CHAPTER - VI

A TABLE FOR OBTAINING MAXIMUM LIKELIHOOD ESTIMATES FOR THE PARAMETERS OF THE SINGLY TRUNCATED NORMAL DISTRIBUTION

6.1 Introduction :

Sometimes, some of the observations below or above a certain value, are discarded in a Normal population say $N(\mu, \sigma^2)$ and a sample is drawn from the remaining population, called a Truncated Normal population. It is then required to estimate μ and σ from the sample values. It is easy to write down equations for estimating μ and σ by the method of maximum likelihood. But the computational part of solving these equations to get estimates of μ and σ presents some difficulty and various workers [13,35,36,37(b),9,12] have given computational methods and tables, graph etc. to obtain easily the maximum likelihood estimates of a Truncated Normal Population. We shall give here a very simple method, with only a single table, for obtaining the maximum likelihood estimates of the parameters of a Truncated Normal Population from a random sample from it. Furthermore, we give their asymptotic variances and the covarience between them. The cases of truncation on the right and of truncation on the left are considered.

6.2 Truncation on the Right :

6.2.1 Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a Truncated Normal distribution, truncated in the right, whose probability density function is given by

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\ \sigma\ \Phi(t)} \exp\left[-(x-\mu)^2/(2\sigma^2)\right] \quad , x \leq x_0 \quad ..(6.1)$$

where (i) x_0 is a certain fixed values of x, called the point of truncation,

(ii) $t = (x_0-\mu)/\sigma$,

(iii) $\Phi(t) = \int_{-\infty}^{t} \phi(u)\ du$,

and (iv) $\phi(u) = (1/\sqrt{2\pi})\exp(-u^2/2)$

Then the logarithm of the likelihood function of the above sample is

$$\text{Log L} = \text{const} - n \log \sigma - \sum(x-\mu)^2/2\sigma^2 - n \log \Phi(t) \quad ...(6.2)$$

and the maximum likelihood equations for estimating μ and σ are as follows:

$$\partial \log L / \partial\mu = \bar{x} - \mu + \lambda\sigma = 0, \quad ...(6.3)$$

$$\partial \log L / \partial\sigma = -n\sigma^2 + n\lambda t\sigma^2 + \sum(x-\mu)^2 = 0; \quad ...(6.4)$$

where $\lambda = \phi(t)/\Phi(t)$. ...(6.5)

Using the notations

$\bar{x} = \Sigma x/n$, $s^2 = \Sigma(x-\bar{x})^2/n$, $d = x_0 - \bar{x}$,

$z = t + \lambda$, ...(6.6)

and $\psi = s^2/(s^2+d^2)$, ...(6.7)

we get, from equations (6.3) and (6.4)

$$\hat{\sigma} = d/z \; , \qquad \text{...(6.8)}$$

$$\psi = (1+tz-z^2)/(1+tz), \qquad \text{...(6.9)}$$

$$\gamma\sigma = (\sigma^2 - s^2)/d. \qquad \text{...(6.10)}$$

Using (6.8) and (6.10), we get from (6.4),

$$\hat{\mu} = \bar{x} + (\hat{\sigma}^2 - s^2)/d \qquad \text{...(6.11)}$$

To facilitate computation, a table is prepared which gives the value of z for a given value of ψ. Table 6.1 gives the values of z satisfying (6.9) for $\psi = 0.06(0.01)0.45$.

6.2.2 The estimation procedure is as follows:

From the given sample, calculate

$$\bar{x} = \sum x/n,$$

$$s^2 = \sum(x-\bar{x})^2/n,$$

$$d = x_0 - \bar{x},$$

$$\psi = s^2/(s^2+d^2).$$

Read from the Table 6.1 the value of z for the sample value of ψ. Substitute this value of z in (6.8) to obtain $\hat{\sigma}$ and then the substitution of $\hat{\sigma}$ in (6.11) yields the estimate of μ.

