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Introduction

Fixed point theory is a very important area of pure and applied mathematics. It is a well-established but still-developing area of mathematical research. A fixed point theorem is a statement that specifies conditions on a mapping and its domain guaranteeing that a mapping has a fixed point. The Banach contractive mapping principle is the most celebrated result in fixed point theory. Thereafter, many researchers proposed coincidence or common fixed point results as one of the methods of generalizing the Banach contractive mapping principle, which can be done by increasing the number of involved mapping. Jungck [13] introduced the concept of common fixed points, which led to the development of the common fixed point theory as a dynamic area of study. Mathematicians subsequently introduced numerous novel concepts, such as coincidence points, compatible mappings, weakly compatible mappings, commuting mappings, and other related ones, over time. Also, this principle has several extensions to various spaces such as D-metric spaces [8], b-metric spaces [6], G-metric spaces [19], G_b-metric spaces [1] and several others. Fixed point theory for G-metric spaces is a specialized area of fixed point theory that deals with the properties of mappings on spaces that are equipped with a generalized metric or G-metric, which allows for a more flexible definition of distance than the traditional metric. The study of common fixed points is an active and important area of research in fixed point theory for G-metric spaces, as it has many open problems and a wide range of applications in various areas, such as optimization, neural network, approximation theory, integral and differential equations, control theory, numerical analysis and several others. In order to unify all the linear contractions, Khojasteh et al. [16] introduced the notion of simulation function, $\mathcal{Z}$-contraction mapping and proved the existence and uniqueness of fixed points. In recent years, the concept of simulation function has been utilized and improved by several authors and accordingly, the literature is well furnished with fixed point results via simulation functions (see [4, 7, 14, 18, 28]). In this thesis, motivated by all the above mention types of generalizations of the Banach contraction principle, many interesting coincidences or common fixed point results in the context of G-metric spaces are cultivated. Further, examine the possibilities of their applications, in the domain of integral equations and neural networks. This thesis consists of six chapters. Here, we need the following definitions and notations as a prerequisite.

Definition 1. [1] Let X be a nonempty set, $s \geq 1$ and $G_b : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

$(GB1)$ $G_b(x, y, z) = 0$, if $x = y = z$,

$(GB2)$ $G_b(x, x, y) > 0,$ for all $x, y \in X$ with $x \neq y$,

$(GB3)$ $G_b(x, x, y) \leq G(x, y, z),$ for all $x, y, z \in X$ with $z \neq y$,

$(GB4)$ $G_b(x, y, z) = G_b(p\{x, y, z\})$, where p is a permutation of x, y, z,

$(GB5)$ $G_b(x, y, z) \leq s[G_b(x, a, a) + G_b(a, y, z)],$ for all $x, y, z, a \in X$.

Then G_b is called generalized b-metric on X and the pair (X, G_b) is called a G_b-metric space. Note that, for $s = 1$, G_b-metric space reduces to G-metric space.

Definition 2. [7] A subset E of a G_b-metric space (X, G_b) is said to be *precomplete* if every Cauchy sequence in E converges to a point of X.

Definition 3. [12] Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function such that the following are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$, for any points $x, y, z \in X$.

Then, d is called a quasi-metric on X and the pair (X, d) is called a quasi-metric space.

Definition 4. [32] Let $(X, \preceq)$ be a partially ordered set and A, B be two closed subsets of X with $X = A \cup B$. Let $f, g : X \to X$ be two mappings. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \preceq gfx, \forall x \in A$ and $gx \preceq fgx, \forall x \in B$.

Definition 5. [16] A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$, for all $t, s > 0$;

(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Definition 6. [2] A function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following conditions:

(i) $F(s, t) \leq s, \ \forall s, t \geq 0$;

(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0, \ \forall s, t \geq 0$.

The collection of all C-class functions is denoted by $\mathcal{C}$.

Definition 7. [18] A function $F : [0, \infty)^2 \to \mathbb{R}$ has the property $\mathcal{C}_F$, if there exists $C_F \geq 0$ such that

(i) $F(s, t) > C_F$ implies $s > t$, $\forall s, t \geq 0$;

(ii) $F(t, t) \leq C_F, \ \forall t \geq 0$.

Definition 8. [18] A C_F-simulation function is a function $\zeta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(i) $\zeta(t, s) < F(s, t) \ \forall t, s > 0$, where $F \in \mathcal{C}$ with property $\mathcal{C}_F$;

(ii) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_F$.

