of $\mathfrak{x}$, $\mathfrak{z}_0$, $\mathfrak{y}$, $\mathfrak{y}_0$ and $\mathfrak{R}_0$ along with $\mathfrak{L}$ and in the limit $\mu_B(\mathfrak{z})$ $\mu_B(\mathfrak{T},\mathfrak{T})$, $\mu_C(\mathfrak{z},\mathfrak{T})$ and $1/\mu^{-1}(\mathfrak{z})$ appear automatically. But still it is not complete without a further explanation.

Now all the hurdles have been crossed. The theory is complete with a last reference to the thesis of Artin - (for the time being). The constant $\rho(\tau)$ can be evaluated for a binary forms and the identity is completely established for binary forms.

$\rho(\tau)$ is proved to be a constant even for the more general case. The proof is a bit involved, still it is incorporated.