CHAPTER 3

BOUNDS FOR THE NUMBER OF COMMON TREATMENTS BETWEEN ANY TWO BLOCKS OF CERTAIN PBIB DESIGNS

3.1 Introduction

In this chapter, we derive bounds for the number of common treatments between any two blocks of (i) SRGD designs, (ii) certain triangular designs, (iii) certain L_2 designs and (iv) certain rectangular designs. The results are established by using the theorems proved by (i) Bose and Connor [6], (ii) Raghavarao [34] and (iii) Vartak [54].

3.2 Bounds for the number of common treatments between any two blocks of SRGD designs

For the description of SRGD design, we refer to Section 2.2 of Chapter 2. We use Theorem 2.2.1 due to Bose and Connor [6] to derive the bounds for the number of treatments common between any two blocks of SRGD design. The result is given in Theorem 3.2.1.

Theorem 3.2.1. If x be the number of treatments common between any two blocks of SRGD design, then

$$\max(0, T_1) \leq x \leq \min(k, T_2),$$

where

$$T_1 = k(r-1)(b-1)^{-1} - A(b-2)^{1/2},$$

$$T_2 = k(r-1)(b-1)^{-1} + A(b-2)^{1/2},$$

$$A^2 = k^2 \left[\frac{Q_1}{v(v-m)} - \frac{(r-1)^2}{(b-1)} \right] \Big/ (b-1)$$

$$= k^2 (v-k)(b-r)(b-v+m-1)/v(v-m)(b-1)^2$$

and

$$Q_1 = (v-k)(b-r) - (v-rk)(v-m).$$

Proof. Let the blocks of the given design be denoted by $B_1, B_2, \ldots, B_b$. Denote the number of treatments common between B_1 and B_i by x_i, $i = 2, 3, \ldots, k$. Let $x_2 = x$. Considering the treatments of the block B_1 singly, we get

$$(3.2.1) \qquad \sum_{i=3}^{b} x_i = k(r-1) - x.$$

Using Theorem 2.2.1 of Chapter 2, due to Bose and

Connor [6], and considering the treatments of the block B_1 pairwise, we get

$$\sum_{i=3}^{b} x_i(x_i-1)$$

(3.2.2)

$$= (k/m)\left[(k-m)\lambda_1+k(m-1)\lambda_2-m(k-1)\right]-x(x-1).$$

In Section 2.2 of Chapter 2, it has been shown that for a SRGD design, $\lambda_1 = r(k-m)/(v-m)$ and $\lambda_2 = rk/v$. Substituting these values of λ_1 and λ_2 in (3.2.2), we get

$$\sum_{i=3}^{b} x_i(x_i-1)$$

(3.2.3)

$$= \left[k^2 Q_1/v(v-m)\right] - k(r-1) - x^2 + x,$$

where $Q_1 = (v-k)(b-r)-(v-rk)(v-m)$.

From (3.2.1) and (3.2.3), we get

$$\sum_{i=3}^{b}(x_i - \bar{x})^2 = \frac{k^2 Q_1}{v(v-m)} - \frac{[k(r-1)-x]^2}{(b-2)} - x^2, \tag{3.2.4}$$

where $\bar{x} = [k(r-1)-x] / (b-2)$. As $\sum_{i=3}^{b}(x_i - \bar{x})^2 \geq 0$, we get the required result.

When $b = v - m + 1$, $A = 0$ and hence

$x = k(r-1)/(b-1) = k(r-1)/(v-m)$. Thus, we get the following corollary.

Corollary 3.2.1. If in a SRGD design $b = v-m+1$, then there are $k(r-1)/(v-m)$ treatments common between any two blocks of this design.

This result is also proved in Theorem 2.2.3 in Chapter 2.

3.3 Bounds for the number of common treatments between any two blocks in certain triangular designs

For the description of triangular design, we refer to Section 2.3 of Chapter 2. We consider here triangular designs in which $rk - v\lambda_1 = n(r - \lambda_1)/2$. We use Theorem 2.3.1 of Chapter 2, due to Raghavarao [34] to derive the bounds for the number of common treatments between any two blocks of this design. The result is given in Theorem 3.3.1.