The family of all C_F-simulation functions is denoted by $\mathcal{Z}_F$.

Definition 9. [30] Let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is triangular α-admissible for g if T satisfies the following conditions:

(i) $\alpha(gx, gy) \geq 1 \implies \alpha(Tx, Ty) \geq 1, \ \forall x, y \in X$;

(ii) $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1 \implies \alpha(gx, gz) \geq 1, \ \forall x, y, z \in X$.

Definition 10. [10] Let $T, g : X \to X$ and $\alpha_w : X^3 \to [0, \infty)$ be mappings. We say that T is a weak α_w-admissible mapping with respect to g, if for all $x, y \in X$, we have

$$\alpha_w(gx, gy, gy) \geq 1 \implies \alpha_w(Tx, Ty, Ty) \geq 1.$$

Notations:

- An altering distance function is a continuous, non-decreasing mapping $\phi : [0, \infty) \to [0, \infty)$ such that $\phi^{-1}(0) = 0$. The family of all altering distance functions is denoted by F_{alt}.
- A function $\psi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is monotone increasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n is n^{th} iterate of ψ. The collection of all comparison functions is denoted by F_{com}.
- Ψ is the family of all mappings $\psi : [0, \infty) \to [0, \infty)$ such that, if $\{t_m\} \subset [0, \infty)$ and $\psi(t_m) \to 0$ then $t_m \to 0$.
- $\Gamma([0, \infty))$ is the set of all non-decreasing functions $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(t) = 0$ if and only if $t = 0$.
- Φ denotes the collection of non-decreasing, continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \phi(t_n) = 0$ if and only if $\lim_{n\to\infty} t_n = 0$.
- For $s > 1$, consider the class $\mathcal{B}$ of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies that } t_n \to 0, \text{ as } n \to \infty.$$

Chapter 1 The first chapter is introductory and provides the development of the subject and necessary background to the rest of the chapters in the thesis.

Chapter 2 In 2003, Kirk et al. [17] introduced cyclic contraction to generalize the Banach contraction principle. An important advantage to this approach is that cyclic contractions, unlike Banach-type contractions, need not be continuous. Such contractions are further generalized by Shatanawi and Postolache [32], by introducing the pair of (A, B)- weakly increasing mappings. Jleli and Samet [12], Samet et al. [29] have shown that G-metric space has a quasi-metric type structure and then many results on such spaces can be derived from quasi-metric spaces. To study fixed point and common fixed point results in G-metric spaces, Shatanawi and Abodayeh [31] introduced a new contractive condition that cannot be reduced to quasi-metric spaces using the methods of Jleli and Samet [12] or Samet et al. [29]. In Chapter 2, we introduce generalized cyclic contractive condition by utilizing the notion of (A, B)-weakly increasing mappings and the notion of altering distance function in the sense of Khan et al. [15]. Also, we establish common fixed point results for generalized cyclic contraction in the setting of G-metric spaces which generalize the previous result of Shatanawi and Abodayeh [31] as follows:

- Let $\preceq$ be an ordered relation in a set X. Let (X,G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let (f,g) be (A,B)-weakly increasing mappings with f or g is continuous, $f(A) \subseteq B$ and $g(B) \subseteq A$. Then there exist two functions $\phi \in F_{alt}, \psi \in \Psi$ such that

$$\phi\big(G(fx,gfx,gy)\big) \leq \phi\big(M(x,y)\big) - \psi\big(M(x,y)\big) \tag{1}$$

holds for all comparative elements $x,y \in X$ with $x \in A$ and $y \in B$ and

$$\phi\big(G(gx,fgx,fy)\big) \leq \phi\big(M'(x,y)\big) - \psi\big(M'(x,y)\big) \tag{2}$$

holds for all comparative elements $x,y \in X$ with $x \in B$ and $y \in A$, where

$$M(x,y) = \max\Big\{G(x,fx,y), G(x,fx,fx), G(y,gy,gy), \frac{1}{2}\Big(G(fx,fx,gy), G(x,gfx,gy), G(fx,gfx,y)\Big)\Big\}$$

and

$$M'(x,y) = \max\Big\{G(x,gx,y), G(x,gx,gx), G(y,fy,fy), \frac{1}{2}\Big(G(gx,gx,fy), G(x,fgx,fy), G(gx,fgx,y)\Big)\Big\}.$$

Then, f and g have a common fixed point in $A \cap B$.

