Chapter 1

Introduction

Fixed point theory is a fundamental and versatile branch of mathematics with wide-ranging applications in various fields. It provides a powerful framework for understanding the existence and uniqueness of fixed points of function, making it a valuable tool in both pure and applied mathematics. It includes classical results for proving the existence and uniqueness theorems in ordinary differential equations, partial differential equations, integral equations, matrix equations, functional equations, iterated function systems, variational inequalities etc. Numerous problems in various branches of mathematics can be recast as fixed point problems, which have roots in functional analysis, topology, operator theory, fractal theory, differential geometry, eigenvalue problems, approximation theory, among others. Significantly, the applications of fixed point theory extend beyond Mathematics such as Statistics, Operation Research, Computer Science, Engineering, Physics, Chemistry, Biology, Medical Science, Economics and several others.

Let T be a self mapping on a non-empty set X. A point $x \in X$ is called a fixed point of the operator T, if $Tx = x$, we denote $x \in Fix(T)$.

The existence of a fixed point for a mapping is guaranteed by a fixed point theorem, which states specific requirements for a mapping and its domain. The most renowned result in fixed point theory is the Banach Contractive Mapping Principle [7], which proclaims "Every contraction mapping of a complete metric space into it self has a unique fixed point." Note that a contraction mapping is continuous, but a continuous mapping need not be a contraction. One limitation of the Banach contraction principle is that the mapping must be continuous on its entire domain. As a solution, Kannan [34] presented a weaker contraction condition to examine the existence of fixed points as follows:

Let (X, d) be a complete metric space and $f : X \to X$ be a self-map. If there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq \lambda[d(x, fx) + d(y, fy)], \forall x, y \in X, \tag{1.1}$$

then f possesses a unique fixed point.

Thereafter, Chatterjea [13] proved the above result, replacing condition (1.1) by the following:

$$d(fx, fy) \leq \lambda[d(x, fy) + d(y, fx)], \forall x, y \in X.$$

However, the Kannan contraction is not an extension of the Banach contraction. Subrahmanyam [59] affirmed that this result characterizes the metric completeness. In the subsequent period, the idea of fixed points was further enhanced by numerous expansions and extensions like almost-contraction, occasional contraction, asymptotic pointwise contraction, etc. These alternative fixed point theorems are valuable because they apply to a broader range of mappings, including those that may not strictly satisfy the conditions of the Banach contraction principle.

Although most of the aforementioned linear contractions do not require the mapping to be continuous across the entire domain, they do require that the mapping to be continuous at the fixed point. With this in mind, Rhoades [52] compared around 250 contractive definitions and proposed the following intriguing open question:

" Is it possible to establish a contractive condition that ensures the existence and uniqueness of a fixed point, without requiring the continuity of the mapping at that fixed point?"

After the first solution given by R. P. Pant [47], several solutions to this open problem have been presented via different approaches [45, 48, 51]. Afterward, the Banach contraction principle and all of the aforementioned findings are expanded. These expansions involve relaxation or modification of the standard Banach contraction principle's assumptions to accommodate various forms of nonlinear mappings or different spaces. The following are three typical methods for enhancing and extending the Banach contraction principle.

(i) Extending the contraction conditions to various general contraction conditions.

(ii) Increasing the number of involved mappings.

(iii) Replacing the metric spaces with the various generalized metric spaces.

1.1 Theory of generalized contractions

The Banach contraction principle employs Lipschitz contraction, in which the contraction constant is less than 1. To extend the Banach contraction principle, many authors have introduced different implicit functions that also cover different types of non-linear contractions of the existing literature. In this direction, Khojasteh et al. [38] introduced the notion of simulation functions and $\mathcal{Z}$-contraction mappings, which unify all the linear contractions. Moreover, the authors also examined the existence and uniqueness of fixed points for such contractions. This makes fixed point theory a powerful tool to study various mathematical, scientific and engineering problems. Later, Roldan et al. [18] sharpened the notion of simulation functions and also proved coincidence and common fixed point results.

Definition 1.1.1. [18, p.346] A simulation function is a function $\zeta : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$, for all $t, s > 0$;

(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Set of all simulation functions is denoted by $\mathcal{Z}$.

