CHAPTER - V

MISCELLANEOUS ASPECTS OF CURTAILED SAMPLING PLANS

5.1 In this chapter we have considered some miscellaneous properties of the maximum likelihood estimates of the fraction defective in Curtailed Sampling Plans in Situation A only, given in earlier chapters. They are: the relation between ASN and the variance of these estimates, bias, sufficiency etc.

5.2 Average Sample Number :

It may be desirable to recall the definitions of the random variables y, z and their probability functions from Section 2.3 to derive the ASN for the various Plans.

Plan 1: ASN is obviously n in this Plan.

Plan 2: Since the sample number takes the value n when a lot is accepted and takes the values associated with y when a lot is rejected, the average sample number will be

$$\text{ASN} = n \sum_{x=0}^{k-1} \binom{n}{x} p^x q^{n-x} + \sum_{y=k}^{n} y \binom{y-1}{k-1} p^k q^{y-k}$$

$$= nB(p,n,k-1) + K\left[1-B(p,n+1,k)\right]/p$$

$$= J_2/p = J_1/p \text{ since } J_2 = J_1 \qquad \ldots(5.1)$$

Plan 3: In this Plan, since the sample number takes the values associated with z when a lot is accepted and takes the values associated with y when a lot is rejected, we obtain the following expression for the average sample number :

$$ASN = \sum_{z=n-k+1}^{n} z\binom{z-1}{n-k} q^{n-k+1} p^{z-(n-k+1)} + \sum_{y=k}^{n} y\binom{y-1}{k-1} p^{k} q^{y-k}$$

$$= (n-k+1)B(p,n+1,k-1)/q + k[1-B(p,n+1,k)]/p$$

$$= J_4/p = J_3/p, \qquad \text{since } J_4 = J_3 \qquad \ldots(5.2)$$

Expressions (5.1) and (5.2) are alternately derivable from expressions (3.8) and (3.10) since

$$(nT_{a,2} + \sum_{y=k}^{n} yr_{y,2})/T$$

and $$[\sum_{i=0}^{k-1} ia_{i,3} + (n-k+1)\, T_{a,3} + \sum_{y=k}^{n} yr_{y,3}]/T$$

are the numbers of articles inspected per lot, respectively in Plans 2 and 3.

Lemma 1 and 2 proved in Section 3.2.3 lead to

$$ASN(\text{plan } 3) \leq ASN(\text{Plan } 2) \leq ASN(\text{Plan } 1) \qquad \ldots(5.3)$$

which is of course an obvious fact.

5.2.1 Relation Between ASN and V(MLE) :

Comparing the expressions of the asymptotic variances of the maximum likelihood estimate of p given by (3.15), (3.16), and (3.17) with those of ASN given by (5.1), (5.2) and (5.3) we find that in all the plans the ASN has a common feature, namely,

$$\text{ASN} = \frac{pq}{TV(\hat{p})} \qquad \ldots(5.4)$$

Thus, for fixed T and p, the variance of the estimate is inversely proportional to the average sample number.

5.2.2 Saving in Inspection Versus Loss in Efficiency:

As one passes from Plan 1 to Plan 2 there is saving in inspection. This saving, in per cent, may be defined as

$$\frac{\text{ASN (Plan 1)} - \text{ASN (Plan 2)}}{\text{ASN (Plan 1)}} 100$$

$$= (1 - J_1/np)100 \qquad \ldots(5.5)$$

Similarly, the percentage saving in the inspection as one switches over to Plan 3 from Plan 1 will be defined as

$$\frac{\text{ASN (Plan 1)} - \text{ASN (Plan 3)}}{\text{ASN (Plan 1)}} 100$$

$$= (1 - J_3/np)100 \qquad \ldots(5.6)$$

Then recalling (3.21) and (3.22) which give the efficiencies in estimation of Plans 2 and 3 with respect to Plan 1, we see that the percentage loss in efficiencies in estimation as one passes from Plan 1 to Plans 2 and 3 are

$$(1 - J_1/np)100$$

and $$(1 - J_3/np)100 .$$

Thus one can state that loss in efficiency in estimation is counter-balanced by saving in inspection. It may be restated that the purpose of these plans is not to estimate the fraction defective, but a reduction in inspection cost. One has always to pay a price for any reduction in inspection and in this case, this price has been paid by a decrease (reduction) in the efficiency of the estimate.