For the next result, we consider C-class [2] functions which cover a large class of contractive conditions.

- The above theorem holds true, if conditions (1) and (2) are replaced by conditions (3) and (4) respectively, as defined below:

$$\phi\big(G(fx,gfx,gy)\big) \leq F\big(\phi(N(x,y)), \psi(N(x,y))\big) \tag{3}$$

and

$$\phi\big(G(gx,fgx,fy)\big) \leq F\big(\phi(N'(x,y)), \psi(N'(x,y))\big), \tag{4}$$

where F is a C-class function,

$$N(x,y) = \max\Big\{G(x,fx,y), \frac{G(fx,fx,y)[1+G(x,x,gy)]}{1+G(x,fx,y)}, \frac{G(gy,gy,y)[1+G(fx,fx,x)]}{1+G(x,fx,y)}\Big\}$$

and

$$N'(x,y) = \max\Big\{G(x,gx,y), \frac{G(gx,gx,y)[1+G(x,x,fy)]}{1+G(x,gx,y)}, \frac{G(fy,fy,y)[1+G(gx,gx,x)]}{1+G(x,gx,y)}\Big\}.$$

Finally, some examples are furnished to support the usability of the results obtained.

Some results of this chapter are published in **Thai Journal of Mathematics (Ref. [23])**.

Chapter 3 To express different contractivity conditions in a simple and unified way, Khojasteh et al. [16] introduced the notion of simulation functions. Thus, it is possible to treat several fixed point problems from a unique common point of view. Later, Roldań et al. [7] investigated the existence and uniqueness of coincidence points via simulation functions in the setting of quasi-metric spaces and deduced corresponding results in the framework of G-metric spaces. Liu et al. [18] extended the class of simulation functions by using C-class functions of Ansari [2] and introduced C_F-simulation functions and proved the existence and uniqueness of coincidence and common fixed point for two operators.

In Chapter 3, we study Ćirić type contractions for compatible mappings via simulation functions in the framework of quasi-metric spaces and its consequences to G-metric spaces.

For that, we introduce $(\mathcal{Z}_{(\alpha,F)}, g)$-quasi-contraction of Ćirić type using α-admissible [30] mappings.

Definition 11. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $f, g : X \to X$ be given mappings. A mapping f is called a $(\mathcal{Z}_{(\alpha,F)}, g)$-quasi-contraction of Ćirić type if there exist $\zeta \in \mathcal{Z}_F$ and $\lambda \in (0, 1)$ such that

$$\zeta\big(\alpha(gx, gy)d(fx, fy), \lambda M(gx, gy)\big) \geq C_F$$

for all $x, y \in X$, where $M(gx, gy) = \max\Big\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\Big\}$.

One of the main results of this chapter is stated below:

- Let (X, d) be a quasi-metric space, f, g be self mappings on X with $f(X) \subset g(X)$. If f is a $(\mathcal{Z}_{(\alpha,F)}, g)$-quasi-contraction of Ćirić type satisfying the following conditions:

  (i) f is triangular α-admissible for g;

  (ii) there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ and $\alpha(fx_0, gx_0) \geq 1$;

  (iii) at least, one of the following conditions hold:

    (a) $f(X)$ is precomplete in $g(X)$.

    (b) (X, d) is a complete quasi-metric space and f and g are continuous and compatible.

  Then, f and g have a point of coincidence.

The above result generalizes the result claimed by Radenovic et al. [25, pp. 147].

Next, the generalized Ćirić type $\mathcal{Z}_F$-contraction for pair of mappings is defined as follows:

Definition 12. Let (X, d) be a quasi-metric space and $f, g : X \to X$ be self-mappings. We say that (f, g) is a generalized Ćirić type $\mathcal{Z}_F$-contractive pair of mappings if there exist $\zeta \in \mathcal{Z}_F$ and $\lambda \in (0, 1)$ such that

$$\zeta\big(d(fx, gy), \lambda M(x, y)\big) \geq C_F,$$

where $M(x,y) = \max\Big\{ d(x,y), d(y,fx), d(x,gy), d(x,fx), d(y,gy) \Big\}$ and

$$\zeta\big(d(gy,fx), \lambda M(y,x)\big) \geq C_F,$$

where $M(y,x) = \max\Big\{ d(y,x), d(fx,y), d(gy,x), d(fx,x), d(gy,y) \Big\}$ for all $x, y \in X$.