Example 1.1.1. Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the function defined by $\zeta(t, s) = \lambda s - t$, where $\lambda \in (0, 1)$. Then, $\zeta \in \mathcal{Z}$.

Alongside simulation functions, the concept of $\mathcal{Z}$-contraction mappings was introduced as a broader class of contraction mappings. $\mathcal{Z}$-contraction mappings expand the standard definition of contractions and offer a framework for discovering fixed points in a broader range of situations. Roldan et al. [18] introduced $\mathcal{Z}$-contraction for pair of mappings as follows:

Definition 1.1.2. [18, p.348] Let (X, d) be a metric space, $T, g : X \to X$ be self mappings. Then T is called a $(\mathcal{Z}, g)$-contraction if there exists $\zeta \in \mathcal{Z}$ such that

$$\zeta(d(Tu, Tv), d(gu, gv)) \geq 0, \text{ for all } u, v \in X \text{ and } gu \neq gv.$$

If g is the identity mapping on X, we say that T is a $\mathcal{Z}$-contraction for ζ.

In recent years, the concept of simulation functions has been utilized and improved by several authors and accordingly, the literature is well furnished with fixed point results via simulation functions (see [11, 17, 36, 41, 19]).The following points highlight the significance of simulation functions.

(i) **Metric space extensions:** Although the Banach contraction principle and other fixed-point theorems are commonly employed in metric spaces, simulation functions can be utilized in more general spaces, including partial metric spaces, G-metric spaces, quasi-metric spaces, b-metric spaces, etc. This expansion enables the extension of fixed-point theory to a broader range of spaces.

(ii) **Nonlinear mappings:** Many practical problems involve nonlinear functions, which do not satisfy the Lipschitz contraction condition. Simulation functions provide a way to address such nonlinear mappings and establish the existence of fixed points.

(iii) **Generalization:** Simulation functions generalize and extend the contraction condition by providing a wider class of nonlinear contractions. Using this notion, many contraction conditions can be addressed from one platform.

(iv) **Overcoming limitations:** The notion of simulation functions overcomes the limitations of standard fixed point problems. If the problem does not satisfy the standard fixed point framework, simulation functions provide a way to redefine and extend the contraction condition to find the solution to a fixed point problem.

On the other hand, the concept of α-admissible mappings was first introduced by Samet et al. [55] to generalize the Banach contraction principle. α-admissible mappings are generally used to solve problems for which the traditional contraction condition might not be applicable. Karapinar [37] extended this to triangular

α-admissible mappings. Later, Shahi [56] introduced this notion for two mappings as follows:

Definition 1.1.3. [56, p.302] For a nonempty set X, let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is α-admissible for g if

$$\alpha(gx, gy) \geq 1 \implies \alpha(Tx, Ty) \geq 1, \text{ for all } x, y \in X.$$

For $g = i_X$ (identity mapping on X), T is an α-admissible mapping.

Definition 1.1.4. [49, p.75] For a nonempty set X, let $T, g : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is triangular α-admissible for g if

(i) T is α-admissible for g;

(ii) $\alpha(gx, gy) \geq 1$ and $\alpha(gy, gz) \geq 1 \implies \alpha(gx, gz) \geq 1$, for all $x, y, z \in X$.

Alghamdi et al. [4] generalized this definition to three variables. Further, Kutbi et al. [40] extended this definition for two mappings as follows:

Definition 1.1.5. [40, p.4] For a nonempty set X, let $T, g : X \to X$ and $\alpha_G : X^3 \to [0, \infty)$ be mappings. We say that T is rectangular α_G-admissible for g, if

(i) $\alpha_G(gx, gy, gz) \geq 1 \implies \alpha_G(Tx, Ty, Tz) \geq 1$, for all $x, y, z \in X$.

(ii) $\alpha_G(gx, gy, gy) \geq 1$ and $\alpha_G(gy, gz, gz) \geq 1 \implies \alpha_G(gx, gz, gz) \geq 1$.

For $g = i_X$ (identity mapping on X), T is a rectangular α_G-admissible mapping in the sense of Alghamdi et al. [4].

Ghosh et al. [24] weakened the concept of rectangular α_G-admissible mappings of Kutbi et al. [40] by introducing weak α_w-admissible mappings for two mappings as follows.

