Chapter 6

Non-linear Contractions via Generalized $\Gamma - C_F$-simulation Functions

6.1 Introduction

Simulation functions have huge applications in non-linear functional analysis. The simplicity and usefulness of these functions have inspired many researchers to diversify it further. Motivated by this dynamic concept, Golshan [25] introduced the notion of generalized simulation functions by modifying the condition (ζ_2) of simulation function of Khojasteh et al. [38]. Further, Golshan introduced the notion of weak ζ-contraction as a generalization of contraction mapping in context of metric spaces and demonstrated fixed point result with a new proof of the main result of Khojasteh et al. [38], under weaker conditions.

Definition 6.1.1. [25, p.2] A function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a (generalized) simulation function of type I if

(ξ_1) There exists function $\phi : [0, \infty) \to \mathbb{R}$ such that

$$\text{if } \xi(t, s) \geq 0 \text{ then } \xi(t, s) \leq \phi(s) - \phi(t), \text{ for all } s, t \geq 0.$$

(ξ_2) If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$.

We say that ξ is a (generalized) simulation function of type II if it satisfies (ξ_1) and the following $(\xi_2)^*$ condition.

$(\xi_2)^*$ If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0, \infty)$ and $\xi(t_n, s_n) \geq 0$ then

$$\lim_{n\to\infty} \xi(t_n, s_n) \to 0 \text{ implies } s_n \to 0.$$

In this chapter, we introduce (generalized) $\Gamma - C_F$-simulation functions using $\Gamma - C$-class functions, which generalize and extend the notion of generalized simulation functions [25]. Subsequently, weak (η_F, T)-contraction for pair of mappings is introduced to establish common fixed point result via such functions in the framework of G-metric spaces. Further, this result is extended to quasi-metric spaces and metric spaces by using the methods of Jleli and Samet [29] and Samet et al. [54].

6.2 Results for weak contraction in G-metric spaces

In this section, firstly, (generalized) $\Gamma - C_F$-simulation function is defined. Subsequently, weak (η_F, T)-contraction is defined for pair of mappings in a G-metric spaces. Further, common fixed point result is established for such contraction in G-metric spaces.

Definition 6.2.1. A function $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a (generalized) $\Gamma - C_F$-simulation function of type I if

(η_1) There exists $C_F \geq 0$ such that

$$\text{if } \eta(t, s) \geq C_F \text{ then } \eta(t, s) < F(s, t), \text{ for all } s, t \geq 0,$$

where $F \in \mathcal{C}_\Gamma$ with property $\Gamma - C_F$.

(η_2) If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \eta(t_n, s_n) < C_F$.

We say that η is a (generalized) $\Gamma - C_F$-simulation function of type II if it satisfies (η_1) and the following $(\eta_2)^*$ condition.

$(\eta_2)^*$ If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0, \infty)$ and $\eta(t_n, s_n) \geq C_F$ then

$$\lim_{n\to\infty} \eta(t_n, s_n) \to C_F \text{ implies } s_n \to 0.$$

Remark 5. Every (generalized) simulation function of type I and type II are (generalized) $\Gamma - C_F$-simulation function of type I and type II respectively.

Proof. Results follows from definition 6.2.1, by considering $F(s,t) = \gamma(s) - \gamma(t)$, where $\gamma \in \Gamma([0,\infty))$ and $C_F = 0$. □

Definition 6.2.2. Let (X, G) be a G-metric space and S, T be self mappings on X. For a function $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, S is called

(i) an (η_F, T)-contraction if

$$\eta(G(Sx, Ty, Ty), G(x, y, y)) \geq C_F, \text{ for all } x, y \in X, \tag{6.1}$$

$$\eta(G(Tx, Sy, Sy), G(x, y, y)) \geq C_F, \text{ for all } x, y \in X; \tag{6.2}$$

(ii) a weak (η_F, T)-contraction if

$$\eta(G(Sx, TSx, TSx), G(x, Sx, Sx)) \geq C_F, \text{ for all } x \in X, \tag{6.3}$$

$$\eta(G(Tx, STx, STx), G(x, Tx, Tx)) \geq C_F, \text{ for all } x \in X; \tag{6.4}$$

