Chapter 2

Non-linear Cyclic Contractions in G-metric Spaces

2.1 Introduction and preliminaries

Many real-world problems involve non-linear relationships and complex interactions. Non-linear cyclic contractions provide a more realistic framework for modeling and analyzing systems where linear contractions may not accurately represent the underlying dynamics.

The study of non-linear cyclic contractions in G-metric spaces is motivated by the need to generalize and extend the classical fixed-point theorems to a more general and flexible mathematical structure known as a G-metric.

In 2003, Kirk et al. [39] introduced cyclic contraction to generalize the Banach contraction principle. An important advantage to this approach is that cyclic contractions, unlike Banach-type contractions, need not be continuous. Such contractions are further generalized by Shatanawi and Postolache [58], by introducing the pair of (A, B)- weakly increasing mappings as follows:

Definition 2.1.1. [58, p.2] Let $(X, \preceq)$ be a partially ordered set and A, B be two closed subsets of X with $X = A \cup B$. Let $S, T : X \to X$ be two mappings. Then the pair (S, T) is said to be (A, B)-weakly increasing if $Sx \preceq TSx$ for all $x \in A$ and $Tx \preceq STx$ for all $x \in B$.

Shatanawi and Abodayeh [57] introduced a new contractive condition by utilizing the notion of (A, B)- weakly increasing mappings and using altering distance functions, proved the common fixed point result in G-metric spaces.

The concept of altering distance functions in fixed-point theory is a powerful tool that arises from the need to provide more flexibility and generality in the study of fixed points. In traditional fixed-point theorems, a single contraction condition, often based on a metric or a norm, is imposed. Altering distance functions allow for the consideration of non-uniform contraction conditions, where the contraction factor may vary for different pairs of points.

An altering distance function is a continuous, non-decreasing mapping $\phi : [0, \infty) \to [0, \infty)$ such that $\phi^{-1}(0) = 0$. The family of all altering distance functions is denoted by F_{alt}.

The flexibility of altering distance function allows for the study of fixed points in a wider variety of spaces and under more diverse contraction conditions, making the theory more applicable to real-world problems across different disciplines.

In numerical methods and algorithms that involve fixed-point iterations, altering distance functions can enhance the robustness of the convergence analysis. They provide a way to tailor the contraction condition to specific regions of interest, improving the convergence properties of the algorithm.

In many applications, especially those involving nonlinear mappings, the contraction condition may not be uniform across the entire space. Altering distance functions provide a flexible framework to accommodate such non-linearities and variations in the contraction behavior.

Shatanawi and Abodayeh [57] proved the following result that cannot be reduced to quasi-metric spaces using the methods of Jleli and Samet [29] or Samet et al. [54].

Theorem A. *[57, Theorem 2.1, p.45] Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let S, T be self mappings on X that satisfy the following conditions:*

(i) The pair (S, T) is (A, B)-weakly increasing.

(ii) $S(A) \subseteq B$ and $T(B) \subseteq A$.

(iii) There exist two functions $\varphi, \psi \in F_{alt}$ such that

$$\varphi(G(Sx, TSx, Ty)) \leq \varphi(G(x, Sx, y)) - \psi(G(x, Sx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$;

$$\varphi(G(Tx, STx, Sy)) \leq \varphi(G((x, Tx, y))) - \psi(G(x, Tx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$.

(iv) *S or T is continuous.*

Then, S and T have a common fixed point in $A \cap B$.

In Chapter 2, the concept of (A, B)-weakly increasing mappings is employed to introduce generalized cyclic contractive conditions and rational type cyclic contractive conditions. These contractions are utilized to establish common fixed point results in the framework of G-metric spaces which generalize Theorem A. Moreover, numerical example is furnished to validate obtained result.

2.2 Results for generalized cyclic contraction in G-metric spaces

In this section, consider functions $\psi \in \Psi$ instead of $\psi \in F_{alt}$ to generalize the contractive condition of Theorem A. Here, the continuity of the auxiliary function ψ is not required to establish common fixed point result.

Ψ is the family of all mappings $\psi : [0, \infty) \to [0, \infty)$ such that, if $\{t_m\} \subset [0, \infty)$ and $\psi(t_m) \to 0$ then $t_m \to 0$. Note that, $F_{alt} \subset \Psi$.

The following lemma is required to prove our main result.

Lemma 2.2.1. *[5, p.143] Let $\phi \in F_{alt}, \psi \in \Psi$ and $t_n \subset [0, \infty)$ be a sequence such that $\phi(t_{n+1}) \leq \phi(t_n) - \psi(t_n)$, for all $n \in \mathbb{N}$, then $t_n \to 0$.*

Now, the main result of this section is stated here.

