Chapter 5

Non-linear Contractions via Extended $\Gamma - C_F$-simulation Functions

5.1 Introduction

Recently, the notion of the simulation functions has been extended and generalized in various ways, like Γ-simulation functions [36], extended simulation functions [19], extended C_F-simulation functions [11] and many others. On the other hand, the Banach contractive principle has been generalized by many authors by modifying the contraction. Some of the generalizations are Geraghty type contractions [23], Suzuki type contraction [60], the notion of almost contraction [8], etc.

This chapter intends to make use of the theories from above mentioned different types of contractions and simulation functions to furnish a couple of related coincidences and common fixed point results. To achieve these results, the notion of $\Gamma - C$-class function and extended $\Gamma - C_F$-simulation functions are introduced. Using these notions, the almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contraction in G-metric spaces and Geraghty type contraction in G_b-metric spaces are introduced. Alongside, some non-trivial examples are illustrated to authenticate the definitions. Moreover, an application for the existence of a solution to a non-linear integral equation is provided.

5.2 Extended $\Gamma - C_F$-simulation functions

In this section, we introduce $\Gamma - C$-class function, extended $\Gamma - C_F$-simulation function.

$\Gamma([0,\infty))$ is the set of all non-decreasing functions $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(t) = 0$ if and only if $t = 0$.

Definition 5.2.1. A function $F : [0,\infty)^2 \to \mathbb{R}$ is called $\Gamma - C$-class function if it is continuous and there exists $\gamma \in \Gamma([0,\infty))$ such that:

(i) $F(s,t) \leq \gamma(s),$ for all $t, s \geq 0;$

(ii) $F(s,t) = \gamma(s)$ implies that either $s = 0$ or $t = 0,$ for all $t, s \geq 0.$

The collection of all $\Gamma - C$-class functions is denoted by $\mathcal{C}_\Gamma$.
Note that, for $\gamma(t) = t$ the set, $\mathcal{C}$ is collection of C-class functions [6].

Example 5.2.1. The following functions $F_i : [0,\infty)^2 \to \mathbb{R}$ are some elements of $\mathcal{C}_\Gamma$.

(i) $F_1(s,t) = \gamma(s) - \gamma(t), \gamma(t) = 2t$ or $\frac{t}{2}$.

(ii) $F_2(s,t) = \gamma(s) - \left(\dfrac{1+\gamma(s)}{2+\gamma(s)}\right)\left(\dfrac{\gamma(t)}{1+\gamma(t)}\right),$

$$\gamma(t) = \begin{cases} 2t, & if \;\; 0 \leq t < 1, \\ 3t, & if \;\; 1 \leq t. \end{cases}$$

(iii) $F_3(s,t) = 2\gamma(s) - \gamma(t), \gamma(t) = 2t.$

Definition 5.2.2. A function $F : [0,\infty)^2 \to \mathbb{R}$ has the property $\Gamma - C_F$, if there exist $\gamma \in \Gamma([0,\infty))$ and $C_F \geq 0$ such that:

($\mathcal{F}_1$) $F(s,t) > C_F$ implies $\gamma(s) > \gamma(t),$ for all $t, s \geq 0;$

($\mathcal{F}_2$) $F(t,t) \leq C_F,$ for all $t \geq 0.$

Example 5.2.2. The following functions $F_i : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}_\Gamma$ with property $\Gamma - C_F$.

(i) $F_1(s,t) = \dfrac{\gamma(s)}{(1+\gamma(t))}, C_F = 1, 2.$

(ii) $F_2(s,t) = \dfrac{\gamma(s)}{(1+\gamma(t))^r}, r \in (0,\infty), C_F = 1.$

Definition 5.2.3. An extended $\Gamma - C_F$-simulation function is a function $\eta : [0,\infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(η_1) $\eta(t,s) < F(s,t),$ for all $t,s > 0$, where $F \in \mathcal{C}_\Gamma$ with property $\Gamma - C_F$;

(η_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l,$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \eta(t_n, s_n) < C_F;$$

(η_3) let $\{t_n\}$ be a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = l \in [0,\infty)$, then

$$\eta(t_n, l) \geq C_F \implies l = 0.$$

The following example validates our definition.

Example 5.2.3. Let $\eta : [0,\infty)^2 \to \mathbb{R}$ be a function defined by $\eta(t,s) = \frac{3}{4}\gamma(s) - \gamma(t),$ for all $t,s \in [0,\infty)$. Taking $F(s,t) = \gamma(s) - \gamma(t)$ with $C_F = 0,$ for all $t,s \in [0,\infty)$ and $\gamma(t) = 2t,$ for all $t \geq 0$. It is easy to verify (η_1).

We now check for (η_2). If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l$ for all $n \in \mathbb{N}$, then we obtain

$$\limsup_{n\to\infty} \eta(t_n, s_n) = \limsup_{n\to\infty} \left[\frac{3}{4}\gamma(s_n) - \gamma(t_n)\right] = \limsup_{n\to\infty} \left[\frac{3}{4}(2s_n) - (2t_n)\right] = -\frac{l}{2} < C_F = 0.$$

Hence (η_2) is satisfied. Now, we check (η_3). Choose a sequence $\{t_n\} \in (0,\infty)$ with

$$\lim_{n\to\infty} t_n = l \geq 0$$

such that, for all $n \in \mathbb{N}$

$$\eta(t_n, l) \geq C_F = 0 \implies \frac{3}{4}(2l) - 2t_n \geq 0.$$

Letting $n \to \infty$, we get

$$\frac{3}{2}l - 2l \geq 0 \implies \frac{-l}{2} \geq 0.$$

Implies that $l = 0$. Hence, η is an extended $\Gamma - C_F$-simulation function.

The class of an extended $\Gamma - C_F$-simulation functions is denoted by $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$. The class of Γ-simulation functions (say $\mathcal{Z}_\Gamma$) and extended C_F-simulation functions (say $\mathcal{E}_{(\mathcal{Z},F)}$) are proper subsets of $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$, which can be easily observed from the following example.

Example 5.2.4. Define $\gamma : [0, \infty) \to [0, \infty)$ by

$$\gamma(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 2t, & \text{if } 1 \leq t, \end{cases}$$

and $\eta_a : [0, \infty)^2 \to \mathbb{R}$ by

$$\eta_a(t, s) = \begin{cases} 1 - \gamma(t), & \text{when } s = 0, \\ \dfrac{k\gamma(s)}{1 + \gamma(t)}, & \text{when } s > 0, \end{cases}$$

for all $t, s \in [0, \infty)$ and $k \in [0, 1)$.
Taking $F(s, t) = \dfrac{\gamma(s)}{1 + \gamma(t)}$ with $C_F = 1$, for all $s, t \in [0, \infty)$.