(iii) a generalize weak non-expansive map if

$$G(Sx, TSx, TSx) \leq G(x, Sx, Sx), \text{ for all } x \in X, \tag{6.5}$$

$$G(Tx, STx, STx) \leq G(x, Tx, Tx), \text{ for all } x \in X. \tag{6.6}$$

If we take $T = S$ in (6.1)-(6.6), we get the following contractions.
A mapping S is called

(a) an η_F-contraction if

$$\eta(G(Sx, Sy, Sy), G(x, y, y)) \geq C_F, \text{ for all } x, y \in X; \tag{6.7}$$

(b) a weak η_F-contraction if

$$\eta(G(Sx, S^2x, S^2x), G(x, Sx, Sx)) \geq C_F, \text{ for all } x \in X; \tag{6.8}$$

(c) a weak non-expansive map if

$$G(Sx, S^2x, S^2x) \leq G(x, Sx, Sx), \text{ for all } x \in X. \tag{6.9}$$

In definition 6.2.2, for $d_G(x,y) = G(x,y,y)$, an (η_F, T)-contraction for G-metric spaces reduced to (η_F, T)-contraction for quasi-metric spaces (X, d_G).

Main result of this chapter is stated here.

Theorem 6.2.3. *Let (X,G) be a complete G-metric space, S and T be self mappings on X and $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a function.*

(i) Let S be an (η_F, T)-contraction. If η satisfies (η_1), then S and T have at most one common fixed point.
Also, if $\gamma \in \Gamma([0,\infty))$ then

$$G(Sx, Ty, Ty) < G(x,y,y), \text{ for all } x \neq y.$$

(ii) Let η be a $\Gamma - C_F$-simulation function of type II, if $S(TS)^{n_0}$ and $(TS)^{n_0}$, $n_0 \in \mathbb{N}$ be a weak (η_F, T)-contraction then S is T-asymptotically regular. The same result holds true if η be a $\Gamma - C_F$-simulation function of type I and S be a generalized weak non-expansive map.

(iii) Let S be an (η_F, T)-contraction with S or T is continuous and η be a $\Gamma - C_F$-simulation function of type II (or type I and S be generalized weak non-expansive map) then S and T have a unique common fixed point.

Proof. (i) Suppose that $Tx = Sx = x$, $Ty = Sy = y$ and $x \neq y$, then $G(x,y,y) = G(Sx,Ty,Ty) = t(say) > 0$.
From (η_1) and $(\mathcal{F}_2)$, we get

$$\eta(t,t) \geq C_F \implies \eta(t,t) < F(t,t) \leq C_F,$$

which is a contradiction. Hence common fixed point of S and T is unique if exists.
Suppose that $0 < s = G(x,y,y) \leq t = G(Sx,Ty,Ty)$, where $x \neq y$.
From (6.1) and (η_1), we have

$$C_F \leq \eta(G(Sx,Ty,Ty), G(x,y,y)) < F(G(x,y,y), G(Sx,Ty,Ty)).$$

From $(\mathcal{F}_1)$, we get

$$\gamma(G(Sx,Ty,Ty)) < \gamma(G(x,y,y)).$$

Since γ is non-decreasing, $G(Sx, Ty, Ty) < G(x, y, y)$, which is a contradiction. Hence $G(Sx, Ty, Ty) < G(x, y, y)$.

(ii) For any fixed x_0 in X construct a sequence $\{x_n\}$ with

$$x_{2n} = (TS)^n(x_0),\ x_{2n+1} = S(x_{2n}),\ \text{for all } n \geq 0.$$

Let $t_i = G(x_i, x_{i+1}, x_{i+1}),$ for all $i \geq 0$. Suppose $t_k = 0$, for some $k \in \mathbb{N}$.
If $x_{2k} = x_{2k+1}$, then x_{2k} is a fixed point of S.
If $x_{2k+1} = x_{2k+2}$, then x_{2k+1} is a fixed point of T.
Thus, at least one of S or T has a fixed point.
Now, assume that $t_k \neq 0,$ for all $k \geq 0$.
Put $x = x_{2n_0+2k} = (TS)^{n_0+k}(x_0), k = 0, 1, \cdots$ in (6.3), we get