Theorem 2.2.2. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let S, T be self mappings on X that satisfy the following conditions:*

(i) *The pair (S, T) is (A, B)-weakly increasing.*

(ii) *$S(A) \subseteq B$ and $T(B) \subseteq A$.*

(iii) There exist two functions $\varphi \in F_{alt}$ and $\psi \in \Psi$ such that

$$\varphi\big(G(Sx, TSx, Ty)\big) \leq \varphi\big(M(x,y)\big) - \psi\big(M(x,y)\big) \tag{2.1}$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$;

$$\varphi\big(G(Tx, STx, Sy)\big) \leq \varphi\big(M'(x,y)\big) - \psi\big(M'(x,y)\big) \tag{2.2}$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$, where

$$\begin{aligned} M(x,y) = \max\Big\{ & G(x,Sx,y), G(x,Sx,Sx), G(y,Ty,Ty), \\ & \frac{1}{2}\Big(G(Sx,Sx,Ty), G(x,TSx,Ty), G(Sx,TSx,y)\Big)\Big\}; \\ M'(x,y) = \max\Big\{ & G(x,Tx,y), G(x,Tx,Tx), G(y,Sy,Sy), \\ & \frac{1}{2}\Big(G(Tx,Tx,Sy), G(x,STx,Sy), G(Tx,STx,y)\Big)\Big\}. \end{aligned}$$

(iv) S or T is continuous.

Then, S and T have a common fixed point in $A \cap B$.

Proof. Since A is nonempty, start with $x_0 \in A$. From condition (ii), we can construct a sequence $\{x_n\}$ in X such that

$$Sx_{2n} = x_{2n+1}, \text{ for } x_{2n} \in A; \; Tx_{2n+1} = x_{2n+2}, \text{ for } x_{2n+1} \in B, n \geq 0.$$

From condition (i), we have $x_n \preceq x_{n+1}$, for all $n \geq 0$. If $x_{2n_0} = x_{2n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{2n_0} is a fixed point of S in $A \cap B$. Since $x_{2n_0} \preceq x_{2n_0+1}$, from (2.1), we have

$$\begin{aligned} \varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big) &= \varphi\big(G(Sx_{2n_0}, TSx_{2n_0}, Tx_{2n_0+1})\big) \\ &\leq \varphi\big(M(x_{2n_0}, x_{2n_0+1})\big) - \psi\big(M(x_{2n_0}, x_{2n_0+1})\big), \end{aligned} \tag{2.3}$$

where

$$
\begin{aligned}
&M(x_{2n_0}, x_{2n_0+1})\\
&= \max\Big\{G(x_{2n_0}, Sx_{2n_0}, x_{2n_0+1}), G(x_{2n_0}, Sx_{2n_0}, Sx_{2n_0}),\\
&\qquad G(x_{2n_0+1}, Tx_{2n_0+1}, Tx_{2n_0+1}), \frac{1}{2}\Big(G(Sx_{2n_0}, Sx_{2n_0}, Tx_{2n_0+1}),\\
&\qquad G(x_{2n_0}, TSx_{2n_0}, Tx_{2n_0+1}), G(Sx_{2n_0}, TSx_{2n_0}, x_{2n_0+1})\Big)\Big\}\\
&= \max\Big\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}),\\
&\qquad \frac{1}{2}\Big(G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+2}), G(x_{2n_0}, x_{2n_0+2}, x_{2n_0+2})\Big)\Big\}.
\end{aligned}
$$

Using Lemma 1.3.3, we obtain

$$G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+2}) \leq 2G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}),$$

and by rectangle inequality (G_5), we get

$$G(x_{2n_0}, x_{2n_0+2}, x_{2n_0+2}) \leq G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) + G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}).$$

Then,

$$
\begin{aligned}
M(x_{2n_0}, x_{2n_0+1}) &= \max\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\}\\
&= G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}).
\end{aligned}
$$

From (2.3), we have

$$
\begin{aligned}
\varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big) \leq &\varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big)\\
&- \psi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big).
\end{aligned}
$$

Implies that $\psi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big) = 0$. Since $\psi \in \Psi$, we have

$$G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) = 0$$

and hence $x_{2n_0+1} = x_{2n_0+2}$. So, we get $x_{2n_0} = x_{2n_0+1} = x_{2n_0+2}$. Therefore, x_{2n_0} is a fixed point of T in $A \cap B$. Hence, x_{2n_0} is a common fixed point of S and T in

$A \cap B$.

Now, we assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$. Since, $x_{2n} \preceq x_{2n+1}$, for all $n \geq 0$, from (2.1), we have

$$\begin{aligned}\varphi\big(G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big) &= \varphi\big(G(Sx_{2n}, TSx_{2n}, Tx_{2n+1})\big)\\ &\leq \varphi\big(M(x_{2n}, x_{2n+1})\big) - \psi\big(M(x_{2n}, x_{2n+1})\big), \qquad (2.4)\end{aligned}$$

where

$$M(x_{2n}, x_{2n+1}) = \max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n+1}, x_{2n+2}, x_{2n+2})\}.$$

Case i: If $M(x_{2n}, x_{2n+1}) = G(x_{2n+1}, x_{2n+2}, x_{2n+2})$, then by (2.4), we get

$$\begin{aligned}\varphi\big(G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big) \leq &\varphi\big(G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big)\\ &- \psi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})).\end{aligned}$$

Therefore, $\psi\big(G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big) = 0$, for all $n \geq 0$. By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \psi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) = 0.$$

Since $\psi \in \Psi$, we have

$$\lim_{n\to\infty} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = 0. \qquad (2.5)$$