5.3 Results for almost Suzuki contraction in G-metric spaces

In this section, the almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contraction is defined by using extended $\Gamma - C_F$-simulation functions for pair of mappings. Further, a common fixed point result involving such mappings is established.

Definition 5.3.1. Let (X, G) be a G-metric space and $S, T : X \to X$ be self mappings on X. We say that (S, T) is the pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contractive maps, if there exist $r \in [0, 1), L \geq 0, C_F \geq 0$ and $\eta \in \mathcal{E}_{(\mathcal{Z},F,\Gamma)}$ such that

$$\frac{1}{1+r}\min\{G(Tx, Sx, Sx), G(Ty, Sy, Sy)\} \leq G(Tx, Ty, Ty),$$

implies

$$\eta(G(Sx,Sy,Sy),M(x,y,y)+L\ N(x,y,y))\geq C_F,\ \text{for all } x,y\in X \tag{5.1}$$

where

$$M(x,y,y)=\max\{G(Tx,Ty,Ty),G(Tx,Sx,Sx),G(Ty,Sy,Sy),\\ \frac{G(Tx,Sy,Sy)+G(Sx,Ty,Ty)}{2}\}$$

and
$N(x,y,y)=\min\{G(Tx,Sx,Sx),G(Ty,Sy,Sy),G(Tx,Sy,Sy),G(Ty,Sx,Sx)\}$.

Now, main result of this section is furnished.

Theorem 5.3.2. *Let (X,G) be a G-metric space and $S,T:X\to X$ be self mappings with $S(X)\subseteq T(X)$. Assume that (S,T) be the pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contractive maps and $S(X)$ is precomplete in $T(X)$. Also, S and T commute at their coincidence point. Then S and T have a unique common fixed point.*

Proof. Let $x_0\in X$ be a point. Define a sequence $\{x_n\}$ in X by $Sx_n=Tx_{n+1}$, for all $n\geq 0$. If there exists $n\in\mathbb{N}$ such that $Tx_n=Tx_{n+1}$ then $Sx_n=Tx_n$; that is, x_n is a coincidence point of S and T. Thus, we assume that $Tx_n\neq Tx_{n+1}$, for all $n\geq 0$.
Hence, we have

$$\begin{aligned}\frac{1}{1+r}\min\{G(Tx_n,Sx_n,Sx_n),G(Tx_{n+1},Sx_{n+1},Sx_{n+1})\}&\leq\frac{1}{1+r}G(Tx_n,Sx_n,Sx_n)\\&=\frac{1}{1+r}G(Tx_n,Tx_{n+1},Tx_{n+1})\\&\leq G(Tx_n,Tx_{n+1},Tx_{n+1}).\end{aligned}$$

Using (5.1) and (η_1), we have

$$\begin{aligned}C_F&\leq\eta(G(Sx_n,Sx_{n+1},Sx_{n+1}),M(x_n,x_{n+1},x_{n+1})+L\ N(x_n,x_{n+1},x_{n+1}))\\&<F(M(x_n,x_{n+1},x_{n+1})+L\ N(x_n,x_{n+1},x_{n+1}),G(Tx_{n+1},Tx_{n+2},Tx_{n+2}))\\&\implies\gamma(G(Tx_{n+1},Tx_{n+2},Tx_{n+2}))<\gamma(M(x_n,x_{n+1},x_{n+1})+L\ N(x_n,x_{n+1},x_{n+1})).\end{aligned}$$

Since $\gamma \in \Gamma([0,\infty))$, we have

$$G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}) < M(x_n, x_{n+1}, x_{n+1}) + L\ N(x_n, x_{n+1}, x_{n+1}), \tag{5.2}$$

where

$$\begin{aligned} N(x_n, x_{n+1}, x_{n+1}) = \min\{&G(Tx_n, Sx_n, Sx_n), G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1}),\\ &G(Tx_n, Sx_{n+1}, Sx_{n+1}), G(Tx_{n+1}, Sx_n, Sx_n)\} = 0 \end{aligned}$$

and

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+1}) &= \max\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_n, Sx_n, Sx_n),\\ &\qquad G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1}),\\ &\qquad \frac{G(Tx_n, Sx_{n+1}, Sx_{n+1}) + G(Sx_n, Tx_{n+1}, Tx_{n+1})}{2}\}\\ &= \max\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}),\\ &\qquad \frac{G(Tx_n, Tx_{n+2}, Tx_{n+2})}{2}\}\\ &= \max\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2})\}, \end{aligned}$$

by rectangle inequality.

If $M(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2})$, then from (5.2), we get the contradiction

$$G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}) < G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}).$$

Therefore, $M(x_n, x_{n+1}, x_{n+1}) = G(Tx_n, Tx_{n+1}, Tx_{n+1})$.
From (5.2), we have

$$G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}) < G(Tx_n, Tx_{n+1}, Tx_{n+1}), \text{ for all } n \geq 0,$$

which implies that the sequence $\{G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$ is a non-negative monotonically decreasing sequence. So, there exists some $l \geq 0$ such that

$$\lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = l \text{ and } G(Tx_n, Tx_{n+1}, Tx_{n+1}) > l, \text{ for all } n \in \mathbb{N}. \tag{5.3}$$

Suppose that $l > 0$, then we consider two sequences (t_n) and (s_n) with same positive limit, where

$$t_n = G(Sx_n, Sx_{n+1}, Sx_{n+1}) > 0$$

and

$$s_n = M(x_n, x_{n+1}, x_{n+1}) + L\ N(x_n, x_{n+1}, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N}.$$

Then from (η_1) and (5.1), we get

$$G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}) < M(x_n, x_{n+1}, x_{n+1}) + L\ N(x_n, x_{n+1}, x_{n+1}),$$

where

$$M(x_n, x_{n+1}, x_{n+1}) = \max\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2});$$
$$N(x_n, x_{n+1}, x_{n+1}) = 0. \tag{5.4}$$