The above fact is numerically illustrated in the Tables 5.1 and 5.2. Columns (3), (4) and (5) of Table 5.1 give ASN of Plans 1, 2, and 3 for n=25, k=3. Columns (6), (7) and (8) give the asymptotic variance of the maximum likelihood estimates for these plans. They are reproduced from Table 3.4 just for ready reference. Columns (9) and (10) give saving in inspection or loss in efficiency in per cent. An entry in Column (9) is

$$\left(1 - \frac{\text{Column (4)}}{\text{Column (3)}} \right) 100$$

or

$$\left(1 - \frac{\text{Column (6)}}{\text{Column (7)}} \right) 100.$$

Table 5.2 is an additional example for n=80, k=5.

It is revealed from columns (9) and (10) that the saving in inspection (and loss in efficiency) increases as p increases. This is in accordance with expectations, since a greater fraction defective means a greater probability of

Table- 5.1

n=25, k=3

p	Prob. of Accept.	ASN			$TV(\hat{p})$			Saving in Insp. Or Loss in Efficiency in % Plan 1=100	
		Plan 1	Plan 2	Plan 3	Plan 1	Plan 2	Plan 3	Plan 2	Plan 3
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0.04	0.92352	25	24.51	23.37	0.001536	0.001567	0.001643	1.98	6.51
0.05	0.87289	25	24.15	23.18	0.001900	0.001967	0.002049	3.41	7.27
0.07	0.74656	25	23.16	22.47	0.002604	0.002811	0.002897	7.36	10.12
0.09	0.60630	25	21.89	21.43	0.003276	0.003741	0.003822	12.43	14.28
0.11	0.47087	25	20.48	20.17	0.003916	0.004780	0.004854	18.08	19.32
0.13	0.35171	25	19.02	18.81	0.004524	0.005948	0.006012	23.94	24.75
0.15	0.25374	25	17.57	17.44	0.005100	0.007255	0.007309	29.70	30.22
0.17	0.17739	25	16.21	16.13	0.005644	0.008706	0.008750	35.17	35.50
0.19	0.12045	25	14.94	14.89	0.006156	0.010300	0.010334	40.23	40.43
0.20	0.09823	25	14.35	14.31	0.006400	0.011147	0.011178	42.58	42.74

Table-5.2

n=80,k=5

p	Prob. of acceptance	ASN			TV($\hat{p}$)			Saving in Insp. or Loss in Efficiency in % Plan 1=100	
		Plan 1	Plan 2	Plan 3	Plan 1	Plan 2	Plan 3	Plan 2	Plan 3
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0.03	0.90721	80	78.42	76.63	.00036375	.00037106	.00037974	1.97	4.21
0.04	0.78358	80	75.94	74.68	.00048000	.00050586	.00051418	5.08	6.65
0.05	0.62888	80	72.19	71.39	.00059375	.00065799	.00066537	9.76	10.76
0.06	0.47174	80	67.63	67.13	.00070500	.00083397	.00084019	15.46	16.09
0.07	0.33333	80	62.67	62.37	.00081375	.00103875	.00104376	21.66	22.04
0.08	0.22350	80	57.70	57.52	.00092000	.00127558	.00127946	27.88	28.09
0.09	0.14311	80	52.97	52.87	.00102375	.00154609	.00154896	33.78	33.91
0.10	0.08797	80	48.64	48.58	.00112500	.00185045	.00185252	39.20	39.27

rejection and thereby curtailingof the inspection in a greater number of cases. Furthermore, it is worth noting that there is no appreciable saving in inspection if one administers Plan 3 instead of Plan 2.

5.2.3 A Remark on Craig's Result [14] :

Craig [14] has stated that Statistical Research Group, Columbia University [56] has considered ASN of both the curtailed sampling plans of this text, namely, Plans 2 and 3, and adds that neither [56] nor Bury [5] , who has considered ASN of these plans, has given any numerical example. It is. clear from Craig's paper that he considers ASN of only Plan 2 and ignores Plan 3, stating merely that the effect on ASN of Plan 3 is small (but surprisingly he does not confirm it numerically). He then gives new formulas for the probability of acceptance and ASN, which he claims are more convenient for calculation, if sample sizes are large and acceptance numbers are small and the existing binomial tables are not adequate.

