CHAPTER II

FRACTIONAL FACTORIAL DESIGNS OF THE TYPE 2^m

2.1 INTRODUCTION

As the number of factors to be considered in a factorial experiment increases the number of treatment combinations increases very rapidly. Along with this increase in the amount of experimentation comes an increase in the number of high-order interactions. Suppose one is interested only in main effects and two-factor interactions, then naturally all the treatment combinations are not needed. Then one has to choose a suitable fraction out of the large number of assemblies, which will be just enough to estimate the main effects and the two-factor interactions providing a reasonable margin for estimating error.

A number of approaches have been made from different directions to solve the problems of fractional factorial designs of type 2^m. In this chapter, a technique has been developed to construct a fractional factorial design of type 2^m with or without blocks, using orthogonal arrays where the main effect and the two-factor interactions (assuming higher order interactions to be absent) can be estimated economically by reducing the total number of runs. It is expected that the use of this technique would result in less complicated computation.

Further, an attempt has been made to construct Group Balanced Fractional Factorial Design (GBFF) of type 2^m. Here each group of main effects and/or some two-factor interactions are estimated with the same variance. This property of having the same variance per group reduces considerably the computational work. Such a design with uniform variance group-wise is defined as GBFF.

Also, in this chapter, a class of $\frac{1}{2^{p-1}}$ fractional designs for 2^{3p} factorial experiments is developed.

As is well known, Daniel [17] the duplicated runs provide an unbiased estimate of error variance and more precise estimates of the effects. Hence, designs with two levels are developed in which some of the treatment combinations are duplicated.

2.2 ESTIMABILITY OF MAIN EFFECTS AND THE TWO-FACTOR INTERACTIONS OF A 2^m FACTORIAL EXPERIMENT

(assuming interactions involving three or more factors to be negligible).

It is known (Rao [35]) that a subset of N assemblies forming an array (N, m, s, d+k-1) yields a fractionally replicated design from which all main effects and interactions involving k or less factors can be obtained when interactions involving d or more factors are absent. Expressions can be obtained for main effects and the interactions from the usual definitions by retaining only the treatment combinations present in the array, the expressions belonging to different contrasts being orthogonal.

Rao [35] has shown that assuming higher order interactions negligible, from an array of strength 4, the main effects and the two-factor interactions of s^m factorial experiment will be estimable orthogonally. It is possible to reduce the number of assemblies, if the estimates are allowed to be correlated. In the following a method in which all main effects are orthogonally estimated, but otherwise their estimates are correlated with those of certain two--factor interaction is given. This method is developed in a series of theorems 2.1, 2.2 and 2.3 and 2.4.

Theorem 2.1. Suppose that $a_{r1}X_1 + a_{r2}X_2 + \dots + a_{rm}X_m = 0$; form the set of $r=1,2,\dots,p,$ largest possible number of linearly independent equations in GF(2) whose solutions constitute an array of strength 2 in EG(m,2).

Let $U_r = (a_{r1}, a_{r2}, \dots, a_{rm})$ and $W(U_r)$ = the number of non-zero co-ordinates of U_r be defined as the weight of vector U_r. Let G_p be the vector space generated by U_r's. Then in G_p, the number of vectors of weight 3 whose i^{th} coordinate is unity is $\leq p, (i=1,2,\dots,m)$.

In proving theorem 2.1, we use the following lemma.

Lemma 2.1. If $U_{r_1}, U_{r_2}, \dots, U_{r_k}$ be the vectors of weight 3 in G_p whose i^{th} coordinate is unity, then they are all linearly independent $(i=1,2,\dots,m)$.

Proof : If not, there exist constants $b_1, b_2, \dots, b_k$ not all zero such that

$$b_1U_{r_1} + b_2U_{r_2} + \dots + b_kU_{r_k} = 0$$

This is possible since no two of the vectors $U_{r_1}, U_{r_2}, \dots, U_{r_k}$ can have unity as coordinate at the same place except i^{th}. For using a well known result

$$W(V_1+V_2) = W(V_1) + W(V_2) - 2W(V_1V_2)$$

where $V_1 = (d_{11}, d_{12}, \dots, d_{1m})$, $V_2 = (d_{21}, d_{22}, \dots, d_{2m})$ and $V_1V_2 = (d_{11}d_{21}, d_{12}d_{22}, \dots, d_{1m}d_{2m})$.

For, if possible suppose that two of the vectors, say U_{r_1} and U_{r_2} have unities at i^{th} and i'^{th} places ($i \neq i' = 1,2,\dots,m$) and the third unity occurs at different places, then

$$W(U_{r_1} + U_{r_2}) = 3 + 3 - (2x2) = 6-4 = 2$$

which implies that the vector $U_{r_1} + U_{r_2}$ does not belong to G_p, a contradiction; since every vector in G_p has weight $\geqslant 3$.