Now, from (5.3) and (5.4), we have $s_n > l$, for all $n \geq 0$. Also $\lim_{n\to\infty} s_n = l$ and $\lim_{n\to\infty} t_n = l$. Using (η_2), we get the contradiction

$$C_F \leq \limsup_{n\to\infty}\ \eta(t_n, s_n) < C_F.$$

So, we conclude that,

$$l = \lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0 \implies \lim_{n\to\infty} G(Tx_n, Tx_n, Tx_{n+1}) = 0, \tag{5.5}$$

from definition of G-metric space. Now, we prove that $\{Tx_n\}$ is a G-Cauchy sequence. Assume that $\{Tx_n\}$ is not Cauchy, then there exists $\varepsilon > 0$ and two sequences $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ such that, for all $k \in \mathbb{N}$, $n_{k+1} > m_k > n_k \geq k$,

$$G(Tx_{n_k}, Tx_{m_k-1}, Tx_{m_k-1}) \leq \varepsilon < G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) \tag{5.6}$$

and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G(Tx_{n_k+p_1}, Tx_{m_k+p_2}, Tx_{m_k+p_3}) = \varepsilon.$$

Further, from (5.5) and (5.6) for all $k \geq n_0$, we obtain

$$\begin{aligned}
&\frac{1}{1+r}\min\{G(Tx_{n_k}, Sx_{n_k}, Sx_{n_k}), G(Tx_{m_k}, Sx_{m_k}, Sx_{m_k})\}\\
&= \frac{1}{1+r}\min\{G(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}), G(Tx_{m_k}, Tx_{m_k+1}, Tx_{m_k+1})\}\\
&\leq \frac{1}{1+r}\varepsilon < \varepsilon\\
&< G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}).
\end{aligned}$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned}
C_F &\leq \eta(G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}), M(x_{n_k}, x_{m_k}, x_{m_k}) + L\ N(x_{n_k}, x_{m_k}, x_{m_k}))\\
&< F(M(x_{n_k}, x_{m_k}, x_{m_k}) + L\ N(x_{n_k}, x_{m_k}, x_{m_k}), G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}))\\
\Longrightarrow &\ \gamma(G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k})) < \gamma(M(x_{n_k}, x_{m_k}, x_{m_k}) + L\ N(x_{n_k}, x_{m_k}, x_{m_k})).
\end{aligned}$$

Since $\gamma \in \Gamma([0,\infty))$, we have

$$G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}) < M(x_{n_k}, x_{m_k}, x_{m_k}) + L\ N(x_{n_k}, x_{m_k}, x_{m_k}),$$

where

$$\begin{aligned}
N(x_{n_k}, x_{m_k}, x_{m_k}) &= \min\{G(Tx_{n_k}, Sx_{n_k}, Sx_{n_k}), G(Tx_{m_k}, Sx_{m_k}, Sx_{m_k}),\\
&\qquad G(Tx_{n_k}, Sx_{m_k}, Sx_{m_k}), G(Tx_{m_k}, Sx_{n_k}, Sx_{n_k})\};\\
M(x_{n_k}, x_{m_k}, x_{m_k}) &= \max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), G(Tx_{n_k}, Sx_{n_k}, Sx_{n_k}),\\
&\qquad G(Tx_{m_k}, Sx_{m_k}, Sx_{m_k}),\\
&\qquad \frac{G(Tx_{n_k}, Sx_{m_k}, Sx_{m_k}) + G(Sx_{n_k}, Tx_{m_k}, Tx_{m_k})}{2}\}\\
&= \max\{G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}), G(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}),\\
&\qquad G(Tx_{m_k}, Tx_{m_k+1}, Tx_{m_k+1}),\\
&\qquad \frac{G(Tx_{n_k}, Tx_{m_k+1}, Tx_{m_k+1}) + G(Tx_{n_k+1}, Tx_{m_k}, Tx_{m_k})}{2}\}.
\end{aligned}$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, x_{m_k}) = \varepsilon \text{ and } \lim_{k\to\infty} N(x_{n_k}, x_{m_k}, x_{m_k}) = 0.$$

Now, consider two sequences $\{t_k\}$ and $\{s_k\}$ with

$$t_k = G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}) > 0;\ s_k = M(x_{n_k}, x_{m_k}, x_{m_k}) + L\ N(x_{n_k}, x_{m_k}, x_{m_k}) > 0,$$

for all $k \in \mathbb{N}$.
Applying (η_2), we get the contradiction

$$C_F \leq \limsup_{k\to\infty}\ \eta(t_k, s_k) < C_F.$$

Hence $\{Tx_n\}$ is a Cauchy sequence in (X, G). Since $S(X)$ is precomplete in $T(X)$, it follows that there exists $u \in X$ such that

$$\lim_{n\to\infty} Tx_n = Tu = \lim_{n\to\infty} Sx_{n+1}.$$

We claim that, u is a coincidence point of S and T.

Suppose $G(Tu, Su, Su) > 0$ and $G(Su, Tu, Tu) > 0$. we have

$$\frac{1}{1+r}\min\{G(Tx_n, Sx_n, Sx_n), G(Tu, Su, Su)\} \leq G(Tx_n, Tu, Tu),\ \text{for all } n \geq n_0.$$

Using (5.1), we get

$$C_F \leq \eta(G(Sx_n, Su, Su), M(x_n, u, u) + L\ N(x_n, u, u)),$$

where

$$\begin{aligned} M(x_n, u, u) &= \max\{G(Tx_n, Tu, Tu), G(Tu, Su, Su), G(Tx_n, Sx_n, Sx_n), \\ &\qquad \frac{G(Tx_n, Su, Su) + G(Sx_n, Tu, Tu)}{2}\}; \\ N(x_n, u, u) &= \min\{G(Tu, Su, Su), G(Tx_n, Sx_n, Sx_n), \\ &\qquad G(Tx_n, Su, Su), G(Tu, Sx_n, Sx_n)\}. \end{aligned}$$

$\lim_{n\to\infty} N(x_n, u, u) = 0$ and $\lim_{n\to\infty} M(x_n, u, u) = G(Tu, Su, Su) = t > 0$.

Hence, we have

$$\lim_{n\to\infty} G(Sx_n, Su, Su) = t = \lim_{n\to\infty} G(Tx_n, Su, Su).$$

Now, using (η_3), for all $n \geq n_0$, we get

$$\eta(G(Sx_n, Su, Su), t) = \eta\big(G(Sx_n, Su, Su), \lim_{n\to\infty} \big(M(x_n, u, u) + L\, N(x_n, u, u)\big)\big) \geq C_F$$

$$\implies t = 0$$

$$\implies G(Tu, Su, Su) = 0.$$

Hence $Tu = Su$, that is, u is a coincidence point of S and T.

