Chapter 3

Non-linear Ćirić Contractions via C_F-Simulation Functions

3.1 Introduction

Various generalizations of contraction mappings have been proposed in the literature, each introducing a broader class of mappings with contraction-like properties. The importance of these generalizations lies in their ability to extend the applicability of fixed point theorems and provide tools for studying fixed points in a wider range of spaces and under more relaxed conditions than traditional contractions. They find applications in diverse areas, including functional analysis, dynamic systems, optimization, and solving equations.

A quasi-contraction map by Ćirić [14] is a type of mapping that exhibits a contraction-like property, although it may not strictly satisfy the conditions of a contraction mapping. Specifically, a self mapping f on a metric space (X, d), is said to be a quasi-contraction if there exists a nonnegative number $q < 1$ such that

$$d(fx, fy) \leq q \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}, \text{ for all } x, y \in X.$$

The Ćirić fixed point theorem is given by the following.

Theorem B. *[14] Let (X, d) be a metric space and $f : X \to X$ be a quasi-contraction mapping with the contractive constant $q < 1$. Then f has a unique fixed point. Moreover, the sequence $\{x_n\}$ in X, which is defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ such that $x_0 \in X$ is an initial point, converges to a fixed point of f.*

Afterwards, the Banach contraction principle and many available results in the literature were extended by replacing the contractive conditions, using some control functions. In this direction, Khojasteh et al. [38] introduced the notion of simulation functions, $\mathcal{Z}$-contractions and presented fixed point theorems for such contractions in complete metric spaces. Later, Roldan et al. [18] modified the notion of simulation functions [38] by removing the symmetry of variables and proved coincidence and common fixed point results.

Further, Roldan et al. [17] investigated the existence and uniqueness of coincidence points via simulation functions in the setting of quasi-metric spaces and deduced corresponding results in the framework of G-metric spaces. Liu et al. [41] extended the class of simulation functions by using C-class functions of Ansari [6] and introduced C_F-simulation functions which reasonably enlarge the collection obtained by Khojasteh et al. [38]. Further, they proved existence and uniqueness of coincidence and common fixed point for two operators.

Definition 3.1.1. [41, p.1104] A function $F : [0, \infty)^2 \to \mathbb{R}$ has the property $\mathcal{C}_F$, if there exists $C_F \geq 0$ such that

$(\tilde{F_1})$ $F(s, t) > C_F$ implies $s > t$, for all $s, t \geq 0$;

$(\tilde{F_2})$ $F(t, t) \leq C_F$, for all $t \geq 0$.

Definition 3.1.2. [41, p.1105] A C_F-simulation function is a function $\zeta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(i) $\zeta(t, s) < F(s, t)$ for all $t, s > 0$, where $F \in \mathcal{C}$ with property $\mathcal{C}_F$;

(ii) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_F$.

The family of all C_F-simulation functions is denoted by $\mathcal{Z}_F$.

On the other hand, Samet et al. [55] introduced the notion of admissible mappings, $\alpha - \psi$ contractive type mappings and extended existing fixed point results in the literature. Shahi et al. [56] generalized this concept for pair of mappings and proved coincidence and common fixed point result in metric spaces.

Definition 3.1.3. [56, p.302] Let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is α-admissible for g if

$$\alpha(gx, gy) \geq 1 \implies \alpha(Tx, Ty) \geq 1, \text{ for all } x, y \in X.$$

For $g = i_X$ (identity mapping on X), T is an α-admissible mapping.

Definition 3.1.4. [49, p.75] Let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is triangular α-admissible for g if T is α-admissible for g and

$$\alpha(gx, gy) \geq 1 \text{ and } \alpha(gy, gz) \geq 1 \implies \alpha(gx, gz) \geq 1, \text{ for all } x, y, z \in X.$$

This Chapter consists of 3 sections. First section contains preliminaries. In the second section, Ćirić type contraction via simulation functions and α-admissible mappings is introduced. Further, we investigate sufficient conditions for the existence and uniqueness of coincidence point and common fixed point for such contraction in quasi-metric spaces. The obtained results give solution to the open problem posed by Radenovic and Chandok [50]. In the third section, Ćirić type $\mathcal{Z}_F$-contraction using C_F-simulation functions is introduced and proved coincidence and common fixed point results for such contractions in quasi-metric spaces. Finally, its consequences to G-metric spaces are discussed.

3.2 Preliminaries

Here, we take account of some basic definitions and results that are prerequisites for this chapter.

Definition 3.2.1. [29, p.2] Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.$$

The limit of a sequence in quasi-metric space is unique.

As d is not necessarily symmetric, authors defined left convergent, right convergent, left Cauchy, right Cauchy sequences and completeness as follows.

Definition 3.2.2. [29, p.2] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

- *left-Cauchy* if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for all $n \geq m > N$.

- *right-Cauchy* if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for all $m \geq n > N$.

A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

The following lemma is sufficient condition to prove Cauchyness of the given sequence.

Lemma 3.2.3. *[30, p.3] Let $\{x_n\}$ be a sequence in a quasi-metric space (X, d) such that*

(i) $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), n \geq 0,$

(ii) $d(x_{n+2}, x_{n+1}) \leq \lambda d(x_{n+1}, x_n), n \geq 0,$

for some $\lambda \in (0, 1)$. Then $\{x_n\}$ is a Cauchy sequence in X.

Definition 3.2.4. [29, p.2] Let (X, d) be a quasi-metric space. We say that (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Roldan et al. [20] introduced precompleteness for metric spaces which is weaker than completeness of the space.

Definition 3.2.5. [20, p.7] A subset E of a metric space (X, d) is said to be *precomplete* if every Cauchy sequence in E converges to a point of X.

Remark 1. (1) The empty subset is precomplete.

(2) Every complete subset of X is precomplete.