Case ii: If $M(x_{2n}, x_{2n+1}) = G(x_{2n}, x_{2n+1}, x_{2n+1})$, then by (2.4), we get

$$\varphi\big(G(x_{2n+1}, x_{2n+2}, x_{2n+2})\big) \leq \varphi(G(x_{2n}, x_{2n+1}, x_{2n+1})) - \psi\big(G(x_{2n}, x_{2n+1}, x_{2n+1})\big).$$

By Lemma 2.2.1, we get

$$\lim_{n\to\infty} G(x_{2n}, x_{2n+1}, x_{2n+1}) = 0. \qquad (2.6)$$

From (2.5) and (2.6), we obtain that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \qquad (2.7)$$

From definition of G-metric space, we have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{2.8}$$

That is, $\{x_n\}$ is asymptotically regular sequence. Now, we prove that $\{x_n\}$ is a G-Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a G-Cauchy sequence. Suppose that $\{x_n\}$ is not a G-Cauchy sequence. Then by (2.7), (2.8) and Lemma 1.3.14 there exist $\varepsilon > 0$ and two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that, for all $k \in \mathbb{N}$, $k \leq 2n_k < 2m_k < 2n_{k+1}$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k\to\infty} G(x_{2n_k+p_1}, x_{2m_k+p_2}, x_{2m_k+p_3}) = \varepsilon. \tag{2.9}$$

Since, $x_{2m_k} \preceq x_{2n_k+1}$, by using (2.1), we get

$$\begin{aligned}\varphi\big(G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})\big) &= \varphi\big(G(Sx_{2m_k}, TSx_{2m_k}, Tx_{2n_k+1})\big)\\ &\leq \varphi\big(M(x_{2m_k}, x_{2n_k+1})\big) - \psi(M(x_{2m_k}, x_{2n_k+1})),\end{aligned}$$

where

$$\begin{aligned}&M(x_{2m_k}, x_{2n_k+1})\\ &= \max\Big\{G(x_{2m_k}, Sx_{2m_k}, x_{2n_k+1}), G(x_{2m_k}, Sx_{2m_k}, Sx_{2m_k}),\\ &\qquad G(x_{2n_k+1}, Tx_{2n_k+1}, Tx_{2n_k+1}), \frac{1}{2}\Big(G(Sx_{2m_k}, Sx_{2m_k}, Tx_{2n_k+1}),\\ &\qquad G(x_{2m_k}, TSx_{2m_k}, Tx_{2n_k+1}), G(Sx_{2m_k}, TSx_{2m_k}, x_{2n_k+1})\Big)\Big\}\\ &= \max\Big\{G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1}), G(x_{2m_k}, x_{2m_k+1}, x_{2m_k+1}),\\ &\qquad G(x_{2n_k+1}, x_{2n_k+2}, x_{2n_k+2}), \frac{1}{2}\Big(G(x_{2m_k+1}, x_{2m_k+1}, x_{2n_k+2}),\\ &\qquad G(x_{2m_k}, x_{2m_k+2}, x_{2n_k+2}), G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+1})\Big)\Big\}.\end{aligned}$$

By using (2.7), (2.8) and (2.9), we get $\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \max\{\varepsilon, 0, \frac{\varepsilon}{2}\} = \varepsilon$.

Take $\{t_k = G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})\}$ and $\{s_k = M(x_{2m_k}, x_{2n_k+1})\}$. Then $\{t_k\}$ and $\{s_k\}$ are sequences converging to the same limit ε and they satisfy $\varphi(t_k) \leq \varphi(s_k) - \psi(s_k)$, for all k. Therefore, $\psi(s_k) \leq \varphi(s_k) - \varphi(t_k)$.

By taking limit as $k \to \infty$, since $\varphi \in F_{alt}$, we have

$$\lim_{k\to\infty} \psi(s_k) \le \varphi(\varepsilon) - \varphi(\varepsilon) = 0.$$

Since $\psi \in \Psi$, $\lim_{k\to\infty} s_k = 0$. This implies that $\varepsilon = 0$, a contradiction. Thus, $\{x_{2n}\}$ is G-Cauchy. So, sequence $\{x_n\}$ is G-Cauchy. Since (X, G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u. Therefore, the subsequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are G-convergent to u.

Since $\{x_{2n}\} \subseteq A$ and A is closed, implies that $u \in A$. Also, $\{x_{2n+1}\} \subseteq B$ and B is closed, implies that $u \in B$. We may assume that S is continuous. So, we have $\lim_{n\to\infty} Sx_{2n} = Su$ and $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$. By uniqueness of the limit we have $Su = u$. Since $u \preceq u$, from (2.1), we have

$$\begin{aligned} \varphi\big(G(u, Tu, Tu)\big) &= \varphi\big(G(Su, TSu, Tu)\big) \\ &\le \varphi\big(M(u, u)\big) - \psi\big(M(u, u)\big), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} M(u, u) = \max &\Big\{ G(u, Su, u), G(u, Su, Su), G(u, Tu, Tu), \\ &\frac{1}{2}\Big(G(Su, Su, Tu), G(u, TSu, Tu), G(Su, TSu, u)\Big)\Big\} \\ = \max &\Big\{ G(u, u, u), G(u, Tu, Tu), \frac{1}{2}\Big(G(u, u, Tu), G(u, Tu, Tu), G(u, Tu, u)\Big)\Big\} \\ = \max &\Big\{ G(u, Tu, Tu), \frac{1}{2}G(u, Tu, Tu)\Big\} \\ = G&(u, Tu, Tu). \end{aligned}$$