6.2.3 Variance of covariance matrix for $(\hat{\mu}, \hat{\sigma})$.

We note the following results :

(i) $E(x) = \mu - \gamma\sigma = \mu^*$,

(ii) $E(x-\mu^*)^2 = \sigma^2(1-\gamma z)$,

(iii) $-\frac{\sigma^2}{n} E\left(\frac{\partial^2 \log L}{\partial \mu^2}\right) = 1+tz-z^2 = v_{11}$,

(iv) $-\frac{\sigma^2}{n} E\left(\frac{\partial^2 \log L}{\partial \mu \partial \sigma}\right) = -(z-t)(1+tz) = v_{12}$,

(v) $-\frac{\sigma^2}{n} E\left(\frac{\partial^2 \log L}{\partial \sigma^2}\right) = 2 - t(z-t)(1+tz) = v_{22}$,

and

(vi) $\begin{vmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{vmatrix} = 2 - z(z-t)(3+tz)$.

Then noting that the matrix of the asymptotic variances and covariance of $\hat{\mu}$ and $\hat{\sigma}$ is given by $\frac{\sigma^2}{n}\left[\sigma_{ij}\right]$ where $\sigma_{ij} = \left[v_{ij}\right]^{-1}$

we have

$$v(\hat{\sigma}) \doteq \frac{\sigma^2}{n} \cdot \frac{1+tz-z^2}{2-z(z-t)(3+tz)} \qquad \ldots(6.12)$$

$$v(\hat{\mu}) \doteq \frac{\sigma^2}{n} \cdot \frac{2-t(z-t)(1+tz)}{2-z(z-t)(3+tz)} \qquad \ldots(6.13)$$

and

$$\mathrm{Cov}(\hat{\sigma}, \hat{\mu}) \doteq \frac{\sigma^2}{n} \cdot \frac{(z-t)(1+tz)}{2-z(z-t)(3+tz)} \qquad \ldots(6.14)$$

6.3 Truncation on the Left :

6.3.1 In the case of truncation on the left, the probability density function is :

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma\,[1-\Phi(t)]} \exp\left[-(x-\mu)^2/2\sigma^2\right], \quad x \geqslant x_0 \qquad \ldots(6.15)$$

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from this population. Then following the same procedure as described in Section 6.2.1 and noting that, in this Case,

$$\gamma = \phi(t)/[1-\Phi(t)] \quad , \qquad \ldots(6.16)$$

$$z = -t+\gamma \quad , \qquad \ldots(6.17)$$

$$\psi = (1-tz-z^2)/(1-tz), \qquad \ldots(6.18)$$

we have the following maximum likelihood estimates of μ and σ.

$$\hat{\sigma} = (-d)/z \quad , \qquad \ldots(6.19)$$

$$\hat{\mu} = \bar{x} + (\hat{\sigma}^2 - s^2)/d \ . \qquad \ldots(6.20)$$

6.3.2 Estimation procedure in this case is exactly the same as that described in Section 6.2.2, except that the estimating equations are (6.19) and (6.20) instead of (6.8) and (6.11) It may be noted that here $-d$ is positive and the value of z for the sample value of $\psi = s^2/(s^2+d^2)$ is to be read from the same table, namely, Table 6.1.

6.3.3 Again following the same procedure as described in Section 6.2.3 we have the following asymptotic variances and covariance of $\hat{\mu}$ and $\hat{\sigma}$:

$$V(\hat{\sigma}) \doteq \frac{\sigma^2}{n} \frac{1-tz-z^2}{2-z(z+t)(3-tz)} \qquad \ldots(6.21)$$

$$V(\hat{\mu}) \doteq \frac{\sigma^2}{n} \frac{2+t(z+t)(1-tz)}{2-z(z+t)(3-tz)} \qquad \ldots(6.22)$$

$$Cov(\hat{\mu},\hat{\sigma}) = -\frac{\sigma^2}{n} \frac{(z+t)(1-tz)}{2-z(z+t)(3-tz)} \qquad \ldots(6.23)$$