(3) Every subset of a complete metric space is also precomplete.

Example 3.2.1. Although $X = (0, 3)$, endowed with the Euclidean metric, is not complete, and $A = (1, 2)$ is not complete, the set A is precomplete.

Proposition 3.2.6. *[20, Prop.23, p.7] If $A \subseteq B \subseteq X$ and B is precomplete, then A is also precomplete.*

Remark 2. If $T(X) \subseteq g(X)$ and one of X or $T(X)$ or $g(X)$ is complete, then $T(X)$ is precomplete.

Definition 3.2.7. [17, p.3] Let (X, d) be a quasi-metric space and $T : X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then for each sequence $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(Tu, Tx_n) = 0.$$

Now, T is continuous if it is continuous at every point of X.

Roldan et al. [17] defined compatible mappings for quasi-metric spaces as follows.

Definition 3.2.8. [17, p.4] Let $T, g : X \to X$ be mappings on a quasi-metric space (X, d). We say that T and g are compatible if and only if

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0 \text{ or } \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0,$$

for all sequences $\{x_n\} \subseteq X$ such that the sequences $\{gx_n\}$ and $\{Tx_n\}$ are convergent and have the same limit.

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta : X \times X \to [0, \infty)$, defined by

$$\delta(x, y) = \max\{d(x, y), d(y, x)\}$$

is a metric on X (see [29]).

The following result follows from the above definition.

Theorem C. *[29, Theorem 2.3, p.3] Let (X, d) be a quasi-metric space. Let $\delta : X^2 \to [0, \infty)$ be the function defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$. Then*

(1) (X, δ) is a metric space;

(2) $\{x_n\} \subset X$ is convergent to x in (X, d) if and only if $\{x_n\}$ is convergent to x in (X, δ);

(3) $\{x_n\} \subset X$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, δ);

(4) (X, d) is complete if and only if (X, δ) is complete.

The following theorem shows the relationship between G-metrics and quasi-metrics.

Theorem D. *[29, Theeorem 2.2, p.3] Let* (X, G) *be a* $G-$*metric space and* $d_G \colon X^2 \to [0, \infty)$ *be the function defined by* $d_G(x, y) = G(x, y, y)$*. Then,*

(1) (X, d_G) *is a quasi-metric space;*

(2) $\{x_n\} \subset X$ *is G-convergent to* $x \in X$ *if and only if* $\{x_n\}$ *is convergent to* x *in* (X, d_G)*;*

(3) $\{x_n\} \subset X$ *is G-Cauchy if and only if* $\{x_n\}$ *is Cauchy in* (X, d_G)*;*

(4) (X, G) *is G-complete if and only if* (X, d_G) *is complete.*

3.3 Results for Ćirić type simulation functions using α-admissible mappings in quasi-metric spaces

This section deals with the common fixed point results related to α-admissible self mappings involving a Ćirić type contraction using C_F-simulation functions. We need the following result as a prerequisite.

Lemma 3.3.1. *Let* (X, d) *be a quasi-metric space and* S, T *are self mappings on* X*. Let* $\{x_n\}$ *be a Picard-Jungck sequence of* (S, T)*. If* S *is triangular* α*-admissible for* T *with* $\alpha(Tx_0, Sx_0) \geq 1$ *and* $\alpha(Sx_0, Tx_0) \geq 1$*, then* $\alpha(Tx_n, Tx_m) \geq 1$*,* $n \neq m$*.*

Proof. Let $\{x_n\}$ be a Picard sequence of (S, T) based at x_0, that is,

$$Sx_n = Tx_{n+1}, \text{ for all } n \geq 0.$$

Since S is α-admissible for T, we have

$$\alpha(Tx_0, Sx_0) = \alpha(Tx_0, Tx_1) \geq 1 \implies \alpha(Sx_0, Sx_1) = \alpha(Tx_1, Tx_2) \geq 1.$$

By induction, we get

$$\alpha(Tx_n, Tx_{n+1}) \geq 1, \text{ for all } n \geq 0.$$

Since S is triangular α-admissible for T, we have

$$\alpha(Tx_0, Tx_1) \geq 1 \text{ and } \alpha(Tx_1, Tx_2) \geq 1 \implies \alpha(Tx_0, Tx_2) \geq 1.$$

Continuing this way, we get

$$\alpha(Tx_n, Tx_m) \geq 1, \text{ for all } m > n.$$

Analogously, for $\alpha(Sx_0, Tx_0) \geq 1$, we get

$$\alpha(Tx_n, Tx_m) \geq 1, \text{ for all } m < n.$$

Hence $\alpha(Tx_n, Tx_m) \geq 1$, for $n \neq m$. □

Now, by using α-admissible mappings of Shahi et al. [56], $(\mathcal{Z}_{(\alpha,F)}, T)$-quasi-contraction of Ćirić type is introduced as follows.

Definition 3.3.2. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and S, T be self mappings on X. A mapping S is called a $(\mathcal{Z}_{(\alpha,F)}, T)$-quasi-contraction of Ćirić type if there exist $\zeta \in \mathcal{Z}_F, C_F \geq 0$ and $\lambda \in (0, 1)$ such that

$$\zeta\big(\alpha(Tx, Ty)d(Sx, Sy), \lambda M(Tx, Ty)\big) \geq C_F \tag{3.1}$$

for all $x, y \in X$, where

$$M(Tx, Ty) = \max\Big\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Tx, Sy), d(Ty, Sx)\Big\}.$$

Remark 3. (i) For $\alpha(x, y) = 1$, inequality (3.1) becomes a $(\mathcal{Z}_F, T)$ -quasi-contraction of Ćirić-Das-Naik type contraction [50].
(ii) For $\alpha(x, y) = 1$, $T = i_X$ and $C_F = 0$, we get a $\mathcal{Z}$-quasi-contraction of Ćirić type.
(iii) For $\alpha(x, y) = 1$ and $\zeta(t, s) < F(s, t) = s - t$, inequality (3.1) becomes a Das-Naik type quasi-contraction [16].