Another main result of this chapter is given below:

- A theorem that ensures the existence and uniqueness of a common fixed point for a pair of generalized Ćirić type $\mathcal{Z}_F$-contractive self mappings on a complete quasi-metric space is established.

Now, in the above both results, by considering $d(x,y) = G(x,y,y)$ and $\alpha(x,y) = \alpha_G(x,y,y)$, we derive common fixed point results in G–metric spaces.

Some results of this chapter are published in **Problemy Analiza-Issues of Analysis (Ref. [22])**.

Chapter 4 Rhoades' problem on discontinuity at fixed points is one of the interesting problems of fixed point theory. Rhoades [27] brought up the issue of whether there is a contractive condition strong enough to produce a fixed point but which does not require the map to be continuous at the fixed point. Following the initial answer provided by R. P. Pant [21], several further solutions to this open problem have been offered using various techniques. In this chapter, we study this problem for three maps in the setting of G_b-metric spaces and establish a condition for which the common fixed point is a point of discontinuity. For that, we introduce a generalized (ψ, ϕ)-Wardowski contraction for three maps and prove a common fixed point result for such maps. Further, we discuss its application to neural networks.

In this work, the concept of a generalized $(\psi, \phi) - G_b$-Wardowski contraction, for three self mappings, is defined as below:

Definition 13. Let f, g, h be self-mappings defined on the G_b-metric space (X, G). Suppose that there exist $\phi \in \Phi$ and $\psi \in F_{com}$, such that

$$G(fx, gy, hz) > 0 \implies \phi\big(2s^4 G(fx, gy, hz)\big) \leq \psi\big(\phi(M(x,y,z))\big)$$

for all $x, y, z \in X$, where

$$M(x,y,z) = \max\Big\{ G(x,y,z), G(x,fx,gy), G(y,gy,hz), G(z,hz,fx), \frac{1}{4s}[G(fx,y,z) + G(x,gy,z) + G(x,y,hz)] \Big\}.$$

Then, we say that (f, g, h) is a generalized $(\psi, \phi) - G_b$-Wardowski contraction.

The core theorem of this chapter is given below:

- Let $f, g, h : X \to X$ be a generalized $(\psi, \phi) - G_b$-Wardowski contraction in a complete G_b-metric space. Then f, g, h have a unique common fixed point $u \in X$, also $f^n x \to u$, $g^n x \to u$ and $h^n x \to u$, for each $x \in X$. Moreover, at least one of f, g and h is not continuous at u if and only if

$$\lim_{x \to u} M(x,u,u) \neq 0, \;\; \lim_{y \to u} M(u,y,u) \neq 0 \; or \; \lim_{z \to u} M(u,u,z) \neq 0.$$

Subsequently, an example is given to affirm the consistency of the obtained theory. Further, some important corollaries of the above statement are given. In fixed point theorems, contractive mappings that admit discontinuity at the fixed point have found applications in neural networks with discontinuous activation functions (e.g. $\ddot{O}zg\ddot{u}r$ and Tas [20] and Rashid et al. [26]). Furthermore, an application of our result by considering discontinuous activation functions occurring in the neural networks is discussed.

This work is published in the journal **Acta et Commentationes Universitatis Tartuensis de Mathematica (ACUTM) (Ref. [24])** .

Chapter 5 Recently, the notion of the simulation functions has been extended and generalized in various ways, like Γ-simulation functions [14], extended simulation functions [28], extended C_F-simulation functions [4] and many others. On the other hand, the Banach contractive principle has been generalized by many authors by modifying the contraction. Some of the generalizations are Geraghty type contractions [9], Suzuki type contraction [33], the notion of almost contraction [3], etc. This chapter intends to make use of the theories from above mentioned different types of contractions and simulation functions to furnish a couple of related coincidences and common fixed point results. To achieve these results, we introduce the notion of $\Gamma - C$-class function, property $\Gamma - C_F$ of function and extended $\Gamma - C_F$-simulation functions and illustrate the definition by some non-trivial examples. As an application, the existence of a solution of a non-linear integral equation is given.

Definition 14. A function $F : [0, \infty)^2 \to \mathbb{R}$ is called $\Gamma - C$-class function if it is continuous and there exists $\gamma \in \Gamma([0, \infty))$ such that:

(i) $F(s, t) \leq \gamma(s), \ \forall s, t \geq 0$;

(ii) $F(s, t) = \gamma(s)$ implies that either $s = 0 \ or \ t = 0, \ \forall s, t \geq 0.$

The collection of all $\Gamma - C$-class functions is denoted by $\mathcal{C}_\Gamma$.