Definition 1.1.6. [24, p.57] For a nonempty set X, let $T, g : X \to X$ and $\alpha_w : X^3 \to [0, \infty)$ be mappings. We say that T is a weak α_w-admissible mapping for g, if for all $x, y \in X$ we have

$$\alpha_w(gx, gy, gy) \geq 1 \implies \alpha_w(Tx, Ty, Ty) \geq 1.$$

In the above definition, if we consider g as an identity mapping, then T is weak α_w-admissible mapping.

1.2 Theory of common fixed points

Let T, g be self mappings on a non-empty set X. A point $x \in X$ is called:

- a coincidence point of T and g, if $Tx = gx$, we denote $x \in C(T, g)$;
- a common fixed point of T and g, if $Tx = gx = x$.

Although the fixed point theorem for one single-valued mapping is a powerful tool, numerous problems in pure and applied mathematics can be transformed into problems involving more than one mapping. In many real world applications like game theory and network theory, there are often multiple players or components involved, which represent mappings. Common fixed points represent equilibrium or stability in the system. When the system involves more than one mapping that has a different roles, common fixed points correspond to analyze whether all these aspects are in balance. Isbell [28] raised a question on the common fixed point as follows:

" Let (T, g) be a pair of two commuting (that is, $Tgx = gTx$, for all $x \in X$) continuous self-mappings on the unit interval. Do they have a common fixed point?"

Boyce [10] and Huneke [27] answered this question negatively by constructing a pair of commuting mappings having no common fixed point. The counter example can be defined more explicitly as follows [27]:

For each real valued function h defined on a subset of the reals, let $h^* = 1 - h(1 - x)$. Now choose any $b \in [0, \frac{1}{2}]$, define

$$s = \frac{3 - 2b + (6 - 4b)^{1/2}}{1 - 2b}$$

and three linear functions:

$$\begin{aligned} h_1(x) &= sx - sb + b; \\ h_2(x) &= 2 - h_1(x); \\ h_3(x) &= -h_2(x). \end{aligned}$$

Let

$$x_1 = h_1^{-1}(1) \quad ; \quad x_2 = h_3^{-1}(0) \quad ; \quad x_3 = h_3^{-1}(1 - b);$$
$$x_4 = h_3^{*-1}(1) \quad ; \quad x_5 = h_2^{*-1}(h_2^{-1}(0)) \quad ; \quad x_6 = h_1^{*-1}(0).$$

Define the functions f and g as follows:

$g(x) = b$ for $x \in [0, b]$; $g(x) = h_1(x)$ for $x \in [b, x_1]$
$g(x) = h_2(x)$ for $x \in [x_1, x_2]$; $g(x) = h_3(x)$ for $x \in [x_2, x_3]$
$g(x) = h_1^{*-1}(g(h_3^*(x)))$ for $x \in [x_3, x_4]$; $g(x) = h_1^{*-1}(g(h_2^*(x)))$ for $x \in [x_4, x_5]$
$g(x) = h_2^{*-1}(g(h_2^*(x)))$ for $x \in [x_5, x_6]$; $g(x) = h_2^{*-1}(g(h_1^*(x)))$ for $x \in [x_6, 1-b]$
$g(x) =$ fixed point of h_2^* for $x \in [1-b, 1]$
and $f = g^*$ for $x \in [0, 1]$.

The functions f and g are commuting and satisfy the Lipschitz condition:
$|f(x) - f(y)| \leq s|x - y|$, for all $x, y \in [0, 1]$ but does not have a common fixed point.

Thus, coincidence and common fixed point theorems for contractive type mappings necessarily require certain suitable hypotheses on the underlying structure and also sometimes, on the mappings. In this direction, Jungck [31] introduced the concept of compatible mapping and proved common fixed point theorem, which led to the development of the common fixed point theory as a dynamic area of study. Over time, mathematicians subsequently introduced numerous novel concepts, such as weakly compatible mappings, subcompatible mappings and other related ones.

Definition 1.2.1. [32, p.772] Let T, g be self mappings on a metric space (X, d). We say that T and g are compatible if and only if

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0,$$

for all sequences $\{x_n\} \subseteq X$ such that the sequences $\{gx_n\}$ and $\{Tx_n\}$ are convergent and have the same limit.