The following is the main result of this section.

Theorem 3.3.3. *Let (X, d) be a quasi-metric space, $S, T : X \to X$ be mappings with $S(X) \subset T(X)$. If S is a $(\mathcal{Z}_{(\alpha,F)}, T)$-quasi-contraction of Ćirić type satisfying the following conditions:*

(i) S is triangular α-admissible for T;

(ii) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$ and $\alpha(Sx_0, Tx_0) \geq 1$;

(iii) at least, one of the following conditions hold:

(a) $S(X)$ is precomplete in $T(X)$.

(b) (X,d) is a complete quasi-metric space, S and T are continuous and compatible.

Then, S and T have a point of coincidence.

Proof. For any $x_0 \in X$, since $S(X) \subset T(X)$, we get a sequence $\{x_n\}$ in X with $Sx_n = Tx_{n+1}$ for all $n \geq 0$. If $Tx_n = Tx_{n+1}$ for some n, then $Sx_n = Tx_n$, that is, x_n is a coincidence point of S and T. Thus, we assume that $d(Tx_{n+1}, Tx_n) > 0$ and $d(Tx_n, Tx_{n+1}) > 0$, for all $n \geq 0$.
In view of condition (i), by Lemma 3.3.1, we get

$$\alpha(Tx_n, Tx_m) \geq 1, \text{ for all } n \neq m.$$

Now,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(Sx_{n-1}, Sx_n) \\ &\leq \alpha(Tx_{n-1}, Tx_n) d(Sx_{n-1}, Sx_n). \end{aligned} \tag{3.2}$$

Since S is a $(\mathcal{Z}_{(\alpha,F)}, T)$-quasi-contraction of Ćirić type,

$$\begin{aligned} C_F &\leq \zeta(\alpha(Tx_{n-1}, Tx_n) d(Sx_{n-1}, Sx_n), \lambda M(Tx_{n-1}, Tx_n)) \\ &< F(\lambda M(Tx_{n-1}, Tx_n), \alpha(Tx_{n-1}, Tx_n) d(Sx_{n-1}, Sx_n)). \end{aligned}$$

Since $F \in \mathcal{C}$, by (F_1), we get

$$\alpha(Tx_{n-1}, Tx_n) d(Sx_{n-1}, Sx_n) \leq \lambda M(Tx_{n-1}, Tx_n), \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

From (3.2) and (3.3), we have

$$d(Tx_n, Tx_{n+1}) \leq \lambda M(Tx_{n-1}, Tx_n),$$

where

$$\begin{aligned} &M(Tx_{n-1}, Tx_n) \\ &= \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Sx_{n-1}), d(Tx_n, Sx_n), d(Tx_{n-1}, Sx_n), \end{aligned}$$

$$
\begin{aligned}
&d(Tx_n, Sx_{n-1})\} \\
&= \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_{n+1})\} \\
&\leq d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq \lambda(d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})), \\
d(Tx_n, Tx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n), \\
d(Tx_n, Tx_{n+1}) &\leq k d(Tx_{n-1}, Tx_n),
\end{aligned}
$$

where $k = \frac{\lambda}{1-\lambda} < 1$.
Similarly, we get

$$
d(Tx_{n+1}, Tx_n) \leq k d(Tx_n, Tx_{n-1}), \text{ for } k < 1.
$$

By Lemma 3.2.3, the sequence $\{Tx_n\}$ is a Cauchy sequence.
Now, consider independently cases (a)-(b) and prove that S and T have a coincidence point.

Case (a): Assume $S(X)$ is precomplete in $T(X)$. The precompleteness of $S(X)$ in $T(X)$ ensures the existence of some $v \in X$ with

$$
\lim_{n\to\infty} Tx_n = Tv = \lim_{n\to\infty} Sx_{n-1}. \tag{3.4}
$$

We claim that v is a coincidence point of S and T. On contrary, assume that $d(Tv, Sv) > 0$ and $d(Sv, Tv) > 0$.
We have

$$
\begin{aligned}
&\lim_{n\to\infty} M(Tx_n, Tv) \\
&= \lim_{n\to\infty} \max\{d(Tx_n, Tv), d(Tx_n, Sx_n), d(Tv, Sv), d(Tx_n, Sv), d(Tv, Sx_n)\} \\
&= d(Tv, Sv) > 0.
\end{aligned} \tag{3.5}
$$

Using (3.1), we get

$$
C_F \leq \zeta(\alpha(Tx_n, Tv) d(Sx_n, Sv), \lambda M(Tx_n, Tv))
$$

$$< F(\lambda M(Tx_n, Tv), \alpha(Tx_n, Tv)d(Sx_n, Sv)).$$

By $(\tilde{F_1})$, we have

$$\alpha(Tx_n, Tv)d(Sx_n, Sv) < \lambda M(Tx_n, Tv), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in above inequality and using (3.5), we get

$$\lim_{n\to\infty} \alpha(Tx_n, Tv)d(Tx_n, Sv) < \lambda d(Tv, Sv).$$

Hence, $d(Tv, Sv) < \lambda d(Tv, Sv)$, a contradiction. Therefore, $d(Tv, Sv) = 0$. So, v is a coincidence point of S and T.