Theorem 3.3.1. If x be the number of treatments common between any two blocks of a triangular design in which $rk - v\lambda_1 = n(r - \lambda_1)/2$, then

$$\max (0, T_1) \leq x \leq \min (k, T_2),$$

where

$$T_1 = k(r-1)(b-1)^{-1} - A(b-2)^{1/2},$$

$$T_2 = k(r-1)(b-1)^{-1} + A(b-2)^{1/2},$$

$$A^2 = k^2\left[\frac{Q_2}{v(v-n)} - \frac{(r-1)^2}{(b-1)}\right] \Big/ (b-1)$$

$$= k^2(v-k)(b-r)(b-v+n-1)/v(v-n)(b-1)^2$$

and

$$Q_2 = (v-k)(b-r)-(v-rk)(v-n).$$

Proof. Using the same notation and argument as in Theorem 3.2.1, we again get

$$\sum_{i=3}^{b} x_i = k(r-1) - x. \tag{3.3.1}$$

Using Theorem 2.3.1 of Chapter 2 due to Raghavarao [34] and the same argument as in Theorem 3.2.1, we get

$$\begin{aligned}\sum_{i=3}^{b} x_i(x_i-1) &= n(2k/n)(2k/n - 1)(\lambda_1 - 1) \\ &+ [k(k-1)-n(2k/n)(2k/n - 1)](\lambda_2 - 1) \\ &- x(x-1).\end{aligned} \tag{3.3.2}$$

Using the method of proving (2.3.3) of Theorem 2.3.1, it can be shown that

$$\sum_{i=3}^{b} x_i(x_i-1)$$

(3.3.3)

$$= \left[k^2 Q_2 \,/\, v(v-n)\right] - k(r-1) - x^2 + x,$$

where $Q_2 = (v-k)(b-r)-(v-rk)(v-n)$.

From (3.3.1) and (3.3.3), we get

$$(3.3.4) \qquad \sum_{i=3}^{b} (x_i - \bar{x})^2 = \frac{k^2 Q_2}{v(v-n)} - \frac{[k(r-1)-x]^2}{(b-2)} - x^2,$$

where $\bar{x} = [k(r-1)-x] \,/\, (b-2)$. As $\sum_{i=3}^{b} (x_i - \bar{x})^2 \geq 0$,

we get from (3.3.4) the required result.

When $b = v - n + 1$, $A = 0$ and hence $x = k(r-1)/(b-1) = k(r-1)/(v-n)$. Thus, we get the following corollary.

Corollary 3.3.1. If in a triangular design in which $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b = v - n + 1$, then there are $k(r-1)/(v-n)$ treatments common between any two blocks of this design.

This result is also proved in Theorem 2.3.3 of Chapter 2.

3.4 Bounds for the number of common treatments between any two blocks in certain L_2 designs

For the description of a L_2 design, we refer to Section 2.4 of Chapter 2. We consider here L_2 designs in which $rk - v\lambda_1 = s(r - \lambda_1)$. We use Theorem 2.4.1 of Chapter 2, due to Raghavarao [34] to derive the bounds for the number of treatments common between any two blocks of a L_2 design in which $rk - v\lambda_1 = s(r - \lambda_1)$. The result is given in Theorem 3.4.1.

Theorem 3.4.1. If x be the number of treatments common between any two blocks of a L_2 design in which $rk - v\lambda_1 = s(r - \lambda_1)$, then

$$\max(0, T_1) \le x \le \min(k, T_2),$$

where

$$T_1 = k(r-1)(b-1)^{-1} - A(b-2)^{1/2},$$

$$T_2 = k(r-1)(b-1)^{-1} + A(b-2)^{1/2},$$

$$A^2 = k^2\left[\frac{Q_3}{v(s-1)^2} - \frac{(r-1)^2}{(b-1)}\right] \Big/ (b-1)$$

$$= k^2(v-k)(b-r)(b-v+2s-2)/v(s-1)^2(b-1)^2$$

and

$$Q_3 = (v-k)(b-r)-(v-rk)(s-1)^2.$$

Proof. Using the same notation and argument as in Theorem 3.2.1, we get

(3.4.1) $$\sum_{i=3}^{b} x_i = k(r-1) - x.$$

Using Theorem 2.4.1 of Chapter 2, due to Raghavarao [34] and the same argument as in Theorem 3.2.1, we get

(3.4.2) $$\sum_{i=3}^{b} x_i(x_i-1) = (k/s)\left[2(k-s)\lambda_1 + (sk+s-2k)\lambda_2 - s(k-1)\right] - x(x-1).$$

We have shown in Theorem 2.4.2 that $\lambda_1 = r(k-s)/s(s-1)$ and $\lambda_2 = r(sk+s-2k)/s(s-1)^2$. Substituting these values of λ_1 and λ_2 in (3.4.2) and after some simplification, we get

(3.4.3) $$\sum_{i=3}^{b} x_i(x_i-1) = \left[k^2 Q_3 \,/\, v(s-1)^2\right] - k(r-1) - x^2 + x,$$

where $Q_3 = (v-k)(b-r)-(v-rk)(s-1)^2$.