$$\begin{aligned} C_F &\leq \eta\big(G(Sx_{2n_0+2k}, TSx_{2n_0+2k}, TSx_{2n_0+2k}), \\ &\qquad G(x_{2n_0+2k}, Sx_{2n_0+2k}, Sx_{2n_0+2k})\big) \\ &= \eta\big(G(x_{2n_0+2k+1}, x_{2n_0+2k+2}, x_{2n_0+2k+2}), \\ &\qquad G(x_{2n_0+2k}, x_{2n_0+2k+1}, x_{2n_0+2k+1})\big) \\ &= \eta(t_{2n_0+2k+1}, t_{2n_0+2k}) \\ &< F(t_{2n_0+2k}, t_{2n_0+2k+1}). \end{aligned} \tag{6.10}$$

Put $x = x_{2n_0+2k+1} = S(TS)^{n_0+k}(x_0), k = 0, 1, \cdots$ in (6.4), we get

$$\begin{aligned} C_F &\leq \eta(G(Tx_{2n_0+2k+1}, STx_{2n_0+2k+1}, STx_{2n_0+2k+1}), \\ &\qquad G(x_{2n_0+2k+1}, Tx_{2n_0+2k+1}, Tx_{2n_0+2k+1})) \\ &= \eta(G(x_{2n_0+2k+2}, x_{2n_0+2k+3}, x_{2n_0+2k+3}), \\ &\qquad G(x_{2n_0+2k+1}, x_{2n_0+2k+2}, x_{2n_0+2k+2})) \\ &= \eta(t_{2n_0+2k+2}, t_{2n_0+2k+1}) \\ &< F(t_{2n_0+2k+1}, t_{2n_0+2k+2}). \end{aligned} \tag{6.11}$$

From (6.10) and (6.11), we get

$$C_F \leq \eta(t_{i+1}, t_i) < F(t_i, t_{i+1}),\ \text{for all } i \geq n_0. \tag{6.12}$$

From $(\mathcal{F}_1)$, we get $\gamma(t_{i+1}) < \gamma(t_i)$. Since, γ is non decreasing $t_{i+1} < t_i$, that is

$G(x_{i+1}, x_{i+2}, x_{i+2}) < G(x_i, x_{i+1}, x_{i+1})$, for all $i \geq n_0$. Hence $\{G(x_i, x_{i+1}, x_{i+1})\}$ is monotonically decreasing sequence of non negative real numbers. Thus there exists $r \geq 0$ such that $\lim_{i\to\infty} G(x_i, x_{i+1}, x_{i+1}) = r$.
To prove $r = 0$, suppose that $r > 0$.
Taking limit as $i \to \infty$ in (6.12) and using $(\mathcal{F}_2)$, we get

$$C_F \leq \lim_{i\to\infty} \eta(t_{i+1}, t_i) \leq F(\lim_{i\to\infty} t_i, \lim_{i\to\infty} t_{i+1}) = F(r, r) \leq C_F.$$

Hence,

$$\lim_{i\to\infty} \eta(t_{i+1}, t_i) = C_F. \tag{6.13}$$

Type II: From $(\eta_2)^*$, we get $r = \lim_{i\to\infty} t_i = 0$, a contradiction.
Type I: From (6.5) and (6.6), we have $t_{i+1} \leq t_i$, for all $i \geq 0$. Using (η_2), we get $\limsup_{i\to\infty} \eta(t_{i+1}, t_i) < C_F$, a contradiction to (6.13). Hence, $r = 0$. Hence

$$\lim_{i\to\infty} G(x_i, x_{i+1}, x_{i+1}) = 0. \tag{6.14}$$

Since $G(x_i, x_i, x_{i+1}) \leq 2G(x_i, x_{i+1}, x_{i+1})$, we get

$$\lim_{i\to\infty} G(x_i, x_i, x_{i+1}) = 0. \tag{6.15}$$

(iii) We shall show that $\{x_n\}$ is a G-Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a G-Cauchy sequence. Assume that $\{x_{2n}\}$ is not a G-Cauchy sequence. Then from Lemma 1.3.14, there exist $\varepsilon > 0$ and two subsequences $\{x_{2n(k)}\}$ and $\{x_{2m(k)}\}$ of $\{x_{2n}\}$ such that, for all $k \in \mathbb{N}$, $k \leq 2n(k) < 2m(k) < 2n(k+1)$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G(x_{2n(k)+p_1}, x_{2m(k)+p_2}, x_{2m(k)+p_3}) = \varepsilon. \tag{6.16}$$