Firstly, we want to state that the formulas (5.2) and (5.3) for ASN of Plans 2 and 3 and the formulas given by [56] are basically the same. Patil [48] also has derived ASN of Plan 3 which is basically the same as (5.3). But the presentation of the formulas given by [56] is rather clumsy.
Secondly we can calculate ASN of Plans 2 and 3 easily by using

(5.2) and (5.3) and the Tables of cumulative Binomial probability Distribution, for all the typical examples he has worked out. At this stage, the importance of the recurrence relation (3.11) giving B(p,n+1,k) in terms of B(p,n,k) should be pointed out. It is very likely that, for large values of n the binomial tables may not give the cumulative probability at an unit interval for n. We have used this recurrence relation (3.11) to over-come this difficulty while calculating J_2 and J_4. Calculations of J_2 and J_4 are required to determine ASN and $V(\hat{p})$. Table 5.3 gives the details of the cauculations of ASN for Plans 2 and 3 for n=100, c=2, k=3. This is one of typical examples considered by Craig. The ASN of Plan 2 given in Column (6) of this table tallys with that given by Craig. As explained earlier, the reduction in inspection when one uses Plan 3 instead of Plan 2 decreases as p increases (or as probability of rejection of a lot, given in Column (2), increases).

5.3 Bivariate Approach :

The probability functions f_j given by (2.2), (2.3), (2.4) associated with the Plans given in Section 2.3 can be thought as bivariate distributions. Let the number of defectives in a group of inspected articles be denoted by X and the number of items inspected be denoted by Y. Then the bivariate probability functions associated with the plans are as follows:

Table 5.3

$n=100,\ c=2,\ k=3$

p	$\sum_{k}^{n}$ *	$\sum_{k+1}^{n}$ *	$p(2)+q(3)$ $=\sum_{k+1}^{n+1}$	$np[1-(2)]+k(4)$ $=J_2$	J_2/p =ASN of Plan 2	$\sum_{k-1}^{n}$ *	$p(7)+q(2)$ $=\sum_{k}^{n+1}$	$\sum_{k+1}^{n+1}$ *	$(n-k+1)p\cdot[1-(8)]+kq(9)$ $=qJ_4$	$(10)/pq$ $=J_4/p$ =ASN of Plan 3
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0.01	.07937	.01837	.01898	0.97757	97.757	.26424	.08122	.01898	0.95678	96.644
0.02	.32331	.14104	.14469	1.78745	89.373	.59673	.32878	.14469	1.74098	88.826
0.03	.58022	.35275	.35957	2.33805	77.935	.80538	.56697	.35957	2.26066	77.686
0.04	.76786	.57052	.57841	2.66379	66.595	.91284	.77366	.57841	2.55307	66.486
0.05	.88174	.74216	.74914	2.83872	56.774	.96292	.88580	.74914	2.69463	56.729
0.06	.94339	.85698	.86216	2.92614	48.769	.98483	.94588	.86216	2.74952	48.750
0.07	.97421	.92559	.92899	2.96750	42.393	.99399	.97559	.92899	2.75934	42.386

Here $\sum_{k}^{n} = \sum_{x=k}^{n} \binom{n}{x} p^x q^{n-x}$ and number in bracket represents column number.

* Read from the Tables of [55] .

Plan 1 :

P (X defectives, Y items inspected)

$$= \begin{cases} \binom{n}{X} p^X q^{n-X}, & 0 \leqslant X \leqslant k-1, \quad Y=n \\ \binom{n}{X} p^X q^{n-X}, & k \leqslant X \leqslant n, \quad Y=n \\ 0, & \text{otherwise} \end{cases} \quad \ldots(5.7)$$

Plan 2:

P (X defectives, Y items inspected)

$$= \begin{cases} \binom{n}{X} p^X q^{n-X}, & 0 \leqslant X \leqslant k-1, \quad Y=n \\ \binom{Y-1}{k-1} p^k q^{Y-k}, & X=k, \quad k \leqslant Y \leqslant n \\ 0, & \text{otherwise} \end{cases} \quad \ldots(5.8)$$

Plan 3:

P (X defectives, Y items inspected)

$$= \begin{cases} \binom{n-k+X}{n-k} q^{n-k+1} p^X & 0 \leqslant X \leqslant k-1, \quad Y=X+n-k+1 \\ \binom{Y-1}{k-1} p^k q^{Y-k} & X = k, \quad k \leqslant Y \leqslant n \\ 0 & \text{otherwise} \end{cases} \quad \ldots.(5.9)$$

In each of the above probability functions, the first part is associated with the acceptance of the lot and the second part is associated with the rejection of the lot. In cases of Plans 2 and 3, the above functions assign probabilities to points on two different lines, whereas in case of Plan 1, the function assigns probabilities to points on a single line, it being a degenerate case.

Furthermore, each accepted or rejected lot will give rise to one observed pair (X,Y). For instance, if Plan 2 is administered on T lots, we will have T pairs of (X, Y) where for accepted lots the pairs will be of the form (X, n), $0 \leq X \leq k-1$, and for rejected lots the pairs will be of the form (k, Y), $k \leq Y \leq n$. One can further proceed with these pairs, as in Sections 3.2 and its subsections to obtain the likelihood functions, estimates of the fraction defective and asymptotic variances. The results will be identical. It is a matter of choice whether to regard the probability function as a bivariate probability function or as a univariate probability function of a hypothetical variable s.

5.4 Bias :

Estimation of fraction defective under curtailed sampling plans was introduced by Girshick, Mosteller and Savage as early as in 1946 [28] . They considered estimation based on a single lot. Their main work was to determine an unique unbiased estimate in a sequence of binomial trials. In Section 3 of [28] , they have obtained the unquie unbiased estimate when the inspection is of curtailed nature which resembles Plan 3 of our work. The estimate given by them may be denoted by $\hat{p}_g$ which is as follows:

$$\hat{p}_g = \frac{\text{Number of defectives observed}}{\text{One less than the number of articles inspected}} \quad \ldots(5.10)$$

if a lot is accepted

and $$\hat{p}_g = \frac{\text{One less than the number of defectives observed}}{\text{One less than the number of articles inspected}}$$

if a lot is rejected.

This estimate in the present notations can be expressed as

$$\hat{p}_g = \begin{cases} \frac{i}{z-1} \left(= \frac{i}{n-k+i}\right), & \text{if a lot is accepted} \\ \frac{k-1}{y-1}, & \text{if a lot is rejected.} \end{cases} \quad \ldots(5.11)$$

Following their theory, the unique unbiased estimate of the fraction defective in the remaining plans, namely, Plans 1 and 2 can be obtained. Before we give these estimates we would like to mention that the theory developed in [28] is well explained in Reliability Management, Methods and Mathematics [42] . Their treatment is lucid.Then the unique unbiased estimates of Plans 1 and 2, are as follows:

Plan 1:

$$\hat{p}_g = \frac{\text{Number of defectives in n inspected articles}}{\text{Number of articles inspected}}$$

$$= \frac{x}{n} \quad \ldots(5.12)$$

Plan 2:

$$\hat{p}_g = \frac{\text{Number of defectives in n inspected articles}}{\text{Number of articles inspected}}$$

if lot is accepted

and $\hat{p}_g = \dfrac{\text{One less than the number of defective observed}}{\text{One less than the number of articles inspected}}$

if lot is rejected

i.e. $$\hat{p}_g = \begin{cases} \dfrac{x}{n} & \text{if lot is accepted} \\ \dfrac{k-1}{y-1} & \text{if lot is rejected} \end{cases} \qquad \ldots(5.13)$$

Now we have noted that the maximum likelihood estimate of the fraction defective derived in Section 3.2.1 has a common feature, namely,

$$\hat{p} = \frac{\text{Total number of defectives noted}}{\text{Total number of articles inspected}}$$

irrespective of the fact whether a lot is accepted or rejected and valid for any number of lots inspected.