Now, to prove that u is a unique coincidence point of S and T. Assume that, there exist z in X such that $Tz = Sz$ and $Tu \neq Tz$.

Since, $\frac{1}{1+r}\min\{G(Tz, Sz, Sz), G(Tu, Su, Su)\} \leq G(Tz, Tu, Tu)$.
From (5.1), we get

$$C_F \leq \eta(G(Sz, Su, Su), M(z, u, u) + L\ N(z, u, u)), \tag{5.7}$$

where

$$\begin{aligned} M(z,u,u) &= \max\{G(Tz,Tu,Tu), G(Tz,Sz,Sz), G(Tu,Su,Su), \\ &\qquad \frac{G(Tz,Su,Su)+G(Sz,Tu,Tu)}{2}\} \\ &= G(Tz,Tu,Tu); \\ N(z,u,u) &= \min\{G(Tz,Sz,Sz), G(Tu,Su,Su), G(Tz,Su,Su), G(Tu,Sz,Sz)\} \\ &= 0. \end{aligned}$$

Hence, from (5.7), (η_1) and $(\mathcal{F}_2)$, we get

$$\begin{aligned} C_F &\leq \eta(G(Sz,Su,Su), G(Tz,Tu,Tu)) \\ &\leq \eta(G(Tz,Tu,Tu), G(Tz,Tu,Tu)) \\ &< F(G(Tz,Tu,Tu), G(Tz,Tu,Tu)) \\ &\leq C_F, \end{aligned}$$

which is not possible. Hence, $Tu = Tz$.

Existence of a common fixed point: Let $u \in C(S,T)$, that is, $Su = Tu$. Due to commutativity of S and T at their coincidence points, we get

$$TTu = TSu = STu \implies Tz^* = Sz^*, \text{ where } z^* = Tu.$$

Thus, z^* is a coincidence point of S and T. By uniqueness of coincidence point, we have $z^* = Tu = Tz^* = Sz^*$. Then, z^* is a common fixed point of S and T.

Uniqueness: Assume that w^* is another common fixed point of S and T. Then, $w^* \in C(S,T)$. Thus, we have $w^* = Tw^* = Tz^* = z^*$. $\square$

Supporting example for Theorem 5.3.2 is as follows.

Example 5.3.1. Let $X = \{1,3,5,7\}$ and define $G : X \times X \times X \to [0,\infty)$ by $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\},$ for all $x \in X$. Then (X,G) is complete G-metric space.
Define maps $S,T : X \to X$ by

$$Sx = \begin{cases} 3 & (x \neq 7), \\ 1 & (x = 7), \end{cases}$$

and $Tx = x$.
Then (S,T) is the pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contractive maps with η_a as defined in Example 5.2.4. Let $L = 1, k = \frac{1}{2}$ and $r = \frac{1}{2}$, then S,T satisfies all the conditions of Theorem 5.3.2. Hence S and T have a unique common fixed point at $x = 3$.

By choosing $Tx = x$ in Theorem 5.3.2, we get the following results.

Corollary 5.3.3. *Let (X,G) be a complete G-metric space and $S : X \to X$ be self mapping. There exist $r \in [0,1)$, $L \geq 0, C_F \geq 0$ and $\eta \in \mathcal{E}_{(\mathcal{Z},F,\Gamma)}$ such that*

$$\frac{1}{1+r}\min\{G(x,Sx,Sx), G(y,Sy,Sy)\} \leq G(x,y,y),$$

implies

$$\eta(G(Sx,Sy,Sy), M'(x,y,y) + L\ N'(x,y,y)) \geq C_F, \text{ for all } x,y \in X,$$

where

$$M'(x,y,y) = \max\{G(x,y,y), G(x,Sx,Sx), G(y,Sy,Sy), \frac{G(x,Sy,Sy)+G(Sx,y,y)}{2}\};$$

$$N'(x,y,y) = \min\{G(x,Sx,Sx), G(y,Sy,Sy), G(x,Sy,Sy), G(y,Sx,Sx)\}.$$

Then S has a unique fixed point.

In Corollary 5.3.3, for $Tx = x$; $M'(x, y, y) = G(x, y, y)$ and $C_F = 0$, we get [12, Theorem 3.1, p. 610] as a particular case.

Now, by taking $L = 0$ and $Tx = x$ in Theorem 5.3.2, we get the following result.

Corollary 5.3.4. *Let (X, G) be a complete G-metric space and $S : X \to X$ be self mapping. There exist $\eta \in \mathcal{E}_{(\mathcal{Z},F,\Gamma)}, C_F \geq 0$ and $r \in [0, \infty)$ such that*

$$\frac{1}{1+r} G(x, Sx, Sx) < G(x, y, y)$$

implies

$$\eta(G(Sx, Sy, Sy), M'(x, y, y)) \geq C_F, \text{ for all } x, y \in X.$$

Then S has a unique fixed point.

If (X, G) is symmetric, then by taking $d_G(x, y) = G(x, y, y)$, above result is a generalization of [46, Theorem 2.4, p. 425] for metric space.

5.4 Results for Geraghty contraction in G_b-metric spaces

In this section, coincidence and common fixed point results for Geraghty type contraction in G_b-metric spaces are established, which involve rectangular α_G-admissible maps. The obtained results extend the existing results in this area and offer new insights into fixed point theory.