Considering two non increasing subequences

$$a_l = G(x_{2n(k)(l)}, x_{2m(k)(l)}, x_{2m(k)(l)})$$

and

$$a'_l = G(x_{2n(k)(l)+2}, x_{2m(k)(l)+2}, x_{2m(k)(l)+2})$$

of $G(x_{2n(k)}, x_{2m(k)}, x_{2m(k)})$ and $G(x_{2n(k)+2}, x_{2m(k)+2}, x_{2m(k)+2})$

such that

$$\lim_{l\to\infty} a_l = \lim_{l\to\infty} a_l' = \varepsilon. \tag{6.17}$$

From (6.1) and (η_1), we have

$$C_F \leq \eta(a_l', a_l) < F(a_l, a_l').$$

Letting $l \to \infty$, we get

$$C_F \leq \lim_{l\to\infty} \eta(a_l', a_l) \leq F(\lim_{l\to\infty} a_l, \lim_{l\to\infty} a_l') = F(\varepsilon, \varepsilon) \leq C_F.$$

This implies,

$$\lim_{l\to\infty} \eta(a_l', a_l) = C_F. \tag{6.18}$$

Type II: From $(\eta_2)^*$, $\lim_{l\to\infty} a_l = 0$, a contradiction to (6.17).
Type I: From (η_2), we get, $\limsup_{l\to\infty} \eta(a_l, a_l') < C_F$, a contradiction to (6.18).
Thus $\{x_{2n}\}$ is a G-Cauchy sequence. Hence $\{x_n\}$ is G-Cauchy sequence. Since (X, G) is complete, $x_n \to u \in X$, implies that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$$

Assume S is continuous, then $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = Su$. Implies that $Su = u$.
From (6.1), we have

$$\begin{aligned} C_F &\leq \eta(G(Su, TSu, TSu), G(u, Su, Su)) \\ &= \eta(G(u, Tu, Tu), G(u, u, u)) \\ &< F(G(u, u, u), G(u, Tu, Tu)). \end{aligned}$$

From ($\mathcal{F}_1$), we get $0 \leq \gamma(G(u, Tu, Tu)) < \gamma(G(u, u, u)) = \gamma(0) = 0$. Since $\gamma \in \Gamma([0, \infty))$, we get $G(u, Tu, Tu) = 0$, implies that $Tu = u$. The uniqueness follows from part (i).

□

The following example validates our result.

Example 6.2.1. Let $X = [0, 1]$. Define $G : X^3 \to [0, \infty)$ as

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases} \tag{6.19}$$

Then (X, G) is a complete G-metric space. Define $S, T : X \to X$ as $S(x) = \frac{x}{2+x}$ and $T(x) = \frac{x}{2}$, for all $x \in X$. Also define $\gamma : [0, \infty) \to [0, \infty)$ by

$$\gamma(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 2t, & \text{if } 1 \leq t \end{cases}$$

and $\eta : [0, \infty)^2 \to \mathbb{R}$ by

$$\eta(t, s) = \frac{\gamma(s)}{1 + \gamma(s)} - \gamma(t), \text{ for all } t, s \in [0, \infty).$$

Taking $F(s, t) = \gamma(s) - \gamma(t)$ with $C_F = 0,$ for all $s, t \in [0, \infty)$. Then η is a (generalized) $\Gamma - C_F$-simulation function type I and all the conditions of Theorem 6.2.3 are satisfied and $x = 0$ is the unique common fixed point of S and T.

6.3 Consequences: Common fixed point results in quasi-metric spaces and metric spaces

In this section, firstly the weak (η_F, T)-contraction for quasi-metric spaces is defined. Further, the common fixed point result is extended to quasi-metric and metric spaces. The result obtained for metric spaces generalizes the result of Golshan [25, Theorem 2.4, p.6].