Note that, if T and g are commuting, then T and g are compatible.

Definition 1.2.2. [33, p.200] Let T, g be self mappings on a metric space (X, d). We say that T and g are weakly compatible mappings if they commute at their coincidence points, that is, for any $x \in X$,

$$Tx = gx \implies Tgx = gTx.$$

In addition to the condition of commutativity, some weaker hypothesis on the underlying space is required. However, an iterated sequence is generally required to start with this algorithm. In this direction, the concept of Picard-Jungck sequence is defined as follows:

Definition 1.2.3. [18, p.349] Let $T, g : X \to X$ be self mappings on X. A sequence $\{x_n\}$ in X is said to be a Picard-Jungck sequence of the pair (T, g) (based on x_0) if $gx_{n+1} = Tx_n$, for all $n \geq 0$.

If $T(X) \subseteq g(X)$, then there exists a Picard-Jungck sequence of (T, g) based on any point $x_0 \in X$.

1.3 Theory of generalized metric spaces

Many mathematicians extended the Banach contraction principle in the setting of metric spaces to other generalized metric spaces, such as D-metric spaces [22], b-metric spaces [15], G-metric spaces [43], G_b-metric spaces [3] and several others.

In 1963, Gähler [26] first proposed the notion of a 2-metric. Later, it is observed that a 2-metric is not a continuous function of its variables compared to a standard metric. Therefore, it was difficult to connect the results from metric spaces and 2-metric spaces. Due to this, Dhage [22] proposed the idea of a D-metric. However, Mustafa and Sims [43] found that most of the topological properties of the D-metric were incorrect. To overcome the drawbacks of a D-metric, Mustafa and Sims [43] introduced the notion of a G-metric. The authors examined the topological properties of this space and illustrated, how G-metric spaces can employ an equivalent concept to the Banach contraction mapping and numerous fixed point theorems on G-metric spaces have been established since then.

Definition 1.3.1. [43, p.290] Let X be a nonempty set and $G : X^3 \to [0, \infty)$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$, if $x = y = z$,

(G_2) $G(x, x, y) > 0$, for all $x, y \in X$ with $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$(rectangle inequality).

The function G is called a G-metric on X and the pair (X,G) is called a G-metric space. In this case, $G(x,y,z)$ can be interpreted as the perimeter of the triangle with vertices x,y,z, that is,

$$G(x,y,z) = |x-y| + |y-z| + |z-x|, \text{ for all } x,y,z \in X.$$

then (X,G) is a G-metric space.

Here are some fundamental definitions and results of G-metric spaces obtained by Mustafa and Sims [43].

Definition 1.3.2. [43, p.290] A G-metric space (X,G) is said to be *symmetric* if $G(x,y,y) = G(y,x,x)$, for all $x,y \in X$.

We list few examples of G-metric spaces below:

Example 1.3.1. Let (X,G) be a G-metric on X, then $G_* : X^3 \to [0,\infty)$ defined by

$$G_*(x,y,z) = \frac{G(x,y,z)}{1+G(x,y,z)},$$

for all $x,y,z \in X$ is G-metric on X.

Example 1.3.2. Let $X = [0,\infty)$ and $G : X^3 \to [0,\infty)$ be defined by

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}, & \text{otherwise,} \end{cases}$$

for all $x,y,z \in X$ is G-metric on X.

The following are the basic properties of G-metric spaces.

Lemma 1.3.3. *[43, p.291] Let (X,G) be a G-metric space. Then for any $x,y,z \in X$ the following properties hold:*

(i) *If $G(x,y,z) = 0$, then $x = y = z$.*

(ii) $G(x,y,y) \leq 2G(y,x,x)$.

(iii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$.

Now, we see relation between metric spaces and G-metric spaces.
Every metric induces G-metric on X in different ways.

Lemma 1.3.4. *[43, p.290] If (X, d) is a metric space, then the following functions $G_m^d, G_s^d : X^3 \to [0, \infty)$ defined by*

$$G_m^d = \max\{d(x, y), d(y, z), d(z, x)\},$$

$$G_s^d = d(x, y) + d(y, z) + d(z, x),$$

for all $x, y, z \in X$ are G-metrics on X. Furthermore,

$$G_m^d(x, y, z) \leq G_s^d(x, y, z) \leq 3G_m^d(x, y, z), \text{ for all } x, y, z \in X.$$

Conversely, any G-metric on X also induces some metrics on X.