Using (2.10), we obtain

$$\begin{aligned} \varphi\big(G(u, Tu, Tu)\big) &= \varphi\big(G(Su, STu, Tu)\big) \\ &\le \varphi\big(G(u, Tu, Tu)\big) - \psi\big(G(u, Tu, Tu)\big). \end{aligned}$$

Therefore, $\psi\big(G(u, Tu, Tu)\big) = 0$. Implies that $G(u, Tu, Tu) = 0$. Hence, $Tu = u$. Thus, u is a common fixed point of S and T in $A \cap B$. □

To support the usability of our result, the following example is stated.

Example 2.2.1. Let $X = [0, 1]$ and $S, T : X \to X$ be given as

$$S(x) = T(x) = \frac{x^2}{1 + x}.$$

Take $A = [0, \frac{1}{2}]$ and $B = [0, 1]$. Define the function $G : X \times X \times X \to [0, \infty)$ as

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Clearly, G is a complete G-metric on X. We introduce a relation on X by $x \preceq y$ if and only if $y \leq x$. Also, define the functions $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = 2t$ and $\psi(t) = \dfrac{t}{1 + 2t}$.

Note that $S(A) = [0, \frac{1}{6}] \subseteq B$ and $T(B) = [0, \frac{1}{2}] \subseteq A$.

To prove (i), given $x \in X$,

$$STx = TSx = \frac{x^2}{(1 + x)} \frac{x^2}{(1 + x + x^2)}.$$

Since $x \in [0, 1], \dfrac{x^2}{(1 + x + x^2)} < 1$. Thus, $TSx \leq Sx$ and $STx \leq Tx$. Hence $Sx \preceq TSx$, for all $x \in A$ and $Tx \preceq STx$, for all $x \in B$.

To prove (iii), given $x, y \in X$ with $x \geq y$. Then,

$$\begin{aligned} G(Sx, TSx, Ty) &= G(Tx, STx, Sy) \\ &= \max\left\{\frac{x^2}{(1 + x)}, \frac{x^2}{(1 + x)} \frac{x^2}{(1 + x + x^2)}, \frac{y^2}{(1 + y)}\right\} \\ &= \frac{x^2}{(1 + x)} \end{aligned}$$

and

$$M(x, y) = M'(x, y) = \max\left\{x, y, \frac{x^2}{2(1 + x)}, \frac{x}{2}\right\} = x.$$

Since

$$\frac{2x^2}{(1 + x)} \leq 2x - \frac{x}{(1 + 2x)},$$

we have

$$\varphi(G(Sx, TSx, Ty)) \leq \varphi(M(x, y)) - \psi(M(x, y))$$

and

$$\varphi(G(Tx, STx, Sy)) \leq \varphi(M'(x, y)) - \psi(M'(x, y)).$$

Hence, all the conditions of Theorem 2.2.2 are satisfied. Notice that 0 is the unique common fixed point of S and T.

The following result is obtained from Theorem 2.2.2 as a generalization of Theorem A. For that, drop the condition $\psi(0) = 0$, continuity of ψ and replace $\psi \in F_{alt}$ with $\psi \in \Psi$ in Theorem A.

Corollary 2.2.3. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let S, T be self mappings on X that satisfy the following conditions:*

(i) The pair (S, T) is (A, B)-weakly increasing.

(ii) $S(A) \subseteq B$ and $T(B) \subseteq A$.

(iii) There exist two functions $\varphi \in F_{alt}$ and $\psi \in \Psi$ such that

$$\varphi(G(Sx, TSx, Ty)) \leq \varphi(G(x, Sx, y)) - \psi(G(x, Sx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$ and

$$\varphi(G(Tx, STx, Sy)) \leq \varphi(G((x, Tx, y))) - \psi(G(x, Tx, y))$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$.

(iv) S or T is continuous.

Then, S and T have a common fixed point in $A \cap B$.

Proof. By taking $M(x, y) = G(x, Sx, y)$ and $M'(x, y) = G(x, Tx, y)$ in Theorem 2.2.2, we get the result. □

2.3 Rational type cyclic contraction in G-metric spaces

This section deals with the rational type cyclic contraction using C-class [6] functions which cover a large class of contractive conditions. Further, existence of the common fixed points of rational type cyclic contraction is studied. Alongside, an example to authenticate the obtained result is given.

Ansari [6] introduced C-class function as follows.

Definition 2.3.1. [6, p.1] A function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following conditions:

(F_1) $F(s, t) \leq s,$ for all $s, t \geq 0;$

(F_2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0,$ for all $s, t \geq 0.$

The collection of all C-class functions is denoted by $\mathcal{C}$.

The main result of this section is stated below.

Theorem 2.3.2. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let S, T be self mappings on X that satisfy the following conditions:*

(i) The pair (S, T) is (A, B)-weakly increasing.

