CHAPTER 7

ESTIMATION IN THE TRUNCATED NEGATIVE
BINOMIAL DISTRIBUTION

7.1 Introduction

Examples of the zero-truncated negative binomial distribution have been given by Sampford $\angle 39\_7$ and Brass $\angle 9\_7$. The estimation of the parameters of this distribution was discussed by David and Johnson $\angle 15\_7$, Rider $\angle 36\_7$, Sampford $\angle 39\_7$, Brass $\angle 9\_7$ and Khatri $\angle 24\_7$. David and Johnson $\angle 15\_7$ have considered the maximum likelihood estimation of one parameter and its three-moments estimator and observed that three-moments estimator was inefficient and so recommended the use of maximum likelihood estimator. Sampford $\angle 39\_7$ has discussed the estimation of the parameters using the first two moments which was also a trial-and-error method but less laborious than the method of maximum likelihood. Brass $\angle 9\_7$ has given simplified methods of estimating the parameters and also their asymptotic

efficiency. Khatri $\underline{/}$24$\underline{/}$ has extended the simplified method to the case of doubling truncated negative binomial distribution.

Rider $\underline{/}$36$\underline{/}$ obtained the estimators of the parameters of the zero-truncated negative binomial distribution in terms of the first three sample moments, but their asymptotic variances and covariance were not given by him. In this chapter, Rider's method has been extended to obtain the method of moments estimators of the parameters of the truncated negative binomial distribution in which k classes from the lower end are truncated. Further, the asymptotic variances and covariance of the three-moments estimators are derived and the three-moments estimators are compared with the maximum likelihood estimators in the case when one class from the lower end is truncated.

7.2 The truncated negative binomial distribution

The probability law of the negative binomial distribution in which k classes from the lower end of the distribution are truncated is

(7.2.1) $\qquad p_x = L_x/S, \qquad (x = k,\ k+1,\ \ldots,\ \infty)$

where $L_x = \binom{m+x-1}{x} p^x (1+p)^{-(m+x)}$, $\quad S = \sum\limits_{x=k}^{\infty} L_x$ and $p > 0$, $m > 0$. The probabilities p_x satisfy the following

recurrence relation.

$$(7.2.2) \qquad (1 + p)(x + 1)p_{x+1} = p(m + x)p_x,$$

$$(x = k, \ k+1, \ \ldots, \ \infty).$$

Let $\alpha_r = \sum\limits_{x=k}^{\infty} x^r p_x$ denote the rth order raw

moment of the truncated negative binomial distribution

(7.2.1). Multiplying (7.2.2) by $(x + 1)^r$ and taking the

sum over $x = k, \ k+1, \ \ldots, \ \infty,$ we derive the following

recurrence relation for the moments.

$$(7.2.3) \quad \alpha_{r+1} = k^{r+1}p_k(1+p) + \left[m \sum_{j=0}^{r} \binom{r}{j}\alpha_j + \sum_{j=0}^{r-1} \binom{r}{j}\alpha_{j+1} \right]p,$$

$$(r = 0, 1, 2, \ldots).$$

Putting $r = 0, 1, 2$ in (7.2.3), we get

$$(7.2.4) \qquad \alpha_1 = kp_k(1+p) + pm,$$

$$(7.2.5) \qquad \alpha_2 = k^2 p_k(1+p) + pm(1+\alpha_1) + p\alpha_1,$$

$$(7.2.6) \qquad \alpha_3 = k^3 p_k(1+p) + pm(1+2\alpha_1+\alpha_2) + p(\alpha_1+2\alpha_2).$$

Multiplying (7.2.4) by k and k^2 and subtracting

from (7.2.5) and (7.2.6) respectively, we get

$$(7.2.7) \qquad G = p(mH + \alpha_1),$$

$$(7.2.8) \qquad L = p(mM + T),$$

where $G = \alpha_2 - k\alpha_1$, $H = \alpha_1 - k + 1$, $L = \alpha_3 - k^2\alpha_1$, $M = \alpha_2 + 2\alpha_1 - k^2 + 1$ and $T = \alpha_1 + 2\alpha_2$. From (7.2.7) and (7.2.8), we obtain

$$(7.2.9) \qquad p = (HL - GM)/(HT - \alpha_1 M),$$

$$(7.2.10) \qquad m = (GT - \alpha_1 L)/(HL - GM).$$

7.3 The three-moments estimators of the parameters of the singly truncated negative binomial distribution

Consider a random sample of size N from the truncated negative binomial distribution (7.2.1). Let $a_r = \sum_{x=k}^{\infty} x^r n_x / N$ denote the sample rth order raw moment $(r = 1, 2, \ldots)$. Let $\hat{G}, \hat{H}, \hat{L}, \hat{M}$ and $\hat{T}$ denote the values of G, H, L, M and T when the population moments α_r are replaced by a_r. The equations (7.2.9) and (7.2.10) suggest that we can estimate p and m by

$$(7.3.1) \qquad \hat{p} = (\hat{H}\hat{L} - \hat{G}\hat{M})/(\hat{H}\hat{T} - a_1\hat{M}),$$

$$(7.3.2) \qquad \hat{m} = (\hat{G}\hat{T} - a_1\hat{L})/(\hat{H}\hat{L} - \hat{G}\hat{M}).$$