Lemma 1.3.5. *[43, p.292] If (X, G) is a G-metric space, then the following functions $d_m^G, d_s^G : X^2 \to [0, \infty)$ defined by*

$$d_m^G = \max\{G(x, y, y), G(y, x, x)\},$$

$$d_s^G = G(x, y, y) + G(y, x, x),$$

for all $x, y \in X$, are metrics on X. Furthermore,

$$d_m^G(x, y) \leq d_s^G(x, y) \leq 2d_m^G(x, y), \text{ for all } x, y \in X.$$

Also, d_m^G and d_s^G are equivalent metrics on X and they generate the same topology on X.

Now, we present some basic definitions and propositions of G-metric space.

Definition 1.3.6. [43, p.292] Let (X, G) be a G-metric space.

- The open ball of center $x_0 \in X$ and radius $r > 0$ is defined as

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

- The closed ball of center $x_0 \in X$ and radius $r > 0$ is defined as

$$\bar{B}_G(x_0, r) = \{y \in X : G(x_0, y, y) \leq r\}.$$

Clearly, $x \in B_G(x_0, r) \subseteq \bar{B}_G(x_0, r)$.

The family of all open balls $\{B_G(x_0, r) : x_0 \in X, r > 0\}$ is the base of a topology $\tau(G)$ on X, which is known as the G-metric topology.

Proposition 1.3.7. *[2, Prop.3.2.1, p.42] Let (X, G) be a G-metric space and d_m^G and d_s^G are the metrics on X as in Lemma 1.3.5, then*

$$B_{d_s^G}(x_0, r) \subseteq B_{d_m^G}(x_0, r) \subseteq B_G(x_0, r) \subseteq B_{d_m^G}(x_0, 2r) \subseteq B_{d_m^G}(x_0, 2r).$$

Consequently, the G-metric topology $\tau(G)$ coincides with the metric topology generated by the equivalent metrics d_m^G or d_s^G as defined in Lemma 1.3.5. We can give a more geometrical definition of an open set in terms of neighborhoods.

Definition 1.3.8. [2, p.44] Let (X, G) be a G-metric space. Then

- a subset $U \subseteq X$ is a G-neighborhood of a point $x \in X$ if there exists $r > 0$ such that $B_G(x, r) \subseteq U$.
- a subset $U \subseteq X$ is G-open if it is empty or it is a G-neighborhood of all its point.
- a subset $U \subseteq X$ is G-closed if its complement $X \backslash U$ is G-open.

Now, we introduce the notions of convergent sequence and Cauchy sequence using the topology $\tau(G)$.

Definition 1.3.9. [43, p.292] Let (X, G) be a G-metric space, $\{x_n\} \subseteq X$ be a sequence and $x \in X$. We say that:

(1) $\{x_n\}$ G-converges to x, and we write $\{x_n\} \to x$, if $\lim\limits_{n,m\to\infty} G(x_n, x_m, x) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x) < \varepsilon$, for all $n, m \geq n_0$.

(2) $\{x_n\}$ is G-Cauchy if $\lim\limits_{n,m,k\to\infty} G(x_n, x_m, x_k) = 0$, that is, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n, x_m, x_k) < \varepsilon$, for all $n, m, k \geq n_0$.

(3) (X, G) is G-complete if every G-Cauchy sequence in X is G-convergent in X.

Note that, the limit of a G-convergent sequence in a G-metric space is unique. Further, every G-convergent sequence in a G-metric space is G-Cauchy.

Proposition 1.3.10. *[43, Prop.6, p.292] Let (X, G) be a G-metric space, $\{x_n\} \subseteq X$ be a sequence and $x \in X$. Then*

(i) $\{x_n\}$ *G-converges to* $x \iff \lim_{n\to\infty} G(x_n, x_n, x) = 0 \iff \lim_{n\to\infty} G(x_n, x, x) = 0.$

(ii) $\{x_n\}$ *is G-Cauchy* $\iff \lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$

Definition 1.3.11. [43, p.293] Let (X, G) be a G-metric space. We say that a mapping $T : X \to X$ is G-continuous at $x \in X$ if $\{Tx_m\} \to Tx$, for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$.