(ii) $S(A) \subseteq B$ and $T(B) \subseteq A$.

(iii) There exist two functions $\varphi \in F_{alt}$ and $\psi \in \Psi$ such that

$$\varphi\big(G(Sx, TSx, Ty)\big) \leq F\big(\varphi\big(M(x, y)\big), \psi\big(M(x, y)\big)\big) \tag{2.11}$$

holds for all comparative elements $x, y \in X$ with $x \in A$ and $y \in B$ and

$$\varphi\big(G(Tx, STx, Sy)\big) \leq F\big(\varphi\big(M'(x, y)\big), \psi\big(M'(x, y)\big)\big) \tag{2.12}$$

holds for all comparative elements $x, y \in X$ with $x \in B$ and $y \in A$, where $F \in \mathcal{C}$,

$$M(x, y) = \max\left\{G(x, Sx, y), \frac{G(Sx, Sx, y)[1 + G(x, x, Ty)]}{1 + G(x, Sx, y)}, \frac{G(Ty, Ty, y)[1 + G(Sx, Sx, x)]}{1 + G(x, Sx, y)}\right\}$$

and

$$M'(x, y) = \max\left\{G(x, Tx, y), \frac{G(Tx, Tx, y)[1 + G(x, x, Sy)]}{1 + G(x, Tx, y)},\right.$$

$$\left.\frac{G(Sy,Sy,y)[1+G(Tx,Tx,x)]}{1+G(x,Tx,y)}\right\}.$$

(iv) S or T is continuous.

Then, S and T have a common fixed point in $A \cap B$.

Proof. Start with $x_0 \in A$. Using condition (ii), we can construct a sequence $\{x_n\}$ in X such that $Sx_{2n} = x_{2n+1}$, for $x_{2n} \in A$ and $Tx_{2n+1} = x_{2n+2}$, for $x_{2n+1} \in B$, $n \geq 0$. Using condition (i), we have $x_n \preceq x_{n+1}$, for all $n \geq 0$.
If $x_{2n_0} = x_{2n_0+1}$, for some $n_0 \in \mathbb{N}$, then x_{2n_0} is a fixed point of S in $A \cap B$. Since $x_{2n_0} \preceq x_{2n_0+1}$, from (2.11), we have

$$\begin{aligned}\varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big) &= \varphi\big(G(Sx_{2n_0}, TSx_{2n_0}, Tx_{2n_0+1})\big)\\ &\leq F\big(\varphi\big(M(x_{2n_0}, x_{2n_0+1})\big), \psi\big(M(x_{2n_0}, x_{2n_0+1})\big)\big),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}&M(x_{2n_0}, x_{2n_0+1})\\ &= \max\Big\{G(x_{2n_0}, Sx_{2n_0}, x_{2n_0+1}), \frac{G(Sx_{2n_0}, Sx_{2n_0}, x_{2n_0+1})[1+G(x_{2n_0}, x_{2n_0}, Tx_{2n_0+1})]}{1+G(x_{2n_0}, Sx_{2n_0}, x_{2n_0+1})},\\ &\qquad \frac{G(Tx_{2n_0+1}, Tx_{2n_0+1}, x_{2n_0+1})[1+G(Sx_{2n_0}, Sx_{2n_0}, x_{2n_0})]}{1+G(x_{2n_0}, Sx_{2n_0}, x_{2n_0+1})}\Big\}\\ &= \max\Big\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), \frac{G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0+1})[1+G(x_{2n_0}, x_{2n_0}, x_{2n_0+2})]}{1+G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1})},\\ &\qquad \frac{G(x_{2n_0+2}, x_{2n_0+2}, x_{2n_0+1})[1+G(x_{2n_0+1}, x_{2n_0+1}, x_{2n_0})]}{1+G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1})}\Big\}\\ &= \max\{G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}), G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\}\\ &= G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}).\end{aligned}$$

From (2.13) and (F_1) , we have

$$\begin{aligned}&\varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big)\\ &\leq F\big(\varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big), \psi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big)\big)\\ &\leq \varphi\big(G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2})\big),\end{aligned}$$

implies that

$$F\big(\varphi\big(G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})\big),\psi\big(G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})\big)\big)$$
$$=\varphi\big(G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})\big).$$

From (F_2), we have

$$\varphi\big(G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})\big)=0 \text{ or } \psi\big(G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})\big)=0.$$

Since $\varphi \in F_{alt}$ and $\psi \in \Psi$, we have $G(x_{2n_0+1},x_{2n_0+2},x_{2n_0+2})=0$. That is, $x_{2n_0}=x_{2n_0+1}=x_{2n_0+2}$. Hence, x_{2n_0} is a common fixed point of S and T in $A\cap B$. Now, we assume that $x_n \neq x_{n+1}$, for all $n\geq 0$. Since $x_{2n}\preceq x_{2n+1}$, from (2.11), we have

$$\begin{aligned}\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)&=\varphi\big(G(Sx_{2n},TSx_{2n},Tx_{2n+1})\big)\\&\leq F\big(\varphi\big(M(x_{2n},x_{2n+1})\big),\psi\big(M(x_{2n},x_{2n+1})\big)\big),\end{aligned}\tag{2.14}$$