For $s > 1$, consider the class $\mathcal{B}$ of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies that } t_n \to 0, \text{ as } n \to \infty.$$

Theorem 5.4.1. *Let (X, G) be a G_b-metric space with parameter $s \geq 1$. Let $\alpha_G : X^3 \to [0, \infty)$ and $S, T : X \to X$ be mappings with $S(X) \subseteq T(X)$. Assume that* *(i) there exists* $\eta \in \mathcal{E}_{(\mathcal{Z},F,\Gamma)}$ *such that* $\alpha_G(Tx,Ty,Tz) \geq 1$ *implies that*

$$\eta\big(G(Sx,Sy,Sz), \beta\big(M(x,y,z)\big)M(x,y,z)\big) \geq C_F, \text{ for all } x,y,z \in X, \tag{5.8}$$

where $\beta \in \mathcal{B}$ *and*

$$\begin{aligned} M(x,y,z) = \max\Big\{ & G(Tx,Ty,Tz), G(Tx,Sx,Sx), G(Ty,Sy,Sy), \\ & G(Tz,Sz,Sz), \frac{1}{3s^2}[G(Tx,Sy,Sy)+G(Ty,Sx,Sx)], \\ & \frac{1}{3s^2}[G(Ty,Sz,Sz)+G(Tz,Sy,Sy)], \\ & \frac{1}{3s^2}[G(Tx,Sz,Sz)+G(Tz,Sx,Sx)], \\ & \frac{1}{6s^2}[G(Sx,Ty,Tz)+G(Tx,Sy,Tz)+G(Tx,Ty,Sz)]\Big\}; \end{aligned}$$

(ii) S *is a rectangular* α_G*-admissible mapping for* T*;*

(iii) there exists $x_0 \in X$ *such that* $\alpha_G(Tx_0,Sx_0,Sx_0) \geq 1$*;*

(iv) $S(X)$ *is* G_b*-precomplete in* $T(X)$*;*

(v) Suppose $\{Tx_n\}$ *is a sequence in* X *with* $\alpha_G(Tx_n,Tx_{n+1},Tx_{n+1}) \geq 1$, *for all* $n \in \mathbb{N}$ *and* $Tx_n \to Tu$ *as* $n \to \infty$, *then there exists a subsequence* $\{Tx_{n_k}\}$ *of* $\{Tx_n\}$ *such that* $\alpha_G(Tx_{n_k},Tu,Tu) \geq 1$, *for all* $k \in \mathbb{N}$.

Then, S *and* T *have a coincidence point.*

Proof. Let $x_0 \in X$ be given such that $\alpha_G(Tx_0,Sx_0,Sx_0) \geq 1$. Since $S(X) \subseteq T(X)$, we get a sequence $\{x_n\}$ in X such that $Sx_n = Tx_{n+1},$ for all $n \geq 0$.

If there exists some $n_0 \in \mathbb{N}$ such that $Tx_{n_0} = Tx_{n_0+1}$ implies that $Sx_{n_0} = Tx_{n_0}$, hence x_{n_0} is a coincidence point of S and T.
Thus, now we assume that $Tx_n \neq Tx_{n+1},$ for all $n \geq 0$, that is,

$$G(Tx_n,Tx_{n+1},Tx_{n+1}) > 0 \text{ and } G(Tx_n,Tx_{n+1},Tx_{n+1}) > 0, \text{ for all } n \geq 0.$$

Now, $\alpha_G(Tx_0,Sx_0,Sx_0) = \alpha_G(Tx_0,Tx_1,Tx_1) \geq 1$. Since, S is rectangular α_G-admissible mapping for T. So, we get

$$\alpha_G(Sx_0,Sx_1,Sx_1) = \alpha_G(Tx_1,Tx_2,Tx_2) \geq 1.$$

Continuing the same procedure, we get

$$\alpha_G(Sx_{n-1}, Sx_n, Sx_n) = \alpha_G(Tx_n, Tx_{n+1}, Tx_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}.$$

From (5.8), we get

$$\begin{aligned} C_F &\leq \eta\big(G(Sx_n, Sx_{n+1}, Sx_{n+1}), \beta\big(M(x_n, x_{n+1}, x_{n+1})\big)M(x_n, x_{n+1}, x_{n+1})\big) \\ &\leq F\big(\beta\big(M(x_n, x_{n+1}, x_{n+1})\big)M(x_n, x_{n+1}, x_{n+1}), G(Sx_n, Sx_{n+1}, Sx_{n+1})\big). \end{aligned}$$

Using (η_1), we get

$$\gamma\big(G(Sx_n, Sx_{n+1}, Sx_{n+1})\big) < \gamma\big(\beta(M(x_n, x_{n+1}, x_{n+1}))M(x_n, x_{n+1}, x_{n+1})\big).$$

Since $\gamma \in \Gamma([0, \infty))$, we have

$$\begin{aligned} G(Sx_n, Sx_{n+1}, Sx_{n+1}) &< \beta\big(M(x_n, x_{n+1}, x_{n+1})\big)M(x_n, x_{n+1}, x_{n+1}) \\ &< M(x_n, x_{n+1}, x_{n+1}), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} &M(x_n, x_{n+1}, x_{n+1}) \\ &= \max\Big\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_n, Sx_n, Sx_n), G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1}), \\ &\qquad \frac{1}{3s^2}[G(Tx_n, Sx_{n+1}, Sx_{n+1}) + G(Tx_{n+1}, Sx_n, Sx_n)], \\ &\qquad \frac{1}{3s^2}[G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1}) + G(Tx_{n+1}, Sx_{n+1}, Sx_{n+1})], \\ &\qquad \frac{1}{6s^2}[G(Sx_n, Tx_{n+1}, Tx_{n+1}) + G(Tx_n, Sx_{n+1}, Tx_{n+1}) \\ &\qquad + G(Tx_n, Tx_{n+1}, Sx_{n+1})]\Big\} \\ &= \max\Big\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}), \\ &\qquad \frac{1}{3s^2}G(Tx_n, Tx_{n+2}, Tx_{n+2}), \frac{2}{3s^2}G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}), \\ &\qquad \frac{1}{3s^2}G(Tx_n, Tx_{n+1}, Tx_{n+2})\Big\} \\ &\leq \max\Big\{G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_{n+1}, Tx_{n+2}, Tx_{n+2}), \end{aligned}$$

$$\frac{1}{3s}[G(Tx_n,Tx_{n+1},Tx_{n+1})+G(Tx_{n+1},Tx_{n+2},Tx_{n+2})],$$
$$\frac{2}{3s^2}G(Tx_{n+1},Tx_{n+2},Tx_{n+2}),$$
$$\frac{1}{3s}[G(Tx_n,Tx_{n+1},Tx_{n+1})+G(Tx_{n+1},Tx_{n+1},Tx_{n+2})]\Big\}$$
$$\leq \max\Big\{G(Tx_n,Tx_{n+1},Tx_{n+1}),G(Tx_{n+1},Tx_{n+2},Tx_{n+2}),$$
$$\frac{1}{3s}[G(Tx_n,Tx_{n+1},Tx_{n+1})+G(Tx_{n+1},Tx_{n+2},Tx_{n+2})],$$
$$\frac{2}{3s^2}G(Tx_{n+1},Tx_{n+2},Tx_{n+2}),$$
$$\frac{1}{3s}[G(Tx_n,Tx_{n+1},Tx_{n+1})+2sG(Tx_{n+1},Tx_{n+2},Tx_{n+2})]\Big\}$$
$$=\max\{G(Tx_n,Tx_{n+1},Tx_{n+1}),G(Tx_{n+1},Tx_{n+2},Tx_{n+2})\}.$$

If $M(x_n,x_{n+1},x_{n+1}) = G(Tx_{n+1},Tx_{n+2},Tx_{n+2})$, for all $n \geq 0$, then from (5.9), we get a contradiction, $G(Tx_{n+1},Tx_{n+2},Tx_{n+2}) < G(Tx_{n+1},Tx_{n+2},Tx_{n+2})$.