Definition 6.3.1. Let (X, d) be a quasi-metric space and S, T be self mappings on X. For a function $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, S is called

(i) an (η_F, T)-contraction if

$$\eta(d(Sx, Ty), d(x, y)) \geq C_F, \text{ for all } x, y \in X, \tag{6.20}$$

$$\eta(d(Tx, Sy), d(x, y)) \geq C_F, \text{ for all } x, y \in X, \tag{6.21}$$

(ii) a weak (η_F, T)-contraction if

$$\eta(d(Sx, TSx), d(x, Sx)) \geq C_F, \text{ for all } x \in X, \tag{6.22}$$

$$\eta(d(Tx, STx), d(x, Tx)) \geq C_F, \text{ for all } x \in X, \tag{6.23}$$

(iii) a generalize weak non-expansive map if

$$d(Sx, TSx) \leq d(x, Sx), \text{ for all } x \in X. \tag{6.24}$$

$$d(Tx, STx) \leq d(x, Tx), \text{ for all } x \in X. \tag{6.25}$$

If we take $T = S$ in (6.20)-(6.25), we get the following contractions.
A mapping S is called

(a) an η_F-contraction if

$$\eta(d(Sx, Sy), d(x, y)) \geq C_F, \text{ for all } x, y \in X, \tag{6.26}$$

(b) a weak η_F-contraction if

$$\eta(d(Sx, S^2x), d(x, Sx)) \geq C_F, \text{ for all } x \in X, \tag{6.27}$$

(c) a weak non-expansive map if

$$d(Sx, S^2x) \leq d(x, Sx), \text{ for all } x \in X. \tag{6.28}$$

Remark 6. In (6.26) and (6.27), $C_F = 0$ reduced to ξ-contraction and weak ξ-contraction of [25] respectively.

Theorem 6.2.3 in context of quasi-metric spaces is stated as follows.

Theorem 6.3.2. *Let (X, d) be a complete quasi-metric space, S and T be self mappings on X and $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function.*

(i) Let S be an (η_F, T)-contraction. If η satisfies (η_1), then S and T have at most one common fixed point.
Also, if $\gamma \in \Gamma([0, \infty))$ then

$$d(Sx, Ty) < d(x, y), \text{ for all } x \neq y.$$

Conditions (ii) and (iii) of Theorem 6.2.3 holds. S and T have a unique common fixed point.

Proof. In Theorem 6.2.3, take $d_G(x, y) = G(x, y, y)$, then result follows from Theorem D. □

Theorem 6.3.2 is also valid in context of metric spaces. Now, if we consider (X, d) as a complete metric space then based on Theorem 6.3.2, Theorem 2.4 in [25] can be improved as follows.

Corollary 6.3.3. *Let (X, d) be a complete metric space, S be a self mapping on X and $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function.*

(i) Let S be an ξ-contraction. If ξ satisfies (ξ_1), then S has at most one common fixed point.
Also, if $\gamma \in \Gamma([0, \infty))$ then

$$d(Sx, Sy) < d(x, y), \text{ for all } x \neq y.$$

(ii) Let ξ be a simulation function of type II, if S^{n_0}, $n_0 \in \mathbb{N}$ be a weak ξ-contraction then S is asymptotically regular. The same result holds true if ξ be a simulation function of type I and f be a weak non-expansive map.

(iii) Let S be an ξ-contraction with S is continuous and ξ be a simulation function of type II (or type I and S be weak non-expansive map) S has a unique fixed point.

Proof. In Theorem 6.3.2, if we take $T = S$, $F(s, t) = \gamma(s) - \gamma(t)$ and $C_F = 0$, then (η_F, T)-contraction reduces to ξ-contraction in [25]. □

Remark 7. Thus, Corollary 6.3.3 generalizes [25, Theorem 2.4, p.6] for weaker hypothesis. In Corollary 6.3.3, we do not require the condition (7) and (8) of Theorem 2.4-(ii) and condition (10) of Theorem 2.4-(iii) of [25].