6.4 Range of Ψ :

The minimum and the maximum values of Ψ are 0 and 1 theoretically. However, minimum and maximum values considered in the Table 6.1 are 0.06 and 0.45. The values of t corresponding to these values are 3.96 and -1.97 approximately; and therefore, the corresponding values of $\Phi(t)$ are 0.99996 and 0.02442. Thus in the first case, namely, Ψ =0.06 the population available is almost complete from the practical point of view, or in other words the portion of the population left over is almost negligible. In the case of Ψ =0.45 the part of the population available is 2.4%. Hence, from practical considerations, the sample value of Ψ will be in the range of 0.06 to 0.45 and tables are prepared for Ψ =0.06(0.01)0.45 only.

6.5 Numerical Examples :

We demonstrate the use of the Table 6.1 for estimation of μ and σ where the sample is drawn from the Normal population which is truncated on the left. We have solved two such examples. The purpose of solving the two examples will be clear in Section 6.7.

6.5.1 Example 1

We take the example which has been worked out by Cohen and Woodward (1953) [13] . The data of their example are summarized as follows:

$n=37$, $\Sigma y=51.8600$, $\Sigma y^2=98.0156$ and $x_0=0.8500$ where y is measured from x_o as origin.

Hence we find that

$\bar{x} = 2.25162$, $s^2= 0.68453$, $d= -1.40162$ and

$\Psi = s^2/(s^2+d^2)=0.25840$.

Then referring to the Table 6.1 we have $z=1.39519$ for $\Psi=0.25840$ by linear interpolation. So estimates of σ and μ using (6.19) and (6.20) are

$$\hat{\sigma} = 1.0046, \quad \hat{\mu} = 2.0200.$$

Then, the estimate of $t=(x_o-\mu)/\sigma$ is $(x_0-\hat{\mu})/\hat{\sigma} = -1.16464$. Substituting the estimates of t and z in (6.21), (6.22) and (6.23) we have $V(\hat{\sigma}) = 0.0361$, $V(\hat{\mu})=0.0690$, $Cov\ (\hat{\mu},\hat{\sigma})= -0.0323$. The value obtained by Cohen and Woodward in [13] are as follows:

$$\hat{\mu} = 2.020, \quad \hat{\sigma} = 1.0047, \quad V(\hat{\sigma}) = 0.0365.$$

6.4.2 Example 2

The data of this example are from Cohen (1961) [12] which are summarized as follows:

$X_0 = 0.1215$, $\bar{x} = 0.124624$, $s^2=(2.1106)10^{-6}$, $n=100$,

Following the same procedure as given in Example 1 we find that

$\psi = 0.1778$, $z=2.01518$, $\hat{\sigma}=0.0015502$.

$\hat{\sigma}^2 = (2.403)10^{-6}$, $\hat{\mu}=0.124530$,

and $V(\hat{\sigma}) = (1.8301)10^{-8}$, $V(\hat{\mu})=(2.9499)10^{-8}$,

$Cov(\hat{\mu},\hat{\sigma}) = -(0.6238)10^{-8}$.

The values obtained by Cohen where he uses his table of [12] are as follows:

$\hat{\sigma} = 0.00155$, $\hat{\sigma}^2=(2.405)10^{-6}$, $\hat{\mu}=0.1245$

and $V(\hat{\sigma})= (1.85)10^{-8}$, $V(\hat{\mu})=(2.98)10^{-8}$, $Cov(\hat{\mu},\hat{\sigma})= -(0.65)10^{-8}$

6.6 Similarity with Gupta's Statistic [32] .

Now the statistic $\psi=s^2/(s^2+d^2)$ used in our method is similar to that proposed by Gupta [32] , which he uses for obtaining the maximum likelihood estimates of the parameters of the normal distribution based on censored sample of Type II. He converts the estimating equation so as to have one auxiliary estimating function. First, he estimates σ by d/z where z is to be read from his tables for the sample value $\psi= s^2/(s^2+d^2)$ and percentage of the censored observation. Our procedure of estimating the parameters of singly truncated normal distribution is similar to this procedure. It is obvious that one cannot use Gupta's Table [32] for our problem, because of the difference between censored samples and samples from a truncated distribution.

