PART I

CHAPTER 1

SOME PROPERTIES OF BALANCED INCOMPLETE BLOCK DESIGNS

1.1 Introduction

Fisher's inequality [19], namely, $v \leq b$ gives the lower bound for the number of blocks in BIB design. In Section 1.2, we obtain an upper bound for the number of blocks in a BIB design.

Majumdar [27] obtained the upper bound for the number of disjoint blocks in BIB design. In Section 1.3, we generalize this result and obtain an upper bound for the number of blocks in a BIB design having a given number of treatments in common with a given block and deduce some known and some new results.

1.2 An upper bound for the number of blocks in a BIB design

The parameters v, b, r, k, and λ of a BIB design satisfy the following relations.

$$(1.2.1) \qquad bk = vr,$$

$$(1.2.2) \qquad r(k - 1) = \lambda(v - 1),$$

$$(1.2.3) \qquad b \geq v, \quad r \geq k.$$

We prove the following lemma.

Lemma 1.2.1. If a BIB design with parameters v, b, r, k and λ exists, then

$$b \leq (r^2 - 1)/\lambda .$$

Proof. From (1.2.2), we get

$$(1.2.4) \qquad rk - \lambda v = r - \lambda .$$

But $r - \lambda \geq 1$, hence we get from (1.2.4),

$$(1.2.5) \qquad rk - \lambda v \geq 1.$$

Multiplying (1.2.5) by v and putting $b = vr/k$, we get

$$(1.2.6) \qquad b \geq \frac{\lambda v^2}{k^2} + \frac{v}{k^2}.$$

Put $v/k = b/r$ in (1.2.6). Then we get

$$(1.2.7) \qquad b \leq \frac{r^2}{\lambda} - \frac{r}{k\lambda} .$$

Now $r \geq k$, hence from (1.2.7), we get

$$b \leq (r^2 - 1)/\lambda .$$

Remark 1. When $r = k$ and $r - \lambda = 1$, i.e. when the design is symmetrical BIB design with $r - \lambda = 1$, then $b = (r^2 - 1)/\lambda = r + 1$.

Remark 2. The upper bound for b can be improved in the case when $v = nk$. This is done in the following lemma.

Lemma 1.2.2. If a BIB design with parameters $v = nk$, b, r, k and λ exists, then

$$b \leq r(r - 1)/\lambda .$$

Proof. From (1.2.4), we get

$$k(r - n\lambda) = r - \lambda . \qquad (1.2.8)$$

Now Roy [38] has proved that when $v = nk$, $r - \lambda \geq k$. Hence from (1.2.8), we get

$$r - n\lambda \geq 1,$$

$$\text{i.e.}\quad r - 1 \geq n\lambda ,$$

$$\text{i.e.}\quad nr(r - 1) \geq bn\lambda ,$$

$$\text{i.e.}\quad b \leq r(r - 1)/\lambda .$$

For resolvable BIB design, $v = nk$ and so for such designs $b \leq r(r - 1)/\lambda$. For affine resolvable BIB design, $v = nk$, and $r - \lambda = k$, and hence for such designs $b = r(r - 1)/\lambda$.

1.3 Upper bound for the number of blocks in a BIB design having a given number of treatments common with a given block

Majumdar [27] gave an upper bound for the number of disjoint blocks in a BIB design. Here, we give a more general result from which the result of Majumdar [27] follows as a particular case. We prove the following theorem.

Theorem 1.3.1. If a given block of a non-symmetrical BIB design with parameters v, b, r, k and λ has d blocks having a given number t $(t \leq v)$ of treatments common with it, then

$$d \leq b - 1 - D^{-1} \cdot [k(r - 1) - t(b - 1)]^2,$$

where $D = k(r - \lambda) + k^2(\lambda - 1) + t^2(b - 1) - 2tk(r - 1)$.

If $d = b - 1 - D^{-1} \cdot [k(r - 1) - t(b - 1)]^2$, then $c = [k(r - \lambda) + (\lambda - 1)k^2 - tk(r - 1)] / [k(r - 1) - t(b - 1)]$ is a positive integer and the given block has c treatments common with each of the remaining $(b - d - 1)$ blocks.

Proof. We denote the blocks as $B_1, B_2, \ldots, B_b$. Let l_i denote the number of common treatments between B_1 and B_i, $(i = 2, 3, \ldots, b)$. Let $l_i = t$, for

$i = 2, 3, \ldots, (d + 1)$. Then, we have

(1.3.1) $$dt + l_{d+2} + \ldots + l_b = k(r - 1),$$

(1.3.2) $$dt^2 + l_{d+2}^2 + \ldots + l_b^2 = k(r - \lambda) + k^2(\lambda - 1).$$

Let $\bar{l} = (l_{d+2} + \ldots + l_b) / (b - d - 1)$. As

$\sum_{i=d+2}^{b} (l_i - \bar{l})^2 \geq 0$, we have

(1.3.3) $$k(r - \lambda) + (\lambda - 1)k^2 - dt^2 \geq \frac{[k(r - 1) - dt]^2}{(b - d - 1)}.$$

From (1.3.3), we get

(1.3.4) $$dD \leq (b-1)D - [k(r-1) - t(b-1)]^2,$$

where $D = k(r-\lambda) + k^2(\lambda-1) + t^2(b-1) - 2tk(r-1)$.