Therefore, if a single lot is inspected, the above formula leads to the following maximum likelihood estimate in the respective plans:

Plan 1:

$$\hat{p} = \frac{x}{n} \qquad \ldots(5.14)$$

Plan 2:

$$\hat{p} = \begin{cases} \dfrac{x}{n} & \text{if a lot is accepted} \\ \dfrac{k}{y} & \text{if a lot is rejected} \end{cases} \qquad \ldots(5.15)$$

Plan 3:

$$\hat{p} = \begin{cases} \dfrac{i}{z} \left(\text{or } = \dfrac{i}{n-k+1 \pm l}\right), & \text{if a lot is accepted} \\ \dfrac{k}{y}, & \text{if a lot is rejected} \end{cases} \qquad \ldots(5.16)$$

Comparing (5.15) and (5.16) with (5.12) and (5.11) and remembering the property of the uniqueness of the unbiased estimates of the latter, we conclude that the maximum likelihood estimates given by (5.14) and (5.15) are biased. Unique unbiased estimate and the maximum likelihood estimate are obviously identical in Plan 1.

5.5 Sufficiency :

The usual criterion for a sufficient statistic is as follows:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the density $f(x;\theta)$, $a < x < b$, where a and b do not involve θ, and let the joint density of these n random variables be $g(x_1,x_2,\ldots,x_n;\theta) = f(x_1;\theta).\ f(x_2;\theta)..f(x_n;\theta)$ If this density factors as

$$g(x_1,x_2,\ldots x_n;\theta) = h(\hat{\theta};\ \theta).\ k(x_1,x_2,\ldots,x_n) \qquad \ldots(5.17)$$

where $k(x_1,x_2,\ldots,x_n)$ does not involve the parameter θ, then $\hat{\theta}$ is a sufficient statistic for θ.

Now, likelihood function L_1 for Plan 1 given in Section 3.2.1 can be expressed as

$$L_1 = \text{Const}\ p^{nT\hat{p}_1}\ q^{nT-nT\hat{p}_1}$$

where $\hat{p}_1$ is the maximum likelihood estimate of p as given in (3.1).

It may be remembered that T in the above expression represents size of the sample which is fixed. Thus, we find that $\hat{p}_1$ is a sufficient statistic for p, which is otherwise obvious.

Likelihood functions L_2, L_3 for Plans 2 and 3 given in the same section, namely, Section 3.2.1 can not be factorized as desired in (5.17); for instance the likelihood function L_2 for Plan 2 can be expressed as

$$L_2 = \text{const}\; p^{(n\sum a_{x,2}+\sum yr_{y,2})\hat{p}_2} \cdot q^{(n\sum a_{x,2}+\sum yr_{y,2})(1-\hat{p}_2)} \qquad \ldots(5.18)$$

where $\hat{p}_2$ is the maximum likelihood estimate given by (3.2).

Comparing (5.18) with (5.17) we find that the factor of (5.18), namely,

$$p^{(n\sum a_{x,2}+\sum yr_{y,2})\hat{p}_2} \cdot q^{(n\sum a_{x,2}+\sum yr_{y,2})(1-\hat{p}_2)}$$

cannot be expressed as $h(\hat{\theta};\theta)$ i.e. as a function of a statistic and the parameter. Hence the maximum likelihood estimate $\hat{p}_2$ of Plan 2 is not a sufficient statistic for p. Same is true in case of Plan 3. Thus maximum likelihood estimates namely $\hat{p}_2$, $\hat{p}_3$ of p under Curtailed Sampling Plans are not sufficient.

5.6 Minimum Variance Bound (MVB) :

Let us verify whether the minimum variance bound unbiased estimator of p exists in the respective plans. If the likelihood function L can be expressed as

$$\frac{\partial \log L}{\partial p} = \frac{TI(p)}{\phi'(p)} \left[H - \phi\ (p) \right] \qquad \ldots(5.19)$$

where (i) H is a statistic,

(ii) $\phi(p)$ is some function of p such that $E(H)=\phi(p)$,

and (iii) $I(p)$ = Amount of information = $-E(\partial^2 \log L/\partial p^2)/T$

then H is the minimum variance bound (MVB) unbiased estimator of $\phi(p)$.

We observed that in all the plans $I(p) = (ASN)/pq$.

In Plan 1 condition (5.19) is fulfilled if we take $\phi(p)=p$, $H=\hat{p}_1$= the maximum likelihood estimate of p given by (3.1). Thus, MVB unbiased estimator of p exists in Plan 1 and is given by the maximum likelihood estimate $\hat{p}_1$. Such is not the case in Plans 2 and 3.