To prove the given sequence is Cauchy, the following condition is the sufficient.

Lemma 1.3.12. *[2, p.52] Let $\{x_n\}$ be a sequence in a G-metric space (X, G) and assume that there exist a function $\varphi \in \mathcal{F}_{KR}$ and $n_0 \in \mathbb{N}$ such that, at least, one of the following conditions holds:*

(a) $G(x_{n+1}, x_{n+2}, x_{n+2}) \le \varphi\big(G(x_n, x_{n+1}, x_{n+1})\big);$

(b) $G(x_{n+1}, x_{n+1}, x_{n+2}) \le \varphi\big(G(x_n, x_n, x_{n+1})\big),$

for all $n \ge n_0$, where $\mathcal{F}_{KR}$ is the set of continuous functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = 0$ if and only if $t = 0$.

Then $\{x_n\}$ is a Cauchy sequence in (X, G).

Definition 1.3.13. [2, p.52] A sequence $\{x_n\}$ in a G-metric space is *asymptotically regular* if

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Following is the necessary conditions that must be verified by any asymptotically regular sequence if it is not Cauchy.

Lemma 1.3.14. *[2, p.58] Let $\{x_n\}$ be an asymptotically regular sequence in a G-metric space (X, G) and suppose that $\{x_n\}$ is not Cauchy. Then there exist a positive real number $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N}$,*

$$k \le n_k < m_k < n_{k+1},$$

$$G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \le \varepsilon < G(x_{n_k}, x_{m_k}, x_{m_k})$$

and also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G(x_{n_k+p_1}, x_{m_k+p_2}, x_{m_k+p_3}) = \varepsilon.$$

Barinde [9] introduced asymptotic regularity for two operators in metric spaces which can be extended to G-metric spaces as follows.

Definition 1.3.15. Let (X, G) be a G-metric space and $T, g : X \to X$ be two operators. Then the operator g is called T-asymptotically regular in (X, G) if

$$G(g^n(x), T(g^n(x)), T(g^n(x))) \to 0 \textit{ as } n \to \infty, \textit{ for all } x \in X.$$

Fixed point theory for G-metric spaces is a specialized area of fixed point theory that deals with the properties of mappings on spaces that are equipped with a generalized metric or G-metric, which allows for a more flexible definition of distance than the traditional metric. Abbas and Rhoades [1] were the first to explore the study of common fixed point theory in the context of G-metric spaces. Subsequently, numerous authors have obtained fixed and common fixed point results in the framework of G-metric spaces. The study of common fixed points is an active and important area of research in fixed point theory for G-metric spaces, as it has many open problems and a wide range of applications in various areas, such as optimization, neural network, approximation theory, integral and differential equations, control theory, numerical analysis and several others.

Thereafter, Jleli and Samet [29] and Samet et al. [54] observed that the structure of G-metric and quasi-metric are similar and many fixed point results of G-metric spaces can be derived from the quasi-metric spaces, defined as follows.

Definition 1.3.16. Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function such that the following are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) \leq d(x, z) + d(z, y)$, for any points $x, y, z \in X$.

Then, d is called a quasi-metric on X and the pair (X, d) is called a quasi-metric space.

Notice that, $d_G(x, y) = G(x, y, y)$ forms a quasi-metric. Hence, if the contraction condition of the fixed point theorems in G-metric can be reduced to two variables, then results can be derived from quasi-metric space or usual metric space. Karapinar and Agarwal [35] proved that the techniques of Jleli and Samet [29] and Samet et al. [54] can't be applicable if the contraction condition in fixed point theorem can't be reduced to two variables and they introduced a new contraction condition in G-metric spaces. In this direction, researchers obtained several fixed point results in G-metric spaces by using the notion of Ω-distance introduced by Saadati et al. [53]. On the other side, Kirk et al. [39] introduced the cyclic mappings to generalize the Banach contraction principle, which was further generalized to (A, B) weakly increasing mappings [58] to derive common fixed point. Moreover, the idea of Jleli and Samet [29] and Samet et al. [54] are not applicable to the above mentioned notions.