Since, D can be written as

$$D = \frac{k(b-r)(v-k)(r-k)}{(v-1)(b-1)} + \frac{[k(r-1) - t(b-1)]^2}{(b-1)},$$

$D > 0$, if $r \neq k$ (i.e. for non-symmetrical BIB design).
Hence, dividing by D in (1.3.4), we get

(1.3.5) $$d \leq (b-1) - D^{-1} [k(r-1) - t(b-1)]^2.$$

If the equality sign holds in (1.3.5), then all l_i' s (i = d+2, ..., b) are equal to c, where

$$c = \frac{[\, k(r-\lambda) + k^2(\lambda-1) - tk(r-1) \,]}{[\, k(r-1) - t(b-1) \,]} .$$

This proves the theorem.

Taking t=0, we get the following result due to Majumdar [27].

Corollary 1.3.1. A given block of a BIB design with parameters v, b, r, k and λ can never have more than

$$b - 1 - (r-1)^2 k\, (r - \lambda - k + k\lambda)^{-1}$$

blocks disjoint with it. If some block has that many disjoint blocks, then $c = (r - \lambda - k + k\lambda)/(r-1)$ is a positive integer and each of the non-disjoint blocks has c treatments common with it.

We now derive from Theorem 1.3.1, the results of Chakrabarti [10], Seiden [40] and Parker [31].

Theorem 1.3.2. If in a BIB design with parameters v, b, r, k and λ, b = v+r-1 and v = 2k, where k is a positive odd integer, then no two blocks of this design are disjoint.

Proof. Let a block of the given BIB design have d disjoint blocks. Since $b = v + r - 1$, it follows from the work of Khatri and Shah [25] that $r = k + \lambda$. Hence, using Theorem 1.3.1, for $t = 0$, $b = v + r - 1$, $r = k + \lambda$ and $v = 2k$, we obtain

$$d \leq 1. \tag{1.3.6}$$

If $d = 1$, then Theorem 1.3.1 shows that $k/2$ is an integer which is a contradiction as k is an odd integer. Hence $d < 1$, i.e. $d = 0$. Hence the result.

This result is proved by Chakrabarti [10].

Corollary 1.3.2. If D is a BIB design with parameters $(v, b, r, k, \lambda) = (2x + 2, 4x + 2, 2x + 1, x + 1, x)$, where x is a positive even integer, then no two blocks of D are disjoint.

This follows from Theorem 1.3.2 by taking $\lambda = x$ and using the relations $b = v + r - 1$, $v = 2k$, k being a positive odd integer.

Theorem 1.3.3. If in a BIB design with parameters v, b, r, k, λ; $b = v + r - 1$ and $v = 2k$, then no two blocks of this design are the same set.

Proof. Let a block of the given design have d blocks each having all the k treatments common with it.

Then, using Theorem 1.3.1 for $t = k$, $b = v + r - 1$, $r = k + \lambda$, $v = 2k$, we get

(1.3.7) $$d \leq (r-1)/(r+1) < 1.$$

Hence, $d = 0$ and so the result follows.

Corollary 1.3.3. If D is a BIB design with parameters $(v, b, r, k, \lambda) = (2x + 2, 4x + 2, 2x + 1, x + 1, x)$, where x is any positive integer, then no two blocks of D are the same set.

This follows from Theorem 1.3.3 by taking $\lambda = x$ and using the relations $b = v + r - 1$, $r = k + \lambda$ and $v = 2k$.

This result is also proved by Seiden [40].

Combining the results of the corollaries 1.3.2 and 1.3.3, we get the following theorem due to Parker [31].

Theorem 1.3.4. If D is a BIB design with parameters $(v, b, r, k, \lambda) = (2x + 2, 4x + 2, 2x + 1, x + 1, x)$, where x is a positive even integer, then no two blocks of D are either (i) disjoint or (ii) the same set.

We now deduce the following theorem proved partly in Theorem 1 of Chakrabarti [10], from Theorem 1.3.1.

Theorem 1.3.5. If in a BIB design with parameters v, b, r, k and λ ; $r = 2k$ and λ is a positive odd integer, then no two blocks of this design are the same set.

Proof. Let a block of the given design have d blocks, each having all the k treatments common with it. Then, using Theorem 1.3.1 for $t = k$, $r = 2k$, we obtain

$$d \leq 1. \tag{1.3.8}$$

If $d = 1$, Theorem 1.3.1 shows that $\lambda/2$ is an integer which contradicts that λ is an odd integer. Hence, $d < 1$, i.e. $d = 0$. Hence the result.

Corollary 1.3.4. If D is a BIB design with parameters $[v, b, r, k, \lambda] = [2x(xy+1)+1, 4x(xy+1)+2, 2(xy+1), (xy+1), x]$, where x is any positive odd integer and y is any positive integer, then no two blocks of D are the same set.

This follows from Theorem 1.3.5 by taking $\lambda = x$, $k = xy+1$ and $r = 2k$.