Hence, $M(x_n,x_{n+1},x_{n+1}) = G(Tx_n,Tx_{n+1},Tx_{n+1})$, for all $n \geq 0$.

From (5.9), we get

$$G(Tx_{n+1},Tx_{n+2},Tx_{n+2}) < \beta\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)G(Tx_n,Tx_{n+1},Tx_{n+1}), \tag{5.10}$$

for all $n > 0$. Hence, $\{G(Tx_n,Tx_{n+1},Tx_{n+1})\}$ is a non-decreasing sequence. So there exists $r \geq 0$, such that

$$\lim_{n\to\infty} G(Tx_n,Tx_{n+1},Tx_{n+1}) = r \text{ and } G(Tx_n,Tx_{n+1},Tx_{n+1}) > r, \text{ for all } n > 0.$$

We claim that $r = 0$. Suppose that $r > 0$, then from (5.10), we get

$$r \leq \limsup_{n\to\infty}\ \beta\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big)r.$$

Then,

$$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty}\ \beta\big(G(Tx_n,Tx_{n+1},Tx_{n+1})\big) \leq \frac{1}{s}.$$

Since $\beta \in \mathcal{B}$, then

$$\lim_{n\to\infty} G(Tx_n,Tx_{n+1},Tx_{n+1}) = 0, \tag{5.11}$$

which is a contradiction, that is, $r = 0$.

Now, we show that $\{Tx_n\}$ is a G_b-Cauchy sequence. Suppose, on the contrary that, $\{Tx_n\}$ is not a G_b-Cauchy sequence, then there exists $\varepsilon > 0$ such that for all $k > 0$, $n_k > m_k > k$ we can find subsequences $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ with

$$G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) \leq \varepsilon < G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}). \tag{5.12}$$

Using (iv) and $\alpha_G(Tx_n, Tx_{n+1}, Tx_{n+1}) \geq 1$, we get

$$\alpha_G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) \geq 1, \text{ for all } k \in \mathbb{N}.$$

From (5.8) and (η_1), we have

$$C_F \leq \eta\big(G(Sx_{m_k-1}, Sx_{n_k-1}, Sx_{n_k-1}), \beta\big(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\big)M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\big)$$

implies that

$$G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) < \beta\big(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\big)M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), \tag{5.13}$$

where

$$\begin{aligned}
&M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\\
&= \max\Big\{G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}), G(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}),\\
&\qquad G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k}),\\
&\qquad \frac{1}{3s^2}[G(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k}) + G(Tx_{n_k-1}, Tx_{m_k}, Tx_{m_k})],\\
&\qquad \frac{1}{3s^2}[G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})],\\
&\qquad \frac{1}{6s^2}[G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) + 2G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k})]\Big\}\\
&\leq \max\Big\{G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}), G(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}),\\
&\qquad G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k}),\\
&\qquad \frac{1}{3s^2}[G(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k}) + 2sG(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{m_k})],\\
&\qquad \frac{1}{3s^2}[G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})],
\end{aligned}$$

$$\frac{1}{6s^2}[G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) + 2G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k})]\Big\}.$$

Now, Using (5.12) and ($GB5$), we get

$$\begin{aligned} \varepsilon &< G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) \\ &\leq s[G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})] \\ &\leq s[\varepsilon + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})]. \end{aligned} \tag{5.14}$$

Considering the upper limit as $k \to \infty$ in (5.14) and using (5.11), we obtain

$$\varepsilon \leq \limsup_{k\to\infty} G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) \leq s\varepsilon. \tag{5.15}$$

Now, we have

$$\begin{aligned} &G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) \\ &\leq s[G(Tx_{m_k}, Tx_{m_k-1}, Tx_{m_k-1}) + G(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k})] \\ &\leq s^2[2G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) \\ &+ G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})]; \end{aligned}$$

$$\begin{aligned} &G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) \\ &\leq s[G(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}) + G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1})]. \end{aligned}$$

Taking upper limit as $k \to \infty$ in above inequalities and using (5.11) and (5.12), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) \leq s\varepsilon.$$

Next, we have

$$\begin{aligned} G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) &\leq 2s^2G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) + sG(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k}); \\ G(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k}) &\leq s[G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})]. \end{aligned}$$

Hence,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} G(Tx_{m_k-1}, Tx_{n_k}, Tx_{n_k}) \leq s^2\varepsilon.$$

Again,

$$G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}) \leq s[G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) + G(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k})]$$

and

$$\begin{aligned}G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) \leq & s[G(Tx_{m_k}, Tx_{m_k-1}, Tx_{m_k-1}) \\ & + G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1})].\end{aligned}$$

Now, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \ G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) \leq s^2\varepsilon.$$

Finally, we have

$$\begin{aligned}G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k}) \leq & s[G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) \\ & + G(Tx_{n_k-1}, Tx_{n_k-1}, Tx_{n_k})];\end{aligned}$$

$$\begin{aligned}& G(Tx_{m_k}, Tx_{n_k-1}, Tx_{n_k-1}) \\ & \leq sG(Tx_{n_k-1}, Tx_{n_k}, Tx_{n_k}) + 2s^3G(Tx_{m_k}, Tx_{m_k}, Tx_{m_k-1}) \\ & \quad + s^2G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k}).\end{aligned}$$

Hence, we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k\to\infty} \ G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k}) \leq s^2\varepsilon. \tag{5.16}$$