Definition 15. A function $F : [0, \infty)^2 \to \mathbb{R}$ has the property $\Gamma - C_F$, if there exist $\gamma \in \Gamma([0, \infty))$ and $C_F \geq 0$ such that:

(F_1) $F(s, t) > C_F$ implies $\gamma(s) > \gamma(t), \ \forall s, t \geq 0$;

(F_2) $F(t, t) \leq C_F, \ \forall t \geq 0.$

Definition 16. An extended $\Gamma - C_F-$ simulation function is a function $\eta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(η_1) $\eta(t, s) < F(s, t) \ \forall t, s > 0$, where $F \in \mathcal{C}_\Gamma$ with property $\Gamma - C_F$;

(η_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l > 0$$

and $s_n > l, \ for \ all \ n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \eta(t_n, s_n) < C_F;$$

(η_3) let $\{t_n\}$ be a sequence in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = l \in [0, \infty),$$

then

$$\eta(t_n, l) \geq C_F \implies l = 0.$$

The class of an extended $\Gamma - C_F$-simulation functions is denoted by $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$. Further, we define the pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contractive mappings via extended $\Gamma - C_F$-simulation functions in the framework of G-metric spaces.

Definition 17. Let (X, G) be a G-metric space and $f, g : X \to X$ be self mappings on X. We say that (f, g) is the pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contractive mappings, if there exist $r \in [0, 1), L \geq 0$ and $\eta \in \mathcal{E}_{(\mathcal{Z},F,\Gamma)}$ such that

$$\frac{1}{1+r} \min\left\{G(gx, fx, fx), G(gy, fy, fy)\right\} \leq G(gx, gy, gy)$$

implies

$$\eta\Big(G(fx, fy, fy), M(x, y, y) + L\, N(x, y, y)\Big) \geq C_F\ \forall x, y \in X,$$

where

$$M(x, y, y) = \max\left\{G(gx, gy, gy), G(gx, fx, fx), G(gy, fy, fy), \frac{G(gx, fy, fy) + G(fx, gy, gy)}{2}\right\}$$

and

$$N(x, y, y) = \min\Big\{G(gx, fx, fx), G(gy, fy, fy), G(gx, fy, fy), G(gy, fx, fx)\Big\}.$$

Two important theorems of this chapter are given below:

- A theorem that ensures the unique existence of a common fixed point of a pair of almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$- contractive mappings in framework of G-metric spaces is stated. This result generalizes Theorem 3.1 of Charry and Reddy [5].
- A theorem that enables us to verify the existence of the unique coincidence and common fixed point for two self mappings using the Geraghty function and weak α_w-admissible maps via extended $\Gamma - C_F$-simulation functions in framework of G_b-metric spaces is stated.

Supporting examples for the above mentioned results are furnished. Finally, the existence and uniqueness of a solution to a class of non-linear integral equations are studied using Geraghty type contraction.

Results of this chapter are **communicated**.

Chapter 6 Recently, Golshan [11] generalized the concept of simulation function of Khojasteh et al. [16] and proved fixed point results for weak ζ-contraction in the context of metric spaces. Motivated by Golshan [11], in this chapter, we introduce a generalized $\Gamma - C_F$-simulation function as an extension of the generalized simulation function using $\Gamma - C$-class function. Also, we define weak (η_F, g)- contraction for pair of mappings and prove common fixed point results for G-metric spaces and consequences to quasi-metric spaces and metric spaces. In this work, the concept of generalized $\Gamma - C_F$-simulation function is defined as below:

Definition 18. A function $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a (generalized) $\Gamma - C_F$-simulation function of type II if

(η_1) There exists $C_F \geq 0$ such that

$$\text{if } \eta(t,s) \geq C_F \text{ then } \eta(t,s) \leq F(s,t), \forall s,t \geq 0,$$

where $F \in \mathcal{C}_\Gamma$ with property $\Gamma - C_F$,

(η_2) $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0,\infty)$ and $\eta(t_n, s_n) \geq C_F$ such that

$$\text{if } \lim_{n\to\infty} \eta(t_n, s_n) \to C_F \text{ then } s_n \to 0.$$

We say that η is a (generalized) $\Gamma - C_F$-simulation function of type I if it satisfies (η_1) and the following $(\eta_2)^*$ condition.