Case (b): Assume that (X, d) is complete, S and T are continuous and compatible. In this case, the sequence $\{Tx_n\}$ is a Cauchy sequence in the complete quasi-metric space (X, d), hence there exists $u \in X$ such that $\lim_{n\to\infty} Tx_n = u$. That is,

$$\lim_{n\to\infty} d(Tx_n, u) = \lim_{n\to\infty} d(u, Tx_n) = 0.$$

Since $Sx_n = Tx_{n+1}$, for all $n \geq 0$, we have

$$\lim_{n\to\infty} d(Sx_n, u) = \lim_{n\to\infty} d(u, Sx_n) = 0.$$

The continuity of S yields that

$$\lim_{n\to\infty} d(STx_n, Su) = \lim_{n\to\infty} d(Su, STx_n) = 0.$$

The continuity of T yields that

$$\lim_{n\to\infty} d(TSx_n, Tu) = \lim_{n\to\infty} d(Tu, TSx_n) = 0.$$

Moreover, as S and T are compatible and the sequences $\{Sx_n\}$ and $\{Tx_n\}$ have the same limit, we deduce that

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \text{ or } \lim_{n\to\infty} d(TSx_n, STx_n) = 0.$$

Now,

$$d(Su, Tu) \leq d(Su, STx_n) + d(STx_n, TSx_n) + d(TSx_n, Tu).$$

By taking limit $n \to \infty$ in above inequality, we get $d(Su, Tu) = 0$.

Similarly, we can show that $d(Tu, Su) = 0$.

In any case, $Su = Tu$ and we conclude that u is a coincidence point of S and T. □

For the uniqueness of a coincidence point and existence and uniqueness of a fixed point of a $(\mathcal{Z}_{(\alpha,F)}, T)$-quasi-contraction of Ćirić type, we propose the following hypothesis.

Theorem 3.3.4. *In addition to the hypotheses of Theorem 3.3.3, suppose that for all $u, v \in C(S,T)$, there exists $w \in X$ such that $\alpha(Tu, Tw) \geq 1$, $\alpha(Tw, Tu) \geq 1$, $\alpha(Tw, Tv) \geq 1$ and $\alpha(Tv, Tw) \geq 1$. Also S, T commute at their coincidence points. Then, S and T have a unique common fixed point.*

Proof. We claim that if $u, v \in C(S,T)$, then $Tu = Tv$.
By hypotheses, there exists $w \in X$ such that

$$\alpha(Tw, Tu) \geq 1 \text{ and } \alpha(Tw, Tv) \geq 1.$$

Let us define the Picard sequence $\{w_n\}$ in X by $Tw_{n+1} = Sw_n$, for all $n \geq 0$ and $w_0 = w$. Reasoning as in the proof of Theorem 3.3.3, we obtain that the sequence $\{Tw_n\}$ converges to Tz.
By condition (i) in Theorem 3.3.3, we have

$$\alpha(Tw_n, Tu) \geq 1 \text{ and } \alpha(Tw_n, Tv) \geq 1, \text{ for all } n \geq 1. \tag{3.6}$$

Using (3.1), we have

$$\begin{aligned} C_F &\leq \zeta(\alpha(Tw_n, Tu)d(Sw_n, Su), \lambda M(Tw_n, Tu)) \\ &< F(\lambda M(Tw_n, Tu), \alpha(Tw_n, Tu)d(Sw_n, Su)) \\ &= F(\lambda M(Tw_n, Tu), \alpha(Tw_n, Tu)d(Tw_{n+1}, Tu)). \end{aligned} \tag{3.7}$$

By (F_1) and (3.6), we have

$$\begin{aligned} d(Tw_{n+1}, Tu) &\le \alpha(Tw_n, Tu) d(Tw_{n+1}, Tu) \\ &< \lambda M(Tw_n, Tu), \text{ for all } n \ge 1, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &M(Tw_n, Tu) \\ &= \max\{d(Tw_n, Tu), d(Tw_n, Sw_n), d(Tu, Su), d(Tw_n, Su), d(Tu, Sw_n)\} \\ &= \max\{d(Tw_n, Tu), d(Tw_n, Tw_{n+1}), d(Tu, Tw_{n+1})\}. \end{aligned}$$

Letting limit $n \to \infty$, we get

$$\lim_{n\to\infty} M(Tw_n, Tu) = \max\{d(Tz, Tu), d(Tu, Tz)\}.$$

Similarly, we get

$$d(Tu, Tw_{n+1}) < \lambda M(Tu, Tw_n), \text{ for all } \ge 1, \tag{3.9}$$

where

$$M(Tu, Tw_n) = \max\{d(Tu, Tw_n), d(Tw_n, Tw_{n+1}), d(Tw_n, Tu)\}.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} M(Tw_n, Tu) = \max\{d(Tu, Tz), d(Tz, Tu)\}.$$

If $Tu \ne Tz$ and we take the limit $n \to \infty$ in (3.8) and (3.9), we get

$$\begin{aligned} d(Tz, Tu) &< \lambda \max\{d(Tz, Tu), d(Tu, Tz)\}, \\ d(Tu, Tz) &< \lambda \max\{d(Tz, Tu), d(Tu, Tz)\}. \end{aligned}$$

If $d(Tz, Tu) < \lambda d(Tz, Tu)$, we get a contradiction.
If $d(Tz, Tu) < \lambda d(Tu, Tz) < \lambda^2 d(Tz, Tu)$, a contradiction.
Thus, $d(Tz, Tu) = 0$. Therefore, $Tu = Tz$.
Similarly, we can prove $Tv = Tz$. This implies $Tu = Tv$. Hence, u is a unique coincidence point of S and T.

Existence of a common fixed point: Let $u \in C(S,T)$, that is, $Su = Tu$. Due to commutativity of S and T at their coincidence points, we get

$$TTu = TSu = STu.$$

Let us denote $Tu = z^*$, then $Tz^* = Sz^*$. Thus z^* is a coincidence point of S and T. By uniqueness of coincidence point, we have $z^* = Tu = Tz^* = Sz^*$. Then, z^* is a common fixed point of S and T.

