CHAPTER 5

ON THE BLOCK STRUCTURE OF EQUI-REPLICATE INCOMPLETE BLOCK DESIGNS

5.1 Introduction

In this chapter, we consider an equi-replicate incomplete block design with parameters (v, b, r, k). An equi-replicate incomplete block design is an arrangement of v treatments in b blocks each of k plots $(k < v)$ such that each treatment occurs atmost once in any block and altogether in r blocks. Such a design is completely characterised by its incidence matrix $N = [n_{ij}]$, where n_{ij} is equal to the number of times the ith treatment occurs in the jth block and

$$
\begin{aligned}
n_{ij} &= 1, \quad \text{if the ith treatment occurs in the jth block,} \\
&= 0, \quad \text{if the ith treatment does not occur in the jth block,}
\end{aligned}
$$

$(i = 1, 2, \ldots, v; \quad j = 1, 2, \ldots, b)$.

For the equi-replicate incomplete block design, we derive two results: (i) the necessary and sufficient condition in order that any two blocks will have the same number of treatments in common and (ii) bounds for the number of disjoint blocks. The main result used to establish these results is due to Agrawal [1] which is as follows.

Theorem 5.1.1. If N is the incidence matrix of an equi-replicate incomplete block design with parameters (v, b, r, k) and $rk > \mu_0 > \mu_1 > \ldots > \mu_s$ are the distinct characteristic roots of NN', then the number of common treatments l_{ij} between the blocks i and j $(i \neq j = 1, 2, \ldots, b)$ satisfies the equivalent

$$\max [0, 2k - v, k - \mu_0]$$

$$\leq l_{ij} \leq \min [k, \mu_0 - k + 2b^{-1}(rk - \mu_0)].$$

5.2 Necessary and sufficient condition for the blocks of an equi-replicate incomplete block design to have the same number of treatments in common

An equi-replicate incomplete block design is called linked block (LB) design, if any two blocks have the same number of treatments in common. The LB designs were introduced by Youden [56]. Roy and Laha [37] derived the necessary and sufficient conditions for two associates PBIB design to be of LB type. Here, we derive the necessary and sufficient conditions for an equi-replicate design to be of LB type.

Let N be the incidence matrix of an equi-replicate incomplete block design with parameters (v, b, r, k) and $rk > \mu_0 > \mu_1 > \ldots > \mu_s$ be the distinct characteristic roots of NN', where N' is the transpose of N. We now prove the following theorem.

Theorem 5.2.1. The necessary and sufficient condition that any two blocks of an equi-replicate incomplete block design with parameters (v, b, r, k) will have the same number of common treatments is that

$$\mu_0 = k(b-r)/(b-1), \text{ and } \quad \mu_1 = \mu_2 = \ldots = \mu_s = 0.$$

Proof. (i) To show that the condition is necessary.

Let $l_{ij} = l$, for $i \neq j = 1, 2, \ldots, b$. Then, we have

$$N'N = (k-l)I_b + lE_{bb}, \tag{5.2.1}$$

where I_b is the identity matrix of order $b \times b$ and E_{bb} is a $b \times b$ matrix with all elements unity. Clearly $l = k(r-1)/(b-1)$ and the characteristic roots of $N'N$

are rk and $k(b-r)/(b-1)$ with multiplicities 1 and $(b-1)$. Since the non-zero characteristic roots of $N'N$ and NN' are same except for multiplicities, it follows that

$$\mu_0 = k(b-r)/(b-1), \text{ and } \mu_1 = \mu_2 = \dots = \mu_s = 0.$$

(ii) To show that the condition is sufficient.

Let $\mu_0 = k(b-r)/(b-1)$ and $\mu_1 = \mu_2 = \dots = \mu_s = 0$. We have now

$$\mu_0 - k + 2b^{-1}(rk - \mu_0) = k - \mu_0 = k(r-1)/(b-1).$$

Hence, applying Theorem 5.1.1, it follows that

$$\max\left[0,\ 2k - v,\ k - \mu_0\right] \le l_{ij} \le \min\left[k,\ k - \mu_0\right]. \tag{5.2.2}$$

From (5.2.2), it follows that

$$l_{ij} = k - \mu_0 = k(r-1)/(b-1) = 1, \tag{5.2.3}$$

for all $i \neq j = 1, 2, \dots, b$. This proves the theorem.