$(\eta_2)^*$ If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \eta(t_n, s_n) < C_F$.

Now, we introduce weak (η_F, g)-contraction for pair of self mappings in G-metric spaces.

Definition 19. Let (X, G) be a G-metric space and f, g be self mappings on X. For a function $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$, f is called

(i) an (η_F, g)-contraction if

$$\eta\big(G(fx, gy, gy), G(x, y, y)\big) \geq C_F, \tag{5}$$
$$\eta\big(G(gx, fy, fy), G(x, y, y)\big) \geq C_F \ \forall x, y \in X; \tag{6}$$

(ii) a weak (η_F, g)-contraction if

$$\eta\big(G(fx, gfx, gfx), G(x, fx, fx)\big) \geq C_F, \tag{7}$$
$$\eta\big(G(gx, fgx, fgx), G(x, gx, gx)\big) \geq C_F \ \forall x \in X; \tag{8}$$

(iii) a generalize weak non-expansive map if

$$G(fx, gfx, gfx) \leq G(x, fx, fx), \tag{9}$$
$$G(gx, fgx, fgx) \leq G(x, gx, gx) \ \forall x \in X. \tag{10}$$

The main results of this chapter are as below:

- Let f be an (η_F, g)-contraction. If η satisfies (η_1), then f and g have at most one common fixed point. Also, if $\gamma \in \Gamma([0,\infty))$ then

$$G(fx, gy, gy) < G(x, y, y) \ \ \forall x \neq y.$$

- A unique common fixed point theorem for an (η_F, g)-contractive mapping via $\Gamma - C_F$-simulation function in a complete G-metric space is proved.

In the above results, for $G(x, y, y) = d_G(x, y)$, an (η_F, g)-contraction for G-metric spaces reduces to an (η_F, g)-contraction for quasi-metric spaces (X, d_G). Thus, we derive common fixed point results for quasi-metric spaces. Also, by considering the symmetry of the space, our results for metric spaces generalize the results of Golshan [11, Theorem 2.4] for a weaker hypothesis. Results of this chapter are **communicated**.

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PUBLISHED/ACCEPTED RESEARCH ARTICLES

1. S. V. Puvar, R. G. Vyas, Coincidence and Common Fixed Point results in G-metric Spaces using Generalized Cyclic Contraction, **Thai Journal of Mathematics**, (2022), 20(3), 1109-1117. ISSN: 1686-0209. **(Scopus, ESCI) (MR4524767).**

2. S. V. Puvar, R. G. Vyas, Ćirić type results in quasi-metric spaces and G-metric spaces using simulation function, **Problemy Analiza-Issues of Analysis**, (2022), 11(2), 72-90. ISSN: 2306-3424. **(Scopus, ESCI) (MR4459168).**
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3. S. V. Puvar, R. G. Vyas, (ψ, ϕ)-Wardowski contraction for three maps in G_b-metric spaces, **Acta et Commentationes Universitatis Tartuensis de Mathematica (ACUTM)**, (2023), 27(1), 69-82. ISSN: 1406-2283. **(Scopus, ESCI).**
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COMMUNICATED RESEARCH WORK

1. S. V. Puvar, R. G. Vyas, Rational type cyclic contraction in G-metric Spaces.

2. S. V. Puvar, R. G. Vyas, Common fixed point results of Ćirić type contraction via C_F-simulation function in quasi-metric space.

3. S. V. Puvar, R. G. Vyas, Common Fixed Point Results via $\Gamma - C_F$-simulation Function in G-metric spaces.

4. S. V. Puvar, R. G. Vyas, Almost Suzuki Type Common Fixed Point Results via extended $\Gamma - C_F$-Simulation Functions in G-metric spaces.

5. S. V. Puvar, R. G. Vyas, Geraghty Type Contraction via Extended $\Gamma - C_F$-simulation Function in G_b-metric spaces: Coincidence and Common Fixed Point Results.

PRESENTED RESEARCH WORK IN CONFERENCES

1. S. V. Puvar, Ćirić type results via Simulation and C-class function in quasi-metric spaces and G-metric spaces, at the 87^{th} Annual Conference of Indian Mathematical Society organized by MGM University, Aurangabad during December 4-7, 2021. (Virtual mode)

2. S. V. Puvar, Common fixed point results in G-metric spaces using rational type cyclic contraction via C-class function, at the 88^{th} Annual Conference of the Indian Mathematical Society organized by BIT Mesra during December 27-30, 2022.