Now, considering upper limit as $k \to \infty$ in $M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})$ and using (5.15)-(5.16), we get

$$\begin{aligned}\frac{\varepsilon}{s^2} &= \max\{\frac{\varepsilon}{s^2}, 0, \frac{\frac{\varepsilon}{s} + \frac{2s\varepsilon}{s}}{3s^2}, \frac{\frac{\varepsilon}{s} + \frac{2\varepsilon}{s^3}}{6s^2}\} \\ & \leq \limsup_{k\to\infty} \ M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) \\ & \leq \max\{s\varepsilon, 0, \frac{s^2\varepsilon + 2s^3\varepsilon}{3s^2}, \frac{3s^2\varepsilon}{6s^2}\} = s\varepsilon.\end{aligned}$$

Now, taking upper limit as $k \to \infty$ in (5.13), we get

$$\begin{aligned}\varepsilon &\leq \limsup_{k\to\infty}\ G(Tx_{m_k}, Tx_{n_k}, Tx_{n_k})\\ &\leq \limsup_{k\to\infty}\ \beta(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}))M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})\\ &\leq s\varepsilon \limsup_{k\to\infty}\ \beta(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}))\end{aligned}$$

implies that, $\frac{1}{s} \leq \limsup_{k\to\infty}\ \beta(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})) \leq \frac{1}{s}$.

Since $\beta \in \mathcal{B}$, so $\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) = 0$. Thus, we can conclude that $\lim_{k\to\infty} G(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}) = 0$. This contradicts (5.12). Which implies that $\{Tx_n\}$ is a G_b-Cauchy sequence.

Since $S(X)$ is G_b-precomplete in $T(X)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Tu.$$

By the condition (vi), we can say that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\alpha_G(Tx_{n_k}, Tu, Tu) \geq 1$, for all $k \in \mathbb{N}$.

Now, we claim that u is a coincidence point of S and T. To prove that, we consider $G(Tu, Su, Su) = l > 0$.

Hence, we have

$$\lim_{k\to\infty} G(Tx_{n_k}, Su, Su) = l$$

$$\text{implies that } \lim_{k\to\infty} G(Sx_{n_k}, Su, Su) = l.$$

Here,

$$\begin{aligned}M(x_{n_k}, u, u) = \max\{&G(Tx_{n_k}, Tu, Tu), G(Tx_{n_k}, Sx_{n_k}, Sx_{n_k}), G(Tu, Su, Su)\\ &\frac{1}{3s^2}[G(Tx_{n_k}, Su, Su) + G(Tu, Sx_{n_k}, Sx_{n_k})],\\ &\frac{1}{3s^2}[G(Tu, Su, Su) + G(Tu, Su, Su)],\\ &\frac{1}{6s^2}[G(Sx_{n_k}, Tu, Tu) + G(Tx_{n_k}, Su, Tu) + G(Tx_{n_k}, Tu, Su)]\}.\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty} M(x_{n_k}, u, u) = \max\{&G(Tu, Su, Su), \frac{1}{3s^2}G(Tu, Su, Su),\\ &\frac{2}{3s^2}G(Tu, Su, Su), \frac{1}{3s^2}G(Su, Tu, Tu)\}\end{aligned}$$

$$
\begin{aligned}
&\leq \max\{G(Tu,Su,Su), \frac{1}{3s^2}G(Tu,Su,Su), \frac{2}{3s^2}G(Tu,Su,Su),\\
&\qquad \frac{2}{3s}G(Tu,Su,Su)\}\\
&= G(Tu,Su,Su) = l.
\end{aligned}
$$

Consider two sequences $\{t_{n_k}\}$ and $\{s_{n_k}\}$ with

$$t_{n_k} = G(Sx_{n_k}, Su, Su) > 0$$

and

$$s_{n_k} = \beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u)$$

and using (5.8) and (η_1), we get

$$
\begin{aligned}
C_F &\leq \eta(G(Sx_{n_k}, Su, Su), \beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u))\\
&< F(\beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u), G(Sx_{n_k}, Su, Su))\\
&\text{implies that } \gamma(G(Sx_{n_k}, Su, Su)) < \gamma(\beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u)).
\end{aligned}
$$

Since, $\gamma \in \Gamma([0,\infty))$, we get

$$G(Sx_{n_k}, Su, Su) < \beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u) < M(x_{n_k}, u, u).$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} \beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u) = l.$$

By using (η_3) and (5.8), we get

$$
\begin{aligned}
\eta(G(Sx_{n_k}, Su, Su), l) &= \lim_{k\to\infty} \eta(G(Sx_{n_k}, Su, Su), \beta(M(x_{n_k}, u, u))M(x_{n_k}, u, u))\\
&> C_F,\\
\implies l = 0 &\implies G(Tu,Su,Su) = 0.
\end{aligned}
$$

Thus, $Tu = Su$. Hence, u is a coincidence point of T and S. □

The next theorem deals with the existence and uniqueness of a common fixed point.

Theorem 5.4.2. *In addition to the hypotheses of Theorem 5.4.1, suppose that* $\alpha_G(Tu, Tw, Tw) \geq 1$ *or* $\alpha_G(Tw, Tu, Tu) \geq 1$*, whenever* $u, w \in C(S,T)$ *and* S, T *commute at their coincidence points. Then* S *and* T *have a unique common fixed point.*

Proof. We claim that if $u, w \in C(S,T)$, then $Tu = Tw$. By hypotheses, there exist $u, w \in X$ such that $Tu = Su$ and $Tw = Sw$. Assume that $Tu \neq Tw$. Hence, $\alpha_G(Tu, Tw, Tw) \geq 1$ or $\alpha_G(Tw, Tu, Tu) \geq 1$. Assume that $\alpha_G(Tu, Tw, Tw) \geq 1$. Hence, we have

$$\begin{aligned} C_F &\leq \eta\big(G(Su, Sw, Sw), \beta\big(M(u,w,w)\big)M(u,w,w)\big) \\ &\leq F\big(\beta\big(M(u,w,w)\big)M(u,w,w), G(Su, Sw, Sw)\big) \\ \text{implies that } &\gamma(G(Su, Sw, Sw)) \leq \gamma\big(\beta\big(M(u,w,w)\big)M(u,w,w)\big). \end{aligned}$$