Uniqueness: Assume that w^* is another common fixed point of S and T. Then, $w^* \in C(S,T)$. Thus, we have $w^* = Tw^* = Tz^* = z^*$. This completes the proof. □

If $S(X) \subset T(X)$, then there exists a Picard-Jungck sequence of (S,T) based on any point $x_0 \in X$. Hence, from Theorem C, the above result is also valid for metric spaces.

Corollary 3.3.5. *Let (X,d) be a metric space, $S,T : X \to X$ be mappings and let $\{x_n\}$ be a Picard-Jungck sequence of (S,T). Assume that S is a $(\mathcal{Z}_{(\alpha,F)},T)$-quasi-contraction of Ćirić type satisfying the following conditions:*

(i) S is triangular α-admissible for T;

(ii) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$;

(iii) for all $u,v \in C(S,T)$, there exists $w \in X$ such that $\alpha(Tu,Tw) \geq 1$, $\alpha(Tv,Tw) \geq 1$, S,T commute at their coincidence points.

(iv) at least, one of the following conditions hold:

(a) $S(X)$ is precomplete in $T(X)$.

(b) (X,d) is a complete metric space and S and T are continuous and compatible.

Then, S and T have a unique common fixed point.

The following result is a solution to an open problem posed by Radenovic and Chandok [50].

Corollary 3.3.6. *[50, p.147] Let (X,d) be a metric space, S,T be self mappings on X and let $\{x_n\}$ be a Picard-Jungck sequence of (S,T). Let S be a*

$(\mathcal{Z}_F, T)$-quasi-contraction of Ćirić-Das-Naik type. Assume that, at least, one of the following conditions hold:

(a) $(T(X), d)$ is complete.

(b) (X, d) is a complete metric space, S and T are continuous and compatible.

Then, S and T have a unique point of coincidence. Moreover, if S and T commute at their coincidence point, then they have a unique common fixed point in X.

Proof. The result follows from Theorem 3.3.4, for $\alpha(x, y) = 1$. □

Corollary 3.3.7. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and S, T be self mappings on X. Let $\{x_n\}$ be a Picard-Jungck sequence of (S, T) and $\lambda \in (0, 1)$ such that*

$$\alpha(Tx, Ty)d(Sx, Sy) \leq \lambda M(Tx, Ty), \text{ for all } x, y \in X.$$

Assume that

(i) S is triangular α-admissible for T;

(ii) there exists $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$;

(iii) for all $u, v \in C(S, T)$, there exists $w \in X$ such that $\alpha(Tu, Tw) \geq 1$, $\alpha(Tv, Tw) \geq 1$ and S, T commute at their coincidence points;

(iv) at least, one of the following conditions hold:

(a) $(T(X), d)$ is complete.

(b) (X, d) is a complete metric space, S and T are continuous and compatible.

Then, S and T have a unique common fixed point.

Proof. The result follows from Corollary 3.3.6, for $\alpha(x, y) = 1$, $F(s, t) = s - t$, $C_F = 0$. □

3.4 Results for Ćirić type contraction using C_F-simulation functions in quasi-metric spaces

This section introduces the generalized Ćirić type $\mathcal{Z}_F$-contraction for pair of mappings. Subsequently, the results of Debnath et al. [21] are extended by proving common fixed point result in the frame work of quasi-metric spaces. Here, it is not necessary for mappings to be continuous to obtain the common fixed point result.

Definition 3.4.1. Let (X, d) be a quasi-metric space and S, T be self mappings on X. The pair (S, T) is called a generalized Ćirić type $\mathcal{Z}_F$-contractive pair of mappings if there exist $\zeta \in \mathcal{Z}_F, C_F \geq 0$ and $\lambda \in (0, 1)$ such that

$$\zeta\big(d(Sx, Ty), \lambda M(x, y)\big) \geq C_F, \tag{3.10}$$

where $M(x, y) = \max\left\{d(x, y), d(y, Sx), d(x, Ty), d(x, Sx), d(y, Ty)\right\}$;

$$\zeta\big(d(Ty, Sx), \lambda M(y, x)\big) \geq C_F, \tag{3.11}$$

where $M(y, x) = \max\left\{d(y, x), d(Sx, y), d(Ty, x), d(Sx, x), d(Ty, y)\right\}$,
for all $x, y \in X$.

Remark 4. (i) Due to the absence of symmetry in quasi-metric spaces, we required two inequalities in Definition 3.4.1.

(ii) By setting $S = T$ in (3.10)-(3.11), the mapping S becomes a $\mathcal{Z}_F$-quasi-contraction of the Ćirić type.

Now, we furnish our main result as follows.

Theorem 3.4.2. *Let (X, d) be a complete quasi-metric space and S, T be self mappings on X. Assume that (S, T) is a generalized Ćirić type $\mathcal{Z}_F$-contractive pair of mappings. Then, S and T have a unique common fixed point.*

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ such that

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \text{ for all } n \geq 0.$$

If there is $n_0 \in \mathbb{N}$ such that $x_{2n_0} = x_{2n_0+1}$, then x_{2n_0} is a fixed point of S.
To show that x_{2n_0} is a common fixed point of S and T.
Since $d(x_{2n_0+1}, x_{2n_0+2}) > 0$, using (3.10), we get

$$\begin{aligned} C_F &\leq \zeta\big(d(Sx_{2n_0}, Tx_{2n_0+1}\big), \lambda M(x_{2n_0}, x_{2n_0+1})) \\ &< F\big(\lambda M(x_{2n_0}, x_{2n_0+1}), d(Sx_{2n_0}, Tx_{2n_0+1})\big). \end{aligned}$$