Hence, the lemma.

Proof of Theorem 2.1

Since the k vectors of weight 3 in G_p are linearly independent, the space generated by them is a subspace of G_p. Hence, k is $\leqslant$ p.

As an extension of Theorem 2.1, we have the following theorem.

Theorem 2.2. In G_p, the number of vectors of weight 4 whose i^{th} and i'^{th} coordinates are both unity is $\leqslant p$, ($i \neq i' = 1,2,\dots,m$).

Proof : The proof is exactly similar to that given in Theorem 2.1 for vectors of weight 3 in G_p, since in this case, no two vectors of weight 4 can have unities as co-ordinates at the same places except i and i'th.

Theorem 2.3 : Let $\Lambda(k \times k) = ((\lambda_{j\ell}))$; j=1,2,...,k; ℓ=1,2,...,k be a non-singular matrix in GF(2).

Let $P(k \times k) = ((p_{jl}))$; j= 1,2,...,k; l=1,2,...,k, denote a (k x k) matrix with elements in the real field where

$p_{jl} = 0$ if $\lambda_{jl} = 0$ is the element of GF(2)

$p_{jl} = 1$ if $\lambda_{j\ell} = 1$ is the element of GF(2)

If $P^*(k \times k) = ((C(\lambda_{jl})))$ where C(0)= -1 and C(1) = 1, then the matrix

$$\begin{array}{c} 1 \\ k \end{array} \begin{bmatrix} 1 & \underline{J}' \\ -\underline{J} & P^* \end{bmatrix}$$
$$\quad\quad 1 \quad\quad k$$

where $\underline{J}$(kx1) is a column vector of unities, is non-singular in the real field.

Proof : It is clear that P is non-singular. The value of the determinant

$$\begin{array}{c} 1 \\ k \end{array} \quad \left|\begin{array}{c|c} 1 & \underline{J}' \\ \hline -\underline{J} & P^* \end{array}\right| \quad \begin{array}{cc} 1 & k \end{array}$$

is $2^k\ |P|$ which is non-zero since P is non-singular. Hence, the theorem is proved.

Consider the p linear independent forms

$$L_r \equiv a_{r1} X_1 + a_{r2} X_2 + \dots + a_{rm} X_m ;$$

$$r = 1,2,\dots,p \text{ in } GF(2).$$

Let S denote the set of treatments $(X_1, X_2, \dots, X_m)$ which satisfy the equations

$$L_r = e_r; \quad r = 1,2,\dots,p$$

where e_r's are elements of GF(2) and all operations are in GF(2). For any linear form L, the corresponding treatment comparison will be denoted by T(L). For example if $L=X_1+X_2$, T(L) will mean the interaction A_1A_2. The estimate of T(L) from the fraction of the 2^m experiment containing only the treatments of S is given by

$$\widehat{T(L)} = \frac{1}{2^{m-p}} \Big[(\{L = 1\} \cap S) - (\{L = 0\} \cap S) \Big] .$$

We now prove the following theorem which provides an unbiased estimate of T(L) from the fraction of the 2^m experiment, containing only the treatments of S.

We now prove the following theorem.

Theorem 2.4.

$$E\left[\hat{T}(L)\right] = \sum (-1)^{w}\, C(\underline{\lambda}'\underline{e})\, A_1^{d_1} A_2^{d_2} \cdots A_m^{d_m}$$

where the summation is over all the 2^p vectors $\underline{\lambda}' = (\lambda_1, \lambda_2, \ldots, \lambda_p)$, d_i is the coefficient of

$X_i (i=1,2,\ldots,m)$ in the linear form

$L + (\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_p L_p)$, w is the weight of the same linear form or of the corresponding coefficient vector $(d_1, d_2, \ldots, d_m)$ and

$\underline{\lambda}'\underline{e} = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_p e_p$ and $C(\underline{\lambda}'\underline{e}) = -1$ if $\underline{\lambda}'\underline{e} = 0$ and $C(\underline{\lambda}'\underline{e}) = 1$ if $\underline{\lambda}'\underline{e} = 1$.