The estimators $\hat{p}$ and $\hat{m}$ are rational functions of the sample moments and hence are asymptotically normally distributed. The asymptotic variances of $\hat{p}$ and $\hat{m}$ and covariance between them can be worked out by the δ-method (Kendall and Stuart $\underline{/}$23$\underline{/}$, §10.6) and are found

to be as

(7.3.3) $$V(\hat{p}) = C'AC/NQ^2,$$

(7.3.4) $$V(\hat{m}) = D'AD/Np^2Q^2,$$

(7.3.5) $$\text{Cov}(\hat{p}, \hat{m}) = C'AD/NpQ^2,$$

where

$$C = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \qquad D = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \quad \text{and}$$

$$P_1 = g\alpha_1 + (k-1)(2b\alpha_1 - mp - k),$$

$$P_2 = -h\alpha_1 + (k-1)(2b + k - \alpha_1 - 1),$$

$$P_3 = \alpha_1 - k + 1,$$

$$R_1 = -\alpha_1(m+1)g' - \alpha_1(k-1)\left[2(m+1)(b+p) + k\right]$$
$$+ m(k-1)\left[g' + (k-1)(2b+p-1) - b\right],$$

$$R_2 = \alpha_1\left[(m+1)h' + (k-1)(m+2)\right]$$
$$- m(k-1)(2mp + 4p + k + 1),$$

$$R_3 = -\alpha_1(m+1) + m(k-1),$$

$$Q = \alpha_1^2\left[p(m+1) + (k-1)\right] - \alpha_1(k-1)(k+3mp+2p)$$
$$+ (k-1)^2 2mp,$$

$$b = mp+p+1, \qquad g = b^2+p+1, \qquad g' = g+bp,$$

$$h = 2b+p+1, \qquad h' = h+p, \qquad \text{and}$$

$$A = [\, \alpha_{ij} \,], \quad i,j = 1,2,3; \qquad \alpha_{ij} = \alpha_{i+j} - \alpha_i \alpha_j .$$

7.4 Comparison with the maximum likelihood estimators

For sake of illustration, we consider the case when one class from the lower end of the negative binomial distribution is truncated. The maximum likelihood estimation of the parameters of the truncated negative binomial distribution, in this case, was discussed by David and Johnson $[15]$ and Sampford $[39]$. The maximum likelihood estimators (denoted by putting asterisks over the parameters) of $\omega = 1/(1+p)$ and m are given by (Sampford $[39]$)

$$(7.4.1) \qquad \frac{Nm^*}{\omega^*(1 - \omega^{*m^*})} - \frac{Na_1}{(1 - \omega^*)} = 0,$$

$$(7.4.2) \qquad \frac{N \log \omega^*}{1 - (\omega^*)^{m^*}} + \sum_{j=1}^{R} (m^* + j - 1) \sum_{x=j}^{R} n_x = 0,$$

where R is the highest observed value of x in the sample. The determinant of the variance-covariance matrix of the estimators is given by (Sampford $[39]$ and Brass $[9]$)

$$(7.4.3) \qquad I_0 = m^2 \eta^3 \omega^2 / N^2 (\eta m + P)^2 F,$$

where $\eta = (1 - \omega)$, $P = m\eta\omega^m/(1 - \omega^m)$,

$F = (1 - P)W - \eta P(1 + \log\omega/\eta)^2$ and

$W = \sum\limits_{r=2}^{\infty} \eta^r / r \binom{m+r-1}{m}$.

The three-moments estimators of p and m are obtained from (7.3.1) and (7.3.2) by putting $k = 1$ and are found to be

$$(7.4.4) \qquad \hat{p} = (a_3 a_1 - a_1 a_2 + a_1^2 - a_2^2)/a_1(a_2 - a_1),$$

$$(7.4.5) \qquad \hat{m} = (2a_2^2 - a_2 a_1 - a_3 a_1)/(a_3 a_1 - a_1 a_2 + a_1^2 - a_2^2),$$

which are same as obtained by Rider $\underline{/}36\underline{}7$. The three-moments estimator of $\omega = 1/(1+p)$ is given by $\hat{\omega} = 1/(1+\hat{p})$. The asymptotic variances and covariance of the three-moments estimators are obtained from (7.3.3), (7.3.4) and (7.3.5) by taking $k = 1$ and are found to be as

$$(7.4.6) \qquad V(\hat{p}) = C_1' A C_1 / \alpha_1^2 p^2 (m+1)^2,$$

$$(7.4.7) \qquad V(\hat{m}) = D_1' A D_1 / \alpha_1^2 p^4,$$

$$(7.4.8) \qquad \text{Cov}(\hat{p},\ \hat{m}) = C_1' A D_1 / \alpha_1^2 p^3 (m+1),$$

where

$$C_1 = \begin{bmatrix} g \\ -h \\ 1 \end{bmatrix}, \qquad\qquad D_1 = \begin{bmatrix} -g' \\ h' \\ -1 \end{bmatrix},$$

and A, g, g', h, h' have the same meanings as in (7.3.3), (7.3.4) and (7.3.5). The asymptotic variance of $\hat{\omega}$ and covariance between $\hat{\omega}$ and $\hat{m}$ are given by

$$(7.4.9) \qquad V(\hat{\omega}) = \omega^4 V(\hat{p}),$$

$$(7.4.10) \qquad \text{Cov}(\hat{\omega},\ \hat{m}) = -\omega^2 \text{Cov}(\hat{p},\ \hat{m}).$$