where

$$\begin{aligned}&M(x_{2n},x_{2n+1})\\&=\max\Big\{G(x_{2n},Sx_{2n},x_{2n+1}),\frac{G(Sx_{2n},Sx_{2n},x_{2n+1})[1+G(x_{2n},x_{2n},Tx_{2n+1})]}{1+G(x_{2n},Sx_{2n},x_{2n+1})},\\&\qquad\frac{G(Tx_{2n+1},Tx_{2n+1},x_{2n+1})[1+G(Sx_{2n},Sx_{2n},x_{2n})]}{1+G(x_{2n},Sx_{2n},x_{2n+1})}\Big\}\\&=\max\Big\{G(x_{2n},x_{2n+1},x_{2n+1}),\frac{G(x_{2n+1},x_{2n+1},x_{2n+1})[1+G(x_{2n},x_{2n},x_{2n+2})]}{1+G(x_{2n},x_{2n+1},x_{2n+1})},\\&\qquad\frac{G(x_{2n+2},x_{2n+2},x_{2n+1})[1+G(x_{2n+1},x_{2n+1},x_{2n})]}{1+G(x_{2n},x_{2n+1},x_{2n+1})}\Big\}\\&=\max\{G(x_{2n},x_{2n+1},x_{2n+1}),G(x_{2n+1},x_{2n+2},x_{2n+2})\}.\end{aligned}$$

If $M(x_{2n},x_{2n+1})=G(x_{2n+1},x_{2n+2},x_{2n+2})$, for all $n\geq 0$, then from (2.14), we have

$$\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)$$
$$\leq F\big(\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big),\psi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)\big).$$

Since F is a C-class function, we have

$$F\big(\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big),\psi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)\big)$$
$$=\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big),$$

implies that

$$\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)=0 \text{ or } \psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))=0,\ \text{for all } n\geq 0.$$

Since $\varphi \in F_{alt}$, we have $G(x_{2n+1},x_{2n+2},x_{2n+2})=0$, for all $n\geq 0$, implies that $x_{2n+1}=x_{2n+2}$, for all $n\geq 0$, a contradiction.
Therefore, $M(x_{2n},x_{2n+1})=G(x_{2n},x_{2n+1},x_{2n+1})$, for all $n\geq 0$.
Now, from (2.14) and (F_1), we get

$$\begin{aligned}\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big) &\leq F\big(\varphi\big(G(x_{2n},x_{2n+1},x_{2n+1})\big),\psi\big(G(x_{2n},x_{2n+1},x_{2n+1})\big)\big)\\ &\leq \varphi\big(G(x_{2n},x_{2n+1},x_{2n+1})\big),\ \text{for all } n\geq 0. \qquad (2.15)\end{aligned}$$

Since $x_{2n+1}\preceq x_{2n+2}$, from (2.12), similarly we can prove

$$\begin{aligned}\varphi\big(G(x_{2n+2},x_{2n+3},x_{2n+3})\big) &\leq F\big(\varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big),\psi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big)\big)\\ &\leq \varphi\big(G(x_{2n+1},x_{2n+2},x_{2n+2})\big),\ \text{for all } n\geq 0. \qquad (2.16)\end{aligned}$$

From (2.15) and (2.16), we conclude that

$$\begin{aligned}\varphi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) &\leq F\big(\varphi\big(G(x_{n},x_{n+1},x_{n+1})\big),\psi\big(G(x_{n},x_{n+1},x_{n+1})\big)\big)\\ &\leq \varphi\big(G(x_{n},x_{n+1},x_{n+1})\big),\ \text{for all } n\geq 0. \qquad (2.17)\end{aligned}$$

Since $\varphi \in F_{alt}$, we get $G(x_{n+1},x_{n+2},x_{n+2})\leq G(x_n,x_{n+1},x_{n+1})$, for all $n\geq 0$, which implies that the sequence $\{G(x_n,x_{n+1},x_{n+1})\}$ is a non-negative monotonically decreasing sequence. So, there exists $r\geq 0$ such that

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1})=r.$$

By taking limit as $n\to\infty$ in (2.17), we get

$$\varphi(r)\leq F\big(\varphi(r),\lim_{n\to\infty}\psi\big(G(x_n,x_{n+1},x_{n+1})\big)\big)\leq \varphi(r),$$

which implies that $F\big(\varphi(r), \lim_{n\to\infty} \psi\big(G(x_n, x_{n+1}, x_{n+1})\big)\big) = \varphi(r)$.
From (F_2), we get $\varphi(r) = 0$ or $\lim_{n\to\infty} \psi\big(G(x_n, x_{n+1}, x_{n+1})\big) = 0$. Since $\varphi \in F_{alt}$ and $\psi \in \Psi$, we get

$$r = \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.18}$$

Now, from definition of G-metric space, we have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{2.19}$$