From (3.4.1) and (3.4.3), we get

(3.4.4) $$\sum_{i=3}^{b} (x_i - \bar{x})^2 = \left[k^2 Q_3/v(s-1)^2\right] - k(r-1) - x^2,$$

where $\bar{x} = [k(r-1)-x]/(b-2)$. As $\sum_{i=3}^{b} (x_i - \bar{x})^2 \geq 0$, the required result follows from (3.4.4).

When $b = v - 2s + 2$, $A = 0$, and hence $x = k(r-1)/(b-1) = k(r-1)/(s-1)^2$. Thus, we obtain the following corollary.

Corollary 3.4.1. If in a L_2 design in which $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + 2$, then there are $k(r-1)/(s-1)^2$ treatments common between any two blocks of this design.

This result is also proved in Theorem 2.4.3 in Chapter 2.

3.5 Bounds for the number of common treatments between any two blocks in certain rectangular designs

For the description of a rectangular design, we refer to Section 2.5 of Chapter 2. We consider here rectangular designs in which $\theta_1 = 0 = \theta_2$. We use Theorems 2.5.1 and 2.5.2 of Chapter 2, due to Vartak [54], to derive the bounds for the number of common treatments between any two blocks of the above design. The result is given in Theorem 3.5.1.

Theorem 3.5.1. If x be the number of treatments common between any two blocks of a rectangular design in which $\theta_1 = 0 = \theta_2$, then

$$\max(0, T_1) \leq x \leq \min(k, T_2),$$

where

$$T_1 = k(r-1)(b-1)^{-1} - A(b-2)^{1/2},$$

$$T_2 = k(r-1)(b-1)^{-1} + A(b-2)^{1/2},$$

$$A^2 = k^2\left[\frac{Q_4}{vp} - \frac{(r-1)^2}{(b-1)}\right] / (b-1)$$

$$= k^2(v-k)(b-r)(b-p-1)/vp(b-1)^2,$$

$$Q_4 = (v-k)(b-r)-p(v-rk),$$

and

$$p = (v_1-1)(v_2-1).$$

Proof. Using the same notation and argument as in Theorem 3.2.1, we get

$$\sum_{i=3}^{b} x_i = k(r-1) - x. \tag{3.5.1}$$

Using Theorems 2.5.1 and 2.5.2 of Chapter 2, due to Vartak [54], and the same argument as in Theorem 2.4.1, we get

$$\sum_{i=3}^{b} x_i(x_i-1)$$

$$(3.5.2) \quad = (k/v)\Big[v_2(k-v_1)(\lambda_1 - \lambda_3) + v_1(k-v_2)(\lambda_2 - \lambda_3) + v(k-1)(\lambda_3 - 1)\Big] - x(x-1).$$

In Theorem 2.5.3, we have shown that

$$\lambda_1 = rv_2(k - v_1)(v_1 - 1)/vp,$$

$$\lambda_2 = rv_1(k - v_2)(v_2 - 1)/vp,$$

$$\lambda_3 = r(v + kv - kv_1 - kv_2)/vp,$$

where $p = (v_1 - 1)(v_2 - 1)$. Substituting these values of λ_1, λ_2, and λ_3 in (3.5.2) and after some simplification, we get

$$\sum_{i=3}^{b} x_i(x_i-1)$$

$$(3.5.3) \quad = \Big[k^2 Q_4/vp\Big] - k(r-1) - x^2 + x,$$

where $Q_4 = (v-k)(b-r)-p(v-rk)$.

From (3.5.1) and (3.5.3), we get

$$(3.5.4) \quad \sum_{i=3}^{b} (x_i - \bar{x})^2 = \frac{k^2 Q_4}{vp} - \frac{[k(r-1)-x]^2}{(b-2)} - x^2,$$

where $\bar{x} = [k(r-1)-x] / (b-2)$. As $\sum_{i=3}^{b} (x_i - \bar{x})^2 \geq 0$, the required result follows from (3.5.4).

When $b = p+1$, $A = 0$, and hence $x = [k(r-1)/(b-1)] = k(r-1)/p$. Thus, we get the following corollary.

Corollary 3.5.1. If in a rectangular design in which $\theta_1 = 0 = \theta_2$, $b = p+1$, then there are $k(r-1)/p$ treatments common between any two blocks of this design.

This result is also proved in Theorem 2.5.4 of Chapter 2.

Remark. After having published the results of this chapter the author came to know about the bounds for the number of common treatments between blocks of incomplete block design, obtained by Agrawal [1], who [2] also showed that the bounds obtained by him are superior to those derived here.