We now apply Theorem 5.2.1 to PBIB designs with two associate classes. The characteristic roots θ_1 and θ_2 of NN' of a two associates PBIB design with parameters $v, b, r, k, \lambda_1, \lambda_2, n_1, n_2$ and (p^i_{jk}); $i, j, k = 1, 2$ are given by

$$\theta_1 = r - (1/2)\left[(\lambda_1-\lambda_2)(-\gamma-\sqrt{\Delta})+(\lambda_1+\lambda_2)\right], \tag{5.2.4}$$

$$\theta_2 = r - (1/2)\left[(\lambda_1-\lambda_2)(-\gamma+\sqrt{\Delta})+(\lambda_1+\lambda_2)\right], \tag{5.2.5}$$

where $\gamma = p^2_{12} - p^1_{12}$, $\beta = p^2_{12} + p^1_{12}$ and $\Delta^2 = \gamma^2+2\beta+1$.

Connor and Clatworthy [14] have shown that for an existent two associates PBIB design, $\theta_1 \geq 0$, $\theta_2 \geq 0$. From (5.2.4) and (5.2.5), we get

$$\theta_1 - \theta_2 = \sqrt{\Delta}(\lambda_1 - \lambda_2), \tag{5.2.6}$$

and we have also

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1). \tag{5.2.7}$$

Let $\theta_1 = 0$, then from (5.2.4), we get

$$\lambda_2(\gamma + \sqrt{\Delta} + 1) - \lambda_1(\gamma + \sqrt{\Delta} - 1) = 2r. \tag{5.2.8}$$

Solving (5.2.7) and (5.2.8) for λ_1 and λ_2, we get

$$\lambda_2 - \lambda_1 = 2k(b-r)/\left[(\gamma+\sqrt{\Delta}+1)(v-1) - 2n_2\right]. \tag{5.2.9}$$

Hence, using (5.2.6) and (5.2.9) and noting that $\theta_1 = 0$, we get

$$\begin{aligned} \theta_2 &= 2k(b-r)\sqrt{\Delta}/\left[(\gamma+\sqrt{\Delta}+1)(v-1) - 2n_2\right] \\ &= \mu_0 = k(b-r)/(b-1). \end{aligned} \tag{5.2.10}$$

Hence, the necessary and sufficient condition for a two associates PBIB design with $\theta_1 = 0$ to be of LB type is that $b = 1 + [(\gamma + \sqrt{\Delta} + 1)(v-1) - 2n_2](2\sqrt{\Delta})^{-1}$.

We can also consider the case for $\theta_2 = 0$ similarly and find the necessary and sufficient condition for a two associates PBIB design to be of LB type. Thus, we have the following corollary.

Corollary 5.2.1. The necessary and sufficient condition for a two associates PBIB design to be of LB type is that (i) $\theta_1 = 0$ and $b = 1 + [(\gamma + \sqrt{\Delta} + 1)(v-1) - 2n_2](2\sqrt{\Delta})^{-1}$ or (ii) $\theta_2 = 0$, and $b = 1 + [(-\gamma + \sqrt{\Delta} + 1)(v-1) - 2n_1](2\sqrt{\Delta})^{-1}$.

The results obtained by Roy and Laha [37] about the necessary and sufficient conditions for (i) singular GD design, (ii) SRGD design, (iii) triangular design and (iv) a Latin square type design with i restraints, to be of LB type follow from Corollary 5.2.1.