By (F_1), we obtain

$$\begin{aligned} d(x_{2n_0+1}, x_{2n_0+2}) &= d(Sx_{2n_0}, Tx_{2n_0+1}) \\ &\leq \lambda M(x_{2n_0}, x_{2n_0+1}), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} M(x_{2n_0}, x_{2n_0+1}) &= \max\{d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0+1}, Sx_{2n_0}), d(x_{2n_0}, Tx_{2n_0+1}), \\ &\qquad d(x_{2n_0}, Sx_{2n_0}), d(x_{2n_0+1}, Tx_{2n_0+1})\} \\ &= \max\{d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0+1}, x_{2n_0+1}), d(x_{2n_0}, x_{2n_0+2}), \\ &\qquad d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0+1}, x_{2n_0+2})\} \\ &\leq d(x_{2n_0}, x_{2n_0+1}) + d(x_{2n_0+1}, x_{2n_0+2}). \end{aligned}$$

From (3.12), we get

$$\begin{aligned} d(x_{2n_0+1}, x_{2n_0+2}) &\leq \lambda[d(x_{2n_0}, x_{2n_0+1}) + d(x_{2n_0+1}, x_{2n_0+2})] \\ &\leq \lambda d(x_{2n_0+1}, x_{2n_0+2}), \text{ a contradiction.} \end{aligned}$$

Thus, $x_{2n_0+1} = x_{2n_0+2}$. Hence, $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$ is a common fixed point of S and T.
Now, we assume that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1}, x_n) > 0$, for all $n \geq 0$.
Claim: $\{x_n\}$ is a Cauchy sequence.
From (3.10), we have

$$\begin{aligned} C_F &\leq \zeta\big(d(Sx_{2n}, Tx_{2n+1}\big), \lambda M(x_{2n}, x_{2n+1})) \\ &< F\big(\lambda M(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\big), \end{aligned}$$

where $M(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})$.

Now, by using $(\tilde{F}_1)$, we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq \lambda[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &\leq \frac{\lambda}{1-\lambda} d(x_{2n}, x_{2n+1}) \\ &= k\ d(x_{2n}, x_{2n+1}),\ \text{for all } n \geq 0, \end{aligned} \tag{3.13}$$

where $k = \frac{\lambda}{1-\lambda} < 1$.
Also, from (3.11), we have

$$\begin{aligned} C_F &\leq \zeta\big(d(Tx_{2n-1}, Sx_{2n}), \lambda M(x_{2n-1}, x_{2n})\big) \\ &< F\big(\lambda M(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n-1}, x_{2n}) &= \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, Sx_{2n}), d(x_{2n}, Tx_{2n-1}), \\ &\qquad d(x_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1})\} \\ &= \max\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n+1}), d(x_{2n}, x_{2n}), \\ &\qquad d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n})\} \\ &\leq d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}). \end{aligned}$$

Now, by using $(\tilde{F}_1)$, we get

$$d(x_{2n}, x_{2n+1}) \leq k\ d(x_{2n-1}, x_{2n}),\ \text{for all } n \in \mathbb{N}, \tag{3.14}$$

where $k = \frac{\lambda}{1-\lambda} < 1$.
From (3.13) and (3.14), we have

$$d(x_n, x_{n+1}) \leq k\ d(x_{n-1}, x_n),\ \text{for all } n \in \mathbb{N}. \tag{3.15}$$

Similarly, we can show that

$$d(x_{n+1}, x_n) \leq k\ d(x_n, x_{n-1}),\ \text{for all } n \in \mathbb{N}. \tag{3.16}$$

Thus, from Lemma 3.2.3, we conclude that $\{x_n\}$ is a Cauchy sequence in X.

Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0 = \lim_{n\to\infty} d(u, x_n).$$

To prove that $Su = Tu = u$.
From (3.10), we obtain

$$C_F \leq \zeta\big(d(Su, Tx_{2n+1}), \lambda M(u, x_{2n+1})\big) < F\big(\lambda M(u, x_{2n+1}), d(Su, Tx_{2n+1})\big).$$

By $(\tilde{F_1})$, we obtain

$$d(Su, x_{2n+2}) \leq \lambda M(u, x_{2n+1}), \tag{3.17}$$

where

$$M(u, x_{2n+1}) = \max\{d(u, x_{2n+1}), d(x_{2n+1}, Su), d(u, Tx_{2n+1}), \\ d(u, Su), d(x_{2n+1}, Tx_{2n+1})\}.$$

Also,

$$d(x_{2n+2}, Su) \leq \lambda M(x_{2n+1}, u), \tag{3.18}$$

where

$$M(x_{2n+1}, u) = \max\{d(x_{2n+1}, u), d(Su, x_{2n+1}), d(Tx_{2n+1}, u), \\ d(Su, u), d(Tx_{2n+1}, x_{2n+1})\}.$$

Taking limit as $n \to \infty$ on both sides of (3.17) and (3.18), we get

$$d(Su, u) \leq \lambda\ d(u, Su)\ and\ d(u, Su) \leq \lambda\ d(Su, u).$$

Hence, $d(Su, u) = d(u, Su) = 0$. Implies, $Su = u$.
Similarly, we can show that $Tu = u$. Thus, u is a common fixed point of S and T.
Uniqueness: Let u' is another common fixed point of S and T. Then

$$\begin{aligned} C_F &\leq \zeta\big(d(Su, Tu'), \lambda M(u, u')\big) \\ &< F\big(\lambda M(u, u'), d(Su, Tu')\big) \end{aligned}$$

where

$$M(u,u') = \max\{d(u,u'), d(u',Su), d(u,Tu'), d(u,Su), d(u',Tu')\}$$
$$= \max\{d(u,u'), d(u',u)\}.$$

From $(\tilde{F_1})$, we get

$$d(u,u') \leq \lambda \ \max\{d(u,u'), d(u',u)\}.$$

Similarly, we get

$$d(u',u) \leq \lambda \ \max\{d(u,u'), d(u',u)\},$$

a contradiction. Hence, $d(u,u') = 0$. Thus, u is a unique common fixed point of S and T. □

Corollary 3.4.3. *Let (X,d) be a complete metric space and $S,T\colon X \to X$ be self mappings. Suppose there exists $\lambda \in (0,1)$ such that*

$$d(Sx,Ty) \leq \lambda M(x,y), \ \textit{for all } x,y \in X. \tag{3.19}$$

Then, S and T have a unique common fixed point.