Now, we prove that $\{x_n\}$ is G-Cauchy. It is sufficient to show that $\{x_{2n}\}$ is a G-Cauchy sequence. Suppose, $\{x_n\}$ is not G-Cauchy. Then, by (2.18), (2.19) and Lemma 1.3.14, there exist $\varepsilon > 0$ and two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that, for all $k \in \mathbb{N}$, $k \leq 2n_k < 2m_k < 2n_{k+1}$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(x_{2n_k+p_1}, x_{2m_k+p_2}, x_{2m_k+p_3}) = \varepsilon. \tag{2.20}$$

Since, $x_{2m_k} \preceq x_{2n_k+1}$, from (2.11), we have

$$\begin{aligned}\varphi\big(G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})\big) &= \varphi\big(G(Sx_{2m_k}, TSx_{2m_k}, Tx_{2n_k+1})\big)\\ &\leq F\big(\varphi\big(M(x_{2m_k}, x_{2n_k+1})\big), \psi\big(M(x_{2m_k}, x_{2n_k+1})\big)\big),\end{aligned} \tag{2.21}$$

where

$$M(x_{2m_k}, x_{2n_k+1}) = \max\Bigg\{ G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1}), \frac{G(x_{2m_k+1}, x_{2m_k+1}, x_{2n_k+1})[1 + G(x_{2m_k}, x_{2m_k}, x_{2n_k+2})]}{1 + G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1})}, \frac{G(x_{2n_k+2}, x_{2n_k+2}, x_{2n_k+1})[1 + G(x_{2m_k+1, 2m_k+1}, x_{2m_k})]}{1 + G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1})}\Bigg\}.$$

By using (2.18), (2.19) and (2.20), we get $\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \varepsilon$.
Taking limit as $k \to \infty$ in (2.21), we get

$$\varphi(\varepsilon) \leq F\big(\varphi(\varepsilon), \lim_{k\to\infty} \psi\big(M(x_{2m_k}, x_{2n_k+1})\big)\big).$$

Since, F is a C-class function, we get

$$\varphi(\varepsilon) \leq F\big(\varphi(\varepsilon), \lim_{k\to\infty} \psi\big(M(x_{2m_k}, x_{2n_k+1})\big)\big) \leq \varphi(\varepsilon),$$

implies that

$$\varphi(\varepsilon) = 0 \text{ or } \lim_{k\to\infty} \psi\big(M(x_{2m_k}, x_{2n_k+1})\big) = 0,$$

implies that $\varepsilon = \lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = 0$, a contradiction. Thus, $\{x_{2n}\}$ is a G-Cauchy sequence in (X, G). So, the sequence $\{x_n\}$ is a G-Cauchy sequence in (X, G). Since, (X, G) is complete, there exists $u \in X$ such that $\{x_n\}$ is G-convergent to u. Therefore, the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are G-convergent to u.

Since $\{x_{2n}\} \subseteq A$ and A is closed, implies that $u \in A$. Also, $\{x_{2n+1}\} \subseteq B$ and B is closed, implies that $u \in B$.

Now, we may assume that S is continuous. So, we have

$$Su = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$$

By uniqueness of the limit we have $Su = u$.

Since $u \preceq u$, from (2.11) we have

$$\begin{aligned} \varphi\big(G(u, Tu, Tu)\big) &= \varphi\big(G(Su, TSu, Tu)\big) \\ &\leq F\big(\varphi\big(M(u, u)\big), \psi\big(M(u, u)\big)\big), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} &M(u, u) \\ &= \max\left\{G(u, Su, u), \frac{G(Su, Su, u)[1 + G(u, u, Tu)]}{[1 + G(u, Su, u)]}, \frac{G(Tu, Tu, u)[1 + G(Su, Su, u)]}{[1 + G(u, Su, u)]}\right\} \\ &= G(u, Tu, Tu). \end{aligned}$$

Using (2.22) and (F_1), we get

$$\begin{aligned} \varphi\big(G(u, Tu, Tu)\big) &\leq F\big(\varphi\big(G(u, Tu, Tu)\big), \psi\big(G(u, Tu, Tu)\big)\big) \\ &\leq \varphi\big(G(u, Tu, Tu)\big). \end{aligned}$$

From (F_2), we get

$$\varphi\big(G(u,Tu,Tu)\big) = 0 \text{ or } \psi\big(G(u,Tu,Tu)\big) = 0.$$

This implies $G(u,Tu,Tu) = 0$. Hence, $Tu = u$. Thus, u is a common fixed point of S and T in $A \cap B$. □

In Theorem 2.3.2, if we replace $\psi \in F_{alt}$ with $\psi \in \Psi$ and take $M(x,y) = G(x,Sx,y)$, $M'(x,y) = G(x,Tx,y)$ and $F(s,t) = s - t$, then we get Theorem A, as a particular case.

The following example shows that the condition (iii) defined in Theorem 2.3.2 is more general than the condition (iii) of Theorem A.

Example 2.3.1. Let $X = \{0,1\}$ and define $G : X \times X \times X \to [0,\infty)$ as

$$G(0,0,0) = G(1,1,1) = 0;\ G(0,0,1) = 1 \text{ and } G(0,1,1) = 2.$$

Then the function G is a G-metric on X.