Hence, the determinant of the variance-covariance matrix of the three-moments estimators is given by

$$(7.4.11) \quad I = \begin{bmatrix} \omega^4 V(\hat{p}) & , & -\omega^2 \text{Cov}(\hat{p},\ \hat{m}) \\ -\omega^2 \text{Cov}(\hat{p},\ \hat{m}), & & V(\hat{m}) \end{bmatrix},$$

where $V(\hat{p})$, $V(\hat{m})$ and $\text{Cov}(\hat{p},\ \hat{m})$ are as given by (7.4.6), (7.4.7) and (7.4.8). The joint asymptotic efficiency E, of the three-moments estimators is then given by

(7.4.12)
$$E = I_0/I.$$

In Table 7.4.1, the percentage asymptotic efficiency of the three-moments estimators is tabulated for selected values of m and U = mp.

TABLE 7.4.1

PERCENTAGE ASYMPTOTIC EFFICIENCY OF THE

THREE-MOMENTS ESTIMATORS   (k = 1)

U \ m	0.5	1	2	5
0.5	24.3	32.9	65.5	84.2
1	15.6	26.1	37.3	65.2
2	7.4	15.2	25.6	36.9
5	3.6	7.1	14.1	29.2

From Table 7.4.1 it is observed that for large· values of m and small values of U the three-moments estimators are reasonably efficient.

7.5 An illustrative example

The data of Table 7.5.1 are taken from the paper of Brass $\underline{/}9\underline{/}$ and were collected by the East African Medical Survey in the Kwimba district of Tanganyika territory. The observations are of the number of children

ever born to a sample of mothers over 40 years of age.

TABLE 7.5.1

No. of children per mother	No. of mothers
1	49
2	56
3	73
4	41
5	43
6	23
7	18
8	18
9	7
10	7
11	3
12	2
Total	340

(a)   The three-moments estimators:

We find that $N = 340$, $a_1 = 3.9912$, $a_2 = 21.8853$, $a_3 = 148.1676$. Using (7.4.4) and (7.4.5), we find that $\hat{p} = 0.5738$ and $\hat{m} = 6.8133$. Hence $\hat{\omega} = 0.6354$ and $\hat{\eta} = 0.3646$.

Using (7.2.3) for $r = 3$, 4, 5 and substituting the values of a_1, a_2, a_3 for α_1, α_2, α_3 and the values of $\hat{p}$ and $\hat{m}$ for p and m, we find that the estimates of α_4, α_5 and α_6 are given by $\alpha_4 = 1181.7578$,

$\alpha_5 = 10791.8446,$ $\alpha_6 = 110675.6518.$ Using the estimates for the population quantities, we find that

$$A = \begin{bmatrix} 5.9556 & , & 60.8190 & , & 590.3913 \\ 60.8190 & , & 702.7914 & , & 7549.1522 \\ 590.3913, & & 7549.1522, & & 88718.339 \end{bmatrix} .$$

Further we find that $b = 5.4833,$ $g = 31.6404,$ $g' = 34.7867,$ $h = 12.5404,$ $h' = 13.1142.$ Using (7.4.6), (7.4.7) and (7.4.8), we find that

$$V(\hat{p}) = 15.4925/N,$$

$$V(\hat{m}) = 2533.5507/N,$$

$$Cov(\hat{p}, \hat{m}) = 197.0041/N.$$

Applying (7.4.11), we find that $I = 71.7837/N^2.$

(b) Maximum likelihood estimators:

The maximum likelihood estimators are given by (Brass $\underline{/9\underline{7}}$), $\omega^* = 0.565$ and $m^* = 4.86.$ Applying (7.4.3), we find that $P = 0.140605,$ $W = 0.017713$ and $F = 0.0092489.$ Hence $I_0 = 13.1995/N^2.$ Thus the estimate of the asymptotic efficiency of the three-moments estimators is $I_0/I = 0.184,$ i.e. 18.4 % .

Remark. As a general case, the three-moments

estimators of the parameters of the singly truncated
negative binomial distribution are inefficient. However,
in the case of zero-truncated negative binomial distri-
bution, when m is large and $U = mp$ is very small, the
three-moments estimators are reasonably efficient. They
are easy to compute and hence are useful (i) for
exploratory work when it is not clear which type of the
distribution should be fitted, and (ii) to provide
first-stage values in the iterative solution of the
maximum likelihood equations.