Proof. If we take $\zeta(t,s) = ks - t, k \in (0,1), C_F = 0$ in (3.10), we get (3.19). Due to symmetry of d the result follows from Theorem 3.4.2. □

In (3.19), if we restrict the value of λ to $0 < \lambda < \frac{1}{2}$ and omit $d(x,y)$, $d(Sx,y)$ and $d(Ty,x)$. This gives us the Kannan type contraction. Similarly, by omitting $d(x,y)$, $d(Sx,x)$ and $d(Ty,y)$ in (3.19), we get the Chatterjea type contraction. From Theorem 3.4.2, we obtain the following result of Debnath et al. [21].

Theorem E. *[21, Theorem 2.3, Theorem 2.5, p.386] Let (X,d) be a complete metric space and S,T be self mappings on X. Suppose there exists $\lambda \in (0,\frac{1}{2})$ such that*

$$d(Sx,Ty) \leq \lambda[d(Sx,x) + d(Ty,y)], \quad (Kannan \ \ type)$$

or

$$d(Sx,Ty) \leq \lambda[d(Sx,y) + d(Ty,x)], \quad (Chattarjea \ \ type)$$

for all $x,y \in X$. Then, S and T have a unique common fixed point.

The contractive condition in (3.19) can be modified to obtain the following, by restricting the value of λ to $0 < \lambda < \frac{1}{3}$ and omitting the terms $d(y, Sx)$ and $d(x, Ty)$. This result corresponds to the Reich type common fixed point result for a pair of self-mappings in metric spaces, as established by Debnath et al. [21].

Theorem F. *[21, Theorem 2.6, p.389] Let (X, d) be a complete metric space and S, T be self mappings on X. Suppose there exists $\lambda \in (0, \frac{1}{3})$ such that*

$$d(Sx, Ty) \leq \lambda[d(x, y) + d(Sx, x) + d(Ty, y)], \quad (Reich \;\; type)$$

for all $x, y \in X$. Then, S and T have a unique common fixed point.

3.5 Consequences: Common fixed point results in G-metric spaces

In this section, Theorem 3.3.3, Theorem 3.3.4 and Theorem 3.4.2 are extended in G-metric spaces.
The following results are consequences of Theorem 3.3.3 and Theorem 3.3.4.

Corollary 3.5.1. *Let (X, G) be a G-metric space, $\alpha_w : X \times X \times X \to [0, \infty)$ and $S, T : X \to X$ be mappings with $S(X) \subset T(X)$. Let $\zeta \in \mathcal{Z}_F, C_F \geq 0$ and $\lambda \in (0, 1)$ such that*

$$\zeta(\alpha_w(Tx, Ty, Ty)G(Sx, Sy, Sy), \lambda M(Tx, Ty, Ty)) \geq C_F, \tag{3.20}$$

for all $x, y \in X$, where

$$M(Tx, Ty, Ty) = \max\{G(Tx, Ty, Ty), G(Tx, Sx, Sx), G(Ty, Sy, Sy), G(Tx, Sy, Sy), G(Ty, Sx, x)\}.$$

Suppose that

(i) S is weak α_w-admissible for T;

(ii) $\alpha_w(Tx, Ty, Ty) \geq 1$ and $\alpha_w(Ty, Tz, Tz) \geq 1 \implies \alpha_w(Tx, Tz, Tz) \geq 1$;

(iii) there exists $x_0 \in X$ such that $\alpha_w(Tx_0, Sx_0, Sx_0) \geq 1$ and $\alpha_w(Sx_0, Tx_0, Tx_0) \geq 1$;

(iv) *for all* $u, v \in C(S, T)$, *there exists* $w \in X$ *such that* $\alpha_w(Tu, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tu, Tu) \geq 1$, $\alpha_w(Tv, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tv, Tv) \geq 1$ *and* S, T *commute at their coincidence points;*

(v) *at least, one of the following conditions hold:*

(a) $S(X)$ *is precomplete in* $T(X)$.

(b) (X, G) *is a complete* G*-metric space,* S *and* T *are continuous and compatible.*

Then, S *and* T *have a unique common fixed point.*

Proof. It suffices to take $d_G(x, y) = G(x, y, y)$ and $\alpha(x, y) = \alpha_w(x, y, y)$. From (3.1), we get (3.20). Since (X, G) is complete, by Theorem D, (X, d_G) is a complete quasi-metric space. Hence, the result follows from Theorem 3.3.3 and Theorem 3.3.4. □

If $S(X) \subset T(X)$, then there exists a Picard-Jungck sequence of (S, T) based on any point $x_0 \in X$. The following result is obtained from Corollary 3.5.1.