Since $\gamma \in \Gamma([0,\infty))$, we get

$$G(Su, Sw, Sw) < \beta\big(M(u,w,w)\big)M(u,w,w), \tag{5.17}$$

where

$$\begin{aligned} &M(u,w,w) \\ &= \max\Big\{G(Tu,Tw,Tw), G(Tu,Su,Su), G(Tw,Sw,Sw), \\ &\quad \frac{1}{3s^2}[G(Tu,Sw,Sw)+G(Tw,Su,Su)], \\ &\quad \frac{1}{3s^2}[G(Tw,Sw,Sw)+G(Tw,Sw,Sw)], \\ &\quad \frac{1}{6s^2}[G(Su,Tw,Tw)+G(Tu,Sw,Tw)+G(Tu,Tw,Sw)]\Big\} \\ &= G(Tu,Tw,Tw). \end{aligned}$$

From (5.17), we get

$$\frac{1}{s} \leq 1 = \frac{G(Su,Sw,Sw)}{M(u,w,w)} < \beta\big(M(u,w,w)\big) < \frac{1}{s}$$

which yields that $\limsup_{n\to\infty} \beta(M(u,w,w)) = \frac{1}{s}$ implies that $\limsup_{n\to\infty} M(u,w,w) = \limsup_{n\to\infty} G(Tu,Tw,Tw) = 0$. Therefore, $Tu = Tw$.

Existence of a common fixed point: Let $u \in C(S,T)$, that is $Su = Tu$. Due to commutativity of S and T at their coincidence points, we get

$$TTu = TSu = STu.$$

Let us denote $Tu = z^*$, then $Tz^* = Sz^*$. Thus z^* is a coincidence point of S and T. Since, we have $z^* = Tu = Tz^* = Sz^*$. Then, z^* is a common fixed point of S and T.

Uniqueness: Assume that w^* is another common fixed point of S and T. Then $w^* \in C(S,T)$. We have $w^* = Tw^* = Tz^* = z^*$. □

5.5 Application to nonlinear integral equations

In this section, we present an application of Theorem 5.4.2 to guarantee the existence of a solution to an integral equation.

Let $X = C[0,l], l > 0$ be the set of all continuous functions defined on $[0,l]$ and let $G : X \times X \times X \to \mathbb{R}$ be defined by

$$G(x,y,z) = \Big(\sup_{t\in[0,l]} \mid x(t) - y(t) \mid + \sup_{t\in[0,l]} \mid y(t) - z(t) \mid + \sup_{t\in[0,l]} \mid z(t) - x(t) \mid \Big)^2.$$

Then (X,G) is a complete G_b-metric space with $s = 2$.

Consider the integral equation,

$$x(t) = \int_0^l H(t,s)K\big(s,T(x(s))\big)ds, \tag{5.18}$$

where $l > 0$, $H : [0,l]\times[0,l] \to \mathbb{R}$ and $K : [0,l]\times\mathbb{R} \to \mathbb{R}$ are continuous functions and $T : X \to X$ is a self mapping on X.

Now we present the following theorem.

Theorem 5.5.1. *Suppose the following hypotheses holds:*

(1) for all $t,s \in [0,l]$ and $x,y \in X$, we have

$$\mid K\big(s,T(x(t))\big) - K\big(s,T(y(t))\big) \mid \leq \frac{\sqrt{e^{-M(x,y,y)}M(x,y,y)}}{2},$$

(2) for all $t, s \in [0, l]$, we have

$$\sup_{t\in[0,l]} \int_0^l H(s,t)ds = \frac{1}{4l}.$$

Then the integral equation (5.18) has a solution.

Proof. Let $S : X \to X$ be a mapping defined by

$$S(x(t)) = \int_0^l H(t,s)K\big(s, T(x(s))\big)ds, \quad t \in [0,l],\ x \in X.$$

From condition (1) and (2), we have

$$\begin{aligned}
G(Sx, Sy, Sy) &= \Big(2 \sup_{t\in[0,l]} \mid S(x(t)) - S(y(t)) \mid \Big)^2 \\
&= \Big(2 \sup_{t\in[0,l]} \mid \int_0^l H(t,s)K\big(s, T(x(s))\big)ds - \int_0^l H(t,s)K\big(s, T(y(s))\big)ds \mid \Big)^2 \\
&\le \Big(2 \sup_{t\in[0,l]} \int_0^l H(t,s) \mid K\big(s, T(x(s))\big) - K\big(s, T(y(s))\big) \mid ds\Big)^2 \\
&\le 4\Big(\sup_{t\in[0,l]} \int_0^l H(t,s)ds\Big)^2 \Big(\sup_{t\in[0,l]} \int_0^l \mid K\big(s, T(x(s))\big) - K\big(s, T(y(s))\big) \mid ds\Big)^2 \\
&\le \frac{1}{4l^2}\Big(\sup_{t\in[0,l]} \int_0^l \frac{\sqrt{e^{-M(x,y,y)}M(x,y,y)}}{2} ds\Big)^2 \\
&= \frac{1}{4}\ \frac{e^{-M(x,y,y)}}{2} M(x,y,y).
\end{aligned}$$

So, we get

$$G(Sx, Sy, Sy) \le \frac{1}{4}\beta(M(x,y,y))M(x,y,y). \tag{5.19}$$

Let $\eta(t,s) = \frac{1}{4}\gamma(s) - \gamma(t)$, $F(s,t) = \gamma(s) - \gamma(t)$, for all $s, t \in [0,\infty), C_F = 0$, $\gamma(t) = 2t$ and $\beta(t) = \frac{e^{-t}}{2}$, for all $t \in [0,\infty)$. Then it is clear that $\beta \in \mathcal{B}$. Also, we take $\alpha(x,y,y) = 1$, for all $x, y \in X$.
Now,

$$\begin{aligned}
&\eta\Big(G(Sx, Sy, Sy), \beta\big(M(x,y,y)\big)M(x,y,y)\Big) \\
&= \frac{1}{4}\gamma\Big(\beta\big(M(x,y,y)\big)M(x,y,y)\Big) - \gamma\big(G(Sx, Sy, Sy)\big)
\end{aligned}$$

$$= \frac{1}{4}\Big(2\beta\big(M(x,y,y)\big)M(x,y,y)\Big) - 2G(Sx,Sy,Sy).$$

Then from (5.19), we have

$$\eta\Big(G(Sx,Sy,Sy), \beta\big(M(x,y,y)\big)M(x,y,y)\Big) \geq 0.$$

Thus all the conditions of Theorem 5.4.2 are satisfied and hence S and T have a unique common fixed point $x \in X$. Thus x is a solution of the integral equation (5.18) □