Motivated by the definitions of b-metric and G-metric spaces, recently Aghajani et al. [3] introduced the notion of G_b-metric spaces, replacing the triangle inequality with a more flexible condition, as follows:

Definition 1.3.17. [3, p.1087] Let X be a nonempty set, $s \geq 1$ and $G_b : X^3 \to [0, \infty)$ be a function satisfying the following properties:

$(GB1)$ $G_b(x, y, z) = 0$, if $x = y = z$,

$(GB2)$ $G_b(x, x, y) > 0$, for all $x, y \in X$ with $x \neq y$,

$(GB3)$ $G_b(x, x, y) \leq G_b(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

$(GB4)$ $G_b(x, y, z) = G_b(p\{x, y, z\})$, where p is a permutation of x, y, z,

$(GB5)$ $G_b(x, y, z) \leq s[G_b(x, a, a) + G_b(a, y, z)]$, for all $x, y, z, a \in X$.

Then G_b is called generalized b-metric on X and the pair (X, G_b) is called a G_b-metric space.

Every G-metric space is a G_b-metric space with $s = 1$ and so the class of G_b-metric spaces is larger than the class of G-metric spaces. The following example shows that a G_b-metric on X need not be a G-metric on X.

Example 1.3.3. Let $X = \mathbb{R}$. Define a mapping $G : X^3 \to [0, \infty)$ by

$$G(x, y, z) = \max\{| x - y |^2, | y - z |^2, | z - x |^2\}.$$

Then (X, G) is a G_b-metric space with $s = 2$.

Further, they established common fixed point results for weak contractive mappings in partially ordered G_b-metric spaces.

In this thesis, motivated by all the above mention types of generalizations of the Banach contraction principle, many interesting coincidence or common fixed point results in the context of G-metric spaces are cultivated. Further, examine the possibilities of their applications in the domain of integral equations and neural networks.

1.4 Synopsis of the thesis

The central theme of the thesis revolves around exploring the existence and uniqueness of common fixed points for self-maps. This exploration involves the introduction of a new contraction on G-metric space, motivated by the promising practical applications of fixed point results. The significance of these findings lies in their ability to provide a valuable advantage in addressing a multitude of nonlinear problems documented in the literature, particularly in the realms of neural networks and integral equations.

In Chapter 2, by utilizing the notion of (A, B)-weakly increasing mappings and altering distance functions, a generalized cyclic contraction and rational type cyclic contraction via C-class function in G-metric spaces are introduced. Both contractions generalize the contractive condition of Shatanawi and Abodayeh [57] for larger class of auxiliary functions. Besides, common fixed point results for such contraction in the setting of G-metric spaces are studied. Some examples are also presented to show that our results are effective.

In Chapter 3, Ćirić type $\mathcal{Z}_F$-contraction and the generalized Ćirić type $(\mathcal{Z}_{(\alpha,F)}, T)$-contraction for two mappings via C_F-simulation functions are introduced to study the existence and uniqueness of coincidence points and common fixed points in the context of quasi-metric spaces. Further, extending both the results to G-metric spaces.

In Chapter 4, (ψ, ϕ)-Wardowski contraction for three maps in the setting of G_b-metric spaces is introduced to establish a condition for which the common fixed point is a point of discontinuity. Further, its application to neural networks is discussed.

In Chapter 5, extended $\Gamma - C_F$-simulation functions are studied by introducing the notion of $\Gamma - C$-class functions and illustrative examples of extended

$\Gamma - C_F$-simulation functions are constructed. Further, Geraghty type and almost Suzuki type contractions are studied for pair of mappings and furnished related coincidence and common fixed point result in precomplete subspace of G_b-metric spaces and G-metric spaces. Furthermore, the implication of Geraghty type contraction for non-linear integral equations is explored and discussed.

In Chapter 6, a generalized $\Gamma - C_F$-simulation functions are introduced as an extension of generalized simulation functions and common fixed point result is studied by defining weak (η_F, T)-contraction for pair of mappings in G-metric spaces. Further, consequences to quasi-metric spaces and metric spaces are discussed.