We now apply Theorem 5.2.1 to three associates PBIB designs with rectangular association scheme (Rectangular designs) defined by Vartak [53]. The primary parameters of a rectangular design are $v = v_1 v_2$, b, r, k, λ_1, λ_2, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$ and $n_3 = n_1 n_2$. The characteristic roots of NN', where N is the incidence matrix of this design, are

$$\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3),$$

$$\theta_2 = r - \lambda_2 + (v_2 - 1)(\lambda_1 - \lambda_3),$$

$$\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3.$$

Vartak [53] has shown that for an existent rectangular design, $\theta_1 \geq 0$, $\theta_2 \geq 0$, $\theta_3 \geq 0$. Applying Theorem 5.2.1, we see that in order that any two blocks of a rectangular design will have the same number of treatments in common, the necessary and sufficient condition is that

(i) $\mu_0 = \theta_1$ and $\theta_2 = 0 = \theta_3$; or

(ii) $\mu_0 = \theta_2$ and $\theta_1 = 0 = \theta_3$; or

(iii) $\mu_0 = \theta_3$ and $\theta_1 = 0 = \theta_2$.

Considering (i), (ii) and (iii) seperately, we obtain respectively the following results.

Corollary 5.2.2. If in a rectangular design, $r = \lambda_2$ and $\lambda_1 = \lambda_3$; then the necessary and sufficient condition that any two blocks will have the same number of treatments in common is that $b = v_2$.

Corollary 5.2.3. If in a rectangular design, $r = \lambda_1$ and $\lambda_2 = \lambda_3$; then the necessary and sufficient condition that any two blocks will have the same number of treatments in common is that $b = v_1$.

Corollary 5.2.4. If in a rectangular design, $\theta_1 = 0 = \theta_2$; then the necessary and sufficient condition that any two blocks will have the same number of treatments in common is that $b = (v_1-1)(v_2-1) + 1$.

Corollary 5.2.4. was also derived in Chapter 2 (Theorem 2.5.4).

5.3 Bounds for the number of disjoint blocks in equi-replicate incomplete block designs

In Chapter 2, upper bounds for the number of disjoint blocks in (i) SRGD designs, (ii) certain triangular designs, (iii) certain L_2 designs and (iv) certain rectangular designs were derived. Here we derive bounds for the number of disjoint blocks in an equi-replicate incomplete block design with parameters (v, b, r, k), using the result (Theorem 5.1.1) due to Agrawal [1] about the bounds of the number of common treatments between any two blocks of an equi-replicate incomplete block design.

Consider an equi-replicate incomplete block design with parameters (v, b, r, k). Let the blocks of this design be denoted by $B_1, B_2, \ldots, B_b$. Let l_j be the number of common treatments between the blocks B_1 and B_j $(j = 2, 3, \ldots, b)$. Let $rk, \mu_0, \mu_1, \ldots, \mu_s$ be the distinct characteristic roots of NN', where N is the incidence matrix of the given design and let $rk > \mu_0 > \mu_1 > \dots > \mu_s$.

Applying Theorem 5.1.1, we have

$$A \leq l_j < B, \qquad (j = 2, 3, \dots, b) \tag{5.3.1}$$

where $A = \max[0, 2k-v, k-\mu_0]$ and $B = \min[k, \mu_0 - k + 2b^{-1}(rk - \mu_0)]$. Assume that $l_j = 0$ for $j = 2, 3, \dots, (d+1)$. Adding the inequalities (5.3.1) over $j = d+2, d+3, \dots, b$, and noting that $\sum_{j=d+2}^{b} l_j = k(r-1)$, we get

$$(b - d - 1)A \leq k(r - 1) \leq (b - d - 1)B, \tag{5.3.2}$$

from which it follows that

$$b - 1 - k(r-1)A^{-1} \leq d \leq b - 1 - k(r-1)B^{-1}, \tag{5.3.3}$$

when $A > 0$ and

$$0 \leq d \leq b - 1 - k(r-1)B^{-1}, \tag{5.3.4}$$

when $A = 0$. Both (5.3.3) and (5.3.4) can be combined and rewritten as

$$\max[0, b - 1 - k(r-1)A^{-1}] \leq d \leq [b - 1 - k(r-1)B^{-1}]. \tag{5.3.5}$$

Thus, we have proved the following theorem.