Corollary 3.5.2. *Let* (X, G) *be a* G*-metric space,* $\alpha_w : X \times X \times X \to [0, \infty)$ *and* $S, T : X \to X$ *be mappings. Let* $\{x_n\}$ *be a Picard-Jungck sequence of* (S, T), $\zeta \in \mathcal{Z}_F, C_F \geq 0$ *and* $\lambda \in (0, 1)$ *such that (3.20) is satisfied. Suppose that*

(i) S *is weak* α_w*-admissible for* T*;*

(ii) $\alpha_w(Tx, Ty, Ty) \geq 1$ *and* $\alpha_w(Ty, Tz, Tz) \geq 1 \implies \alpha_w(Tx, Tz, Tz) \geq 1$;

(iii) *there exists* $x_0 \in X$ *such that* $\alpha_w(Tx_0, Sx_0, Sx_0) \geq 1$ *and* $\alpha_w(Sx_0, Tx_0, Tx_0) \geq 1$*;*

(iv) *for all* $u, v \in C(S, T)$, *there exists* $w \in X$ *such that* $\alpha_w(Tu, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tu, Tu) \geq 1$, $\alpha_w(Tv, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tv, Tv) \geq 1$ *and* S, T *commute at their coincidence points;*

(v) *at least, one of the following conditions hold:*

(a) $S(X)$ *is precomplete in* $T(X)$.

(b) (X, G) *is a complete* G*-metric space,* S *and* T *are continuous and compatible.*

Then, S and T have a unique common fixed point.

Corollary 3.5.3. *Let (X, G) be a G-metric space, $\alpha_w : X \times X \times X \to [0, \infty)$ and $S, T : X \to X$ be mappings. Let $\{x_n\}$ be a Picard-Jungck sequence of (S, T) and $\lambda \in (0, 1)$ such that*

$$\alpha_w(Tx, Ty, Ty)G(Sx, Sy, Sy) \leq \lambda M(Tx, Ty, Ty), \text{ for all } x, y \in X.$$

Suppose that

(i) S is weak α_w-admissible for T;

(ii) $\alpha_w(Tx, Ty, Ty) \geq 1$ and $\alpha_w(Ty, Tz, Tz) \geq 1 \implies \alpha_w(Tx, Tz, Tz) \geq 1$;

(iii) there exists $x_0 \in X$ such that $\alpha_w(Tx_0, Sx_0, Sx_0) \geq 1$ and $\alpha_w(Sx_0, Tx_0, Tx_0) \geq 1$;

(iv) for all $u, v \in C(S, T)$, there exists $w \in X$ such that $\alpha_w(Tu, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tu, Tu) \geq 1$, $\alpha_w(Tv, Tw, Tw) \geq 1$, $\alpha_w(Tw, Tv, Tv) \geq 1$ and S, T commute at their coincidence points;

(v) at least, one of the following conditions hold:

(a) $S(X)$ is precomplete in $T(X)$.

(b) (X, G) is a complete G-metric space, S and T are continuous and compatible.

Then, S and T have a unique common fixed point.

Proof. The result follows from Corollary 3.5.2, if we consider $\alpha(x, y) = 1$, $F(s, t) = s - t$, $C_F = 0$. □

Corollary 3.5.4. *Let (X, G) be a G-metric space and $S, T : X \to X$ be mappings. Let $\{x_n\}$ be a Picard-Jungck sequence of (S, T), $\zeta \in \mathcal{Z}_F, C_F \geq 0$ and $\lambda \in (0, 1)$ such that*

$$\zeta(G(Sx, Sy, Sy), \lambda M(Tx, Ty, Ty)) \geq C_F, \text{ for all } x, y \in X. \tag{3.21}$$

Also assume that at least, one of the following conditions hold:

(a) $S(X)$ is precomplete in $T(X)$.

(b) (X, G) *is a complete* G*-metric space,* S *and* T *are continuous and compatible.*

Then, S *and* T *have unique point of coincidence. Moreover, if* S, T *commute at their coincidence points, then* S *and* T *have a unique common fixed point in* X*.*

Proof. In (3.20), if we take $\alpha_w(x, y, y) = 1$, we get (3.21). Then the result follows from Corollary 3.5.2. □

The following results are obtained from Theorem 3.4.2.

Corollary 3.5.5. *Let* (X, G) *be a complete* G*-metric space and* $S, T \colon X \to X$ *be self mappings. Suppose there exist* $\zeta \in \mathcal{Z}_F, C_F \geq 0$ *and* $\lambda \in (0, 1)$ *such that*

$$\zeta\big(G(Sx, Ty, Ty), \lambda M'(x, y)\big) \geq C_F; \tag{3.22}$$

$$\zeta\big(G(Ty, Sx, Sx), \lambda M'(y, x)\big) \geq C_F, \tag{3.23}$$

where

$$M'(x, y) = \max\{G(x, y, y), G(y, Sx, Sx), G(x, Ty, Ty), G(x, Sx, Sx), G(y, Ty, Ty)\};$$

$$M'(y, x) = \max\{G(y, x, x), G(Sx, y, y), G(Ty, x, x), G(Sx, x, x), G(Ty, y, y)\},$$

for all $x, y \in X$*. Then,* S *and* T *have a unique common fixed point.*

Proof. It suffices to take $d_G(x, y) = G(x, y, y)$, from (3.10) and (3.11) we get (3.22) and (3.23) respectively. Since (X, G) is complete, then by Theorem D, (X, d_G) is a complete quasi-metric space. Then, from Theorem 3.4.2 proof follows. □

Corollary 3.5.6. *Let* (X, G) *be a complete* G*-metric space and* $S, T \colon X \to X$ *be self mappings. Suppose there exists* $\lambda \in (0, 1)$ *such that*

$$G(Sx, Ty, Ty) \leq \lambda M'(x, y); \tag{3.24}$$

$$G(Ty, Sx, Sx), \leq \lambda M'(y, x), \tag{3.25}$$

for all $x, y \in X$*. Then,* S *and* T *have a unique common fixed point.*

Proof. If we take $\zeta(t, s) = ks - t, k \in (0, 1), C_F = 0$ in (3.22) and (3.23), we get (3.24) and (3.25) respectively, then result follows from Corollary 3.5.5. □

For $S = T$ in Corollary 3.5.6, obtained result is a generalization of Theorem 4.2.1 in [2].