Executive Summary of the Ph.D. Thesis

Common Fixed Point Theorems in G-metric Spaces

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Chapter 1: Introduction

In mathematics and science, a non-linear phenomenon is a system in which the change of the output is not proportional to the change of the input. Although linearity plays an important role in mathematics and its applications, non-linearity is still more important and useful, since many real-world problems are non-linear and also non-smooth in nature. In non-linear analysis, fixed point theory is a fundamental and versatile branch of mathematics with wide-ranging applications in various fields. This deals with the restrictions and limitations of the underlying spaces and also offers challenges while investigating for fixed point properties of these mappings. Indeed in fixed point theory, the contractive conditions on mappings enact the utmost important role in obtaining the solution of fixed point problems. Thus, many mathematicians have extended, complemented and generalized the existing contraction conditions and then employed it to conceive fixed point results in different abstract spaces.

In the course of this thesis, we endow several major common fixed point theorems in the framework of G-metric spaces and demonstrate some fundamental results in an analysis by reducing non-linear problems to fixed point problems.

Chapter 2: Non-linear Cyclic Contractions in G-metric Spaces

Non-linear cyclic contractions provide a mathematical framework for modeling complex systems with non-trivial dynamics. Many real-world phenomena, such as biological systems, economic dynamics, and ecological interactions, exhibit non-linear behavior. Cyclic contractions help characterize the convergence properties of iterative methods used to approximate fixed points.

Combining non-linear contraction and cyclic behavior in the context of G-metric spaces provides a framework for analyzing the dynamics of mappings on sets endowed with generalized metrics. This framework is applicable in various mathematical and applied contexts where traditional metric spaces may not adequately capture the underlying geometry or structure of the space involved. Applications include the study of dynamical systems, optimization problems, and iterative algorithms in settings where non-linear and cyclic phenomena play a crucial role.

In this direction, the concept of (A, B)-weakly increasing mappings is employed to introduce generalized cyclic contractive conditions and rational type cyclic contractive conditions in G-metric spaces. These contractions are utilized to establish common fixed point results in the framework of G-metric spaces which generalize Theorem 2.1 of Shatanawi and Abodayeh [19]. Moreover, numerical example is furnished to validate obtained result.

Chapter 3: Non-linear Ċiriċ Contractions via C_F-Simulation Functions

Banach contraction principle is generalized in many ways, each introducing a broader class of mappings with contraction-like properties. These generalizations are important due to their applicability in diverse areas, including functional analysis, dynamic systems, optimization, and solving equations. Further, these contractions are extended by using different auxiliary functions. One of them is simulation function by Khojasteh et al. [7]. They introduced $\mathcal{Z}$-contractions and presented fixed point theorems for such contractions in complete metric spaces. Later, the class of simulation functions is extended to C_F-simulation functions [8].

On the other hand, Samet et al. [18] introduced the notion of admissible mappings, $\alpha - \psi$ contractive type mappings and extended existing fixed point results in the literature.

In this chapter, Ćirić type contraction via simulation functions and C_F-simulation functions using α-admissible mappings is introduced. Further, we investigate sufficient conditions for the existence and uniqueness of coincidence point and common fixed point for such contraction in quasi-metric spaces. The obtained results give solution to the open problem posed by Radenovic and Chandok [14].

Chapter 4: (ψ, ϕ)-Wardowski Contractions

In order to generalize the contraction condition, the concept of metric spaces has been extended in many aspects. In particular, Czerwik [3] introduced the concept of b-metric spaces and established the Banach contraction principle in this framework. Recently, Aghajani et al. [1] defined the G_b -metric spaces by using the notions of b-metric spaces and G-metric spaces, and they discussed some basic properties of G_b -metric. On the other hand, Rhoades [15] proposed the following problem:

" Is it possible to establish a contractive condition that ensures the existence and uniqueness of a fixed point, without requiring the continuity of the mapping at that fixed point?"

After the first solution given by R. P. Pant [9], several solutions to this open problem have been presented via different approaches. Currently, fixed points of discontinuous mappings in neural networks have gained attention of researchers. The objective of this chapter is to find out contractive condition which does not force the mapping to be continuous at their common fixed points. For this, generalized $(\psi, \phi)-G_b$-Wardowski contraction for three mappings is introduced to establish a common fixed point theorem in setting of complete G_b-metric spaces. Further, its application to neural networks is discussed.

Chapter 5: Non-linear Contractions via Extended $\Gamma - C_F$-simulation Functions

The class of simulation functions has been extended and generalized in various ways, like Γ-simulation functions [6], extended simulation functions [16], extended C_F-simulation functions [2] and many others. In this chapter, we have introduced $\Gamma - C$-class function and extended $\Gamma - C_F$-simulation functions. Further, the almost Suzuki type $\mathcal{E}_{(\mathcal{Z},F,\Gamma)}$-contraction in G-metric spaces and Geraghty type contraction in G_b-metric spaces are introduced using $\Gamma - C_F$-simulation functions.

Furthermore, the implication of Geraghty type contraction for non-linear integral equations is explored and discussed.

Chapter 6: Non-linear Contractions via Generalized $\Gamma - C_F$-simulation Functions

The intent of this chapter is to make use of the theories from [7] and generalized simulation functions [4] to furnish a common fixed point result in the framework of G-metric spaces. To achieve this result, we conceive the notion of generalized $\Gamma - C_F$-simulation functions and illustrate the definition by some non-trivial examples. Besides, we construct pertinent examples and deduce several related and existing results to demonstrate the applicability of our obtained theorem. Further, this result is extended to quasi-metric spaces and metric spaces by using the methods of Jleli and Samet [5] and Samet et al. [17].

Conclusion

In this discourse, we have studied the existence of common fixed points for generalized cyclic contractions [11] and rational type cyclic contractions [13] in G-metric spaces by utilizing the notion of (A, B)-weakly increasing mappings and α-admissible mappings. Furthermore, by utilizing the simulation functions and C-class functions, Ćirić type contractions [10] are studied to examine existence and uniqueness of common fixed points in the context of quasi-metric spaces and its consequences to G-metric spaces are discussed. Also constructive examples are provided to check validity of the conceived results.

Moreover, generalized (ψ, ϕ)-Wardowski contractions [12] for three mappings is introduced to establish a condition for discontinuity of common fixed point for three mappings in G_b-metric spaces. Further, its application to neural networks is discussed.

Finally, extended $\Gamma - C_F$-simulation functions and generalized $\Gamma - C_F$-simulation functions are introduced to study common fixed point results for Geraghty type contractions, almost Suzuki type contractions and weak contractions for pair of mappings. Application to integral equation is provided. These obtained results correlate, generalize and unify several results in the exiting literature.

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