Theorem 5.3.1. If rk, μ_0, μ_1, ..., μ_s are the distinct characteristic roots of NN', where N is the incidence matrix of an equi-replicate incomplete block design with parameters (v, b, r, k) and $rk > \mu_0 > \mu_1 > \ldots > \mu_s$ and if a given block has d disjoint blocks, then

$$\max \left[0, b - 1 - k(r-1)A^{-1} \right]$$
$$\leq d \leq \left[b - 1 - k(r-1)B^{-1} \right],$$

where $A = \max \left[0, 2k - v, k - \mu_0 \right]$ and $B = \min \left[k, \mu_0 - k + 2b^{-1}(rk - \mu_0) \right]$.

The bounds for the number of disjoint blocks in certain PBIB designs obtained in Chapter 2 can be put in the form

$$(5.3.6) \qquad 0 \leq d \leq b - 1 - v\alpha(r - 1)^2 P^{-1},$$

where $P = (v-k)(b-r) - \alpha(v-rk)$ and $\alpha+1$ is the number of non-zero characteristic roots of NN'. We shall now make comparison between the bounds (5.3.5) and (5.3.6) for the four classes of designs considered in Chapter 2. In the four classes of designs considered in Chapter 2, NN' has one characteristic root rk and the other only one characteristic root μ_0 with multiplicity α. Using the fact that the trace of a matrix is equal to the sum of its characteristic roots, we get $rk + \alpha\mu_0 = bk$, which gives

$$\mu_0 = k(b - r)/\alpha. \tag{5.3.7}$$

Substituting the value of μ_0 in the lower bound for d given in (5.3.5), we note that

(i) when $A = k - \mu_0 > 0$,

$$b - 1 - A^{-1}k(r-1) = (b-r)(\alpha-b+1)/(\alpha-b+r)$$
$$\leq 0, \text{ as } b \geq \alpha+1,$$

and

(ii) when $A = 2k - v > 0$,

$$b - 1 - A^{-1}k(r-1) = (v-k)(r-b+1)/(2k-v)$$
$$\leq 0, \text{ as } b \geq r+1.$$

Hence, for the four classes of PBIB designs considered in Chapter 2, the two kinds of lower bound given in (5.3.5) and (5.3.6) are same, i.e., the lower bound is zero. We shall now prove that the upper bound given in (5.3.6) is superior to the one given in (5.3.5).

Theorem 5.3.2. The upper bound given in (5.3.6) is superior to the one given in (5.3.5).

Proof. To establish this result, we have to prove that

(i) when $\alpha + 1 \leq b \leq 2(\alpha + 1)$, $\quad \phi_1 \geq 0$,

(ii) when $2(\alpha + 1) \leq b \leq r(\alpha + 1)$, $\quad \phi_2 \geq 0$

and

(iii) when $b > r(\alpha + 1)$, $\phi_2 < 0$, where

$$\phi_1 = [\, b - 1 - k(r-1)\{\mu_0 - k + 2b^{-1}(rk - \mu_0)\}^{-1} \,] - [\, b - 1 - v\alpha(r - 1)^2P^{-1} \,],$$

$$\phi_2 = [(b-1) - (r-1)] - [\, b - 1 - v\alpha(r - 1)^2P^{-1} \,]$$

and $P = (v-k)(b-r)-\alpha(v-rk)$.

Substituting the value of μ_0 as given by (5.3.7) in ϕ_1 and after some simplification, we get

$$\phi_1 = v\alpha(r-1)(r-2)(b-r)(b-\alpha-1)/PQ,$$

$$\phi_2 = (r-1)(v-k)(\alpha r+r-b)/P,$$

where P has the same meaning as in (5.3.6) and $Q = (b-r)(b-2)-\alpha(b-2r)$. We, then, note that

(i) $\phi_1 \geq 0$ for $\alpha+1 \leq b \leq 2(\alpha+1)$,

(ii) $\phi_2 \geq 0$ for $2(\alpha+1) \leq b \leq r(\alpha+1)$,

(iii) $\phi_2 < 0$ for $b > r(\alpha+1)$.

Hence, we get the required result.