Take $A = B = \{0,1\}$ and $x \preceq y$ if and only if $x \leq y$. Define the mappings $S,T : X \to X$ as follows:

$$S(0) = 1, S(1) = 0 \text{ and } T(0) = 0, T(1) = 1.$$

Let $\varphi,\psi : [0,\infty) \to [0,\infty)$ and $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

$$\varphi(t) = \frac{t}{2}, \psi(t) = t \text{ and } F(s,t) = \frac{s}{1+t},$$

for all $s,t \in [0,\infty)$.

For $x = 0,\ y = 1$,

$$\begin{aligned} M(0,1) &= \max\left\{G(0,S0,1), \frac{G(S0,S0,1)[1+G(0,0,T1)]}{1+G(0,S0,1)},\right. \\ &\qquad \left.\frac{G(T1,T1,1)[1+G(S0,S0,0)]}{1+G(0,S0,1)}\right\} \\ &= \max\{2,0\} = 2. \end{aligned}$$

Now,

$$F(\varphi(M(0,1)),\psi(M(0,1))) = F(\varphi(2),\psi(2)) = F(1,2) = \frac{1}{3} \geq \varphi(G(S0,TS0,T1)) = \varphi(0).$$

For $x = 1,\ y = 0$,

$$M(1,0) = \max\left\{G(1,S1,0), \frac{G(S1,S1,0)[1+G(1,1,T0)]}{1+G(1,S1,0)}, \frac{G(T0,T0,0)[1+G(S1,S1,1)]}{1+G(1,S1,0)}\right\} = \max\{1,0\} = 1.$$

Now,

$$F(\varphi(M(1,0)),\psi(M(1,0))) = F(\varphi(1),\psi(1)) = F\big(\frac{1}{2},1\big) = \frac{1}{4} \geq \varphi(G(S1,TS1,T0)) = \varphi(0).$$

Hence, the condition (iii) of Theorem 2.3.2 is satisfied.

But, $\varphi(G(0,S0,1)) - \psi(G(0,S0,1)) = \varphi(2) - \psi(2) = 1 - 2 = -1 \leq 0$. Which shows that the condition (iii) of Theorem A does not hold.

Now, the following example validates Theorem 2.3.2.

Example 2.3.2. Let $X = [0, \frac{1}{2}]$ and $S, T : X \to X$ be given as $S(x) = \dfrac{x^2}{1+x}$ and $T(x) = \frac{x}{2}$. Take $A = [0, \frac{1}{2}]$ and $B = [0, \frac{1}{2}]$. Define the function $G : X^3 \to [0, \infty)$ as

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}, & \text{otherwise.} \end{cases}$$

Clearly, G is a complete G-metric on X. We introduce a relation on X by $x \preceq y$ if and only if $y \leq x$. Also, define the functions $F : [0,\infty)^2 \to \mathbb{R}$ by $F(s,t) = s - t$ and $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = 2t$ and $\psi(t) = \dfrac{t}{1+2t}$.

Note that $S(A) = [0, \frac{1}{6}] \subseteq B$ *and* $T(B) = [0, \frac{1}{2}] \subseteq A$.

To prove (i), given $x \in X$,

$$TSx = \frac{x^2}{2(1+x)}.$$

Since $x \in [0, \frac{1}{2}]$, $\dfrac{x^2}{2(1+x)} < \dfrac{x^2}{(1+x)}$. Thus, $TSx \leq Sx$ and hence $Sx \preceq TSx$, for all $x \in X$.

To prove (iii), given $x \in A$ and $y \in B$ with $y \geq x$. Then,

$$G(Sx, TSx, Sy) = \max\left\{\frac{x^2}{(1+x)}, \frac{x^2}{2(1+x)}, \frac{y}{2}\right\} = \frac{y}{2}$$

and

$$M(x,y) = \max\left\{y, \frac{y(1+x)}{(1+y)}, \ \frac{y(1+\frac{y}{2})}{(1+y)}\right\} = y.$$

Since

$$\frac{2y}{2} \leq 2y - \frac{y}{(1+2y)},$$

we have

$$\varphi(G(Sx, TSx, Sy)) \leq F(\varphi(M(x,y)), \psi(M(x,y))).$$

Hence, all the conditions of Theorem 2.3.2 are satisfied. Notice that 0 is the unique common fixed point of S and T.

Corollary 2.3.3. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space. and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let S be a continuous self map on X that satisfies the following conditions:*

(i) $Sx \preceq S^2x$, *for all* $x \in X$.

(ii) $S(A) \subseteq B$ *and* $S(B) \subseteq A$.

(iii) There exist two functions $\varphi \in F_{alt}, \psi \in \Psi$ *such that*

$$\varphi(G(Sx, S^2x, Sy)) \leq F(\varphi(M(x,y)), \psi(M(x,y))) \tag{2.23}$$

holds for all comparative elements $x, y \in X$, *where*

$$M(x,y) = \max\left\{G(x, Sx, y), \frac{G(Sx, Sx, y)[1 + G(x, x, Sy)]}{1 + G(x, Sx, y)},\right.$$

$$\left.\frac{G(Sy,Sy,y)[1+G(Sx,Sx,x)]}{1+G(x,Sx,y)}\right\}.$$

Then, S has a fixed point in $A \cap B$.

Proof. The proof follows from Theorem 2.3.2 by taking $T = S$. □