6.7 References to Other Tables :

Hald (1949) [34] and also Cohen and Woodward (1953) [13] had prepared Tables to facilitate the computation of the maximum likelihood estimates of the parameters of a singly truncated Normal population. In Cohen and Woodward's method one has to interpolate twice: an inverse interpolation for t and a direct interpolation for 1/z. Hald's method is the same as that of Cohen and Woodward, except that the first step is a direct interpolation. Hald demonstrated the use of his Tables is Statistical Theory with Engineering Applications (1952) [35] and reproduced these Tables in Statistical Tables and Formulas (1952) [36] . The Table given by us in this chapter was prepared by us in 1960 and the work was submitted to the Journal of American Statistical Association. They did not accept the paper, with the remark that the present Table did not make a substantial improvement. It is, however, clear that our metnod is simpler than these two methods, as it requires only one interpolation, which being a direct one, is easy to make and is more accurate.

We later on found that Cohen independently improved his earlier Tables of 1953 [13] on our lines in two papers of 1959 [9] and 1961 [12] . The improvement was in the direction to have only one auxiliary estimating function. In the first paper of 1959 [9] he took into account all the cases -

samples from singly truncated population and singly censored samples of both the types, Types I and II. It appears that the improvement was not complete. The presentation of the tables in this paper [9] did not take a proper form from the users point of view. He, therefore, remarked in the next paper which appeared in 1961 [12] that the estimates as well as their asymptotic variances are relatively easy to calculate when the necessary tables are available, but unfortunately the tables originally provided in [35,36,32, 13, 9] failed to prove adequate in all cases. He, therefore, presented extensive tables in this paper (1961) [12] to meet all the cases. In order to use these Tables, one has to compute $\gamma = s^2/d^2$ instead of our $\psi = s^2/(s^2+d^2)$. Clearly the relation between γ and ψ is $\psi = \gamma/(1+\gamma)$. Table 1 of (1961) [12] which is to be used for estimating the parameters of singly truncated normal distribution has finer grid for γ, namely $\gamma = 0.050(0.001)0.859$. For these values of γ the values of ψ range from 0.047 to 0.462.

Coming back to Section 6.5 we see that we have solved examples worked out by Cohen from his (1953) paper [13] and (1961) paper [12] by using our Table 6.1. The results obtained by using Table 6.1 do not differ materially from the corresponding results in the examples worked out by Cohen. Even though the grid in Table 6.1 is coarse we can estimate μ and σ with reasonable accuracy. If one is desirous to print a collection of tables, the Table 6.1 would save space.

It may be further noted that Left Hand Part of Table 3 of Cohen (1961) [12] gives the coefficient of σ^2/n in the various expressions (6.21), (6.22), (6.23) useful in calculating the asymptotic variances and covariance of the estimates of μ and σ .

Table 6.1

ψ	z	ψ	z
0.06	3.95688	0.26	1.38460
0.07	3.64145	0.27	1.32052
0.08	3.38365	0.28	1.25838
0.09	3.16628	0.29	1.19803
0.10	2.97847	0.30	1.13933
0.11	2.81299	0.31	1.08215
0.12	2.66486	0.32	1.02636
0.13	2.53053	0.33	0.97185
0.14	2.40744	0.34	0.91853
0.15	2.29362	0.35	0.86628
0.16	2.18761	0.36	0.81501
0.17	2.08823	0.37	0.76462
0.18	1.99458	0.38	0.71499
0.19	1.90590	0.39	0.66602
0.20	1.82161	0.40	0.61758
0.21	1.74118	0.41	0.56952
0.22	1.66422	0.42	0.52165
0.23	1.59036	0.43	0.47378
0.24	1.51931	0.44	0.42556
0.25	1.45079	0.45	0.37662