Proof : The expectation of the observed response

$Y(X_1, X_2, \ldots, X_m)$ of the treatment $(X_1 X_2 \ldots X_m)$ is given by

$$E\left[Y(X_1, X_2, \ldots, X_m)\right] = \prod_{i=1}^{m} \left[I + C(X_i)\, A_i\right] \qquad \ldots(2.2.1)$$

which follows from (1.7.6), With the help of this expectation equation, we shall determine the coefficient of $A_1^{d_1} A_2^{d_2} \ldots A_m^{d_m}$ in the expectation of $\hat{T}(L)$ for any arbitrary linear form

$$d_1 X_1 + d_2 X_2 + \ldots + d_m X_m.$$

First we notice that if the linear form $d_1X_1 + d_2X_2 + \dots + d_mX_m$ is not of the form $L + (\lambda_1L_1 + \lambda_2L_2 + \dots + \lambda_pL_p)$, then in the coefficient of $A_1^{d_1} A_2^{d_2} \dots A_m^{d_m}$, there will be 2^{m-p-1} plus signs and 2^{m-p-1} minus signs and hence, the required coefficient is zero. Consider a linear form

$$d_1X_1 + d_2X_2 + \dots + d_mX_m = L+(\lambda_1L_1 + \lambda_2L_2 + \dots + \lambda_pL_p)$$

Case (1) : $\underline{\lambda}'\underline{e} = 0$

Let the weight of the linear form $d_1X_1+d_2X_2+\dots+d_mX_m$ is W. For the sake of definiteness, suppose $d_1=d_2=\dots=d_w=1$. The remaining d's are zero. Then for any treatment $(X_1,X_2,\dots,X_m)$ belonging to $\{L = 1\} \cap S$, we shall have $X_1+X_2+X_3 + \dots + X_w = 1$. So among $X_1,X_2,\dots,X_w$ an odd number say, $2q-1$, would be equal to 1. Therefore, from (2.2.1) it follows that the contribution to the coefficient of $A_1^{d_1} A_2^{d_2} \dots A_m^{d_m} = A_1^{d_1} A_2^{d_2} \dots A_w^{d_w}$ in $E\left[\widehat{T(L)}\right]$ from the response $Y(X_1,X_2,\dots,X_m)$ of any treatment $(X_1,X_2,\dots X_m)$ belonging to $\{L=1\} \cap S$ in $\widehat{T(L)}$ would be

$$+ \frac{1}{2^{m-p}} (-1)^{w-(2q-1)} = \frac{1}{2^{m-p}} c(\underline{\lambda}'\underline{e})(-1)^{w},$$

Similarly, for any treatment $(X_1, X_2, \ldots, X_m)$ belonging to $\{L=0\} \cap S$, we have $X_1+X_2+ \ldots + X_m = 0$. Hence, an even number of x's say 2q would be 1. This means that the response of any treatment belonging to $\{L=0\} \cap S$ would attribute,

$$- \frac{1}{2^{m-p}} (-1)^{w-2q} = \frac{1}{2^{m-p}} C(\underline{\lambda}' \underline{e})(-1)^w,$$ to the coefficient of $A_1^{d_1} A_2^{d_2} \ldots A_w^{d_w}$ in the expectation of $\hat{T}(L)$. Finally there are 2^{m-p} treatments in S, we obtain $C(\underline{\lambda}' \underline{e})(-1)^w$ as the required coefficient of $A_1^{d_1} A_2^{d_2} \ldots A_w^{d_w}$ in the expression for the expectation of $\hat{T}(L)$.

Case (2) : $\underline{\lambda}' \underline{e} = 1$

The coefficient of $A_1^{d_1} A_2^{d_2} \ldots A_w^{d_w}$ in this case can be derived by arguments exactly similar to that in case (1).

This completes the proof.

<u>Corollary 2.4.1</u>

If all interactions involving (t+1) or more factors are assumed to be zero and the linear form L is not aliased with any main effect or interaction involving t or less factors, then $E\left[\hat{T}(L)\right] = T(L)$.

<u>Corollary 2.4.2</u>

If all interactions involving (t+1) or more factors are assumed to be zero and for every $(\lambda_1, \lambda_2, \ldots, \lambda_p) \neq (0, 0, \ldots, 0)$, the weight of $\lambda_1 L_1 + \lambda_2 L_2 + \ldots + \lambda_p L_p$ is at least (2t+1), then for any linear form L of weight not greater than t(which represents a main effect or an interaction involving t or less factors).

$$E\left[\widehat{T(L)}\right] = T(L)$$

This is the same as Rao's [35] Theorem. In this case the fractional replication based on the set S is actually an orthogonal array of strength 2t.

Theorem 2.5

Let p linearly independent forms

$$L_r \equiv a_{r1} X_1 + a_{r2} X_2 + \ldots + a_{rm} X_m ;$$

$$r = 1, 2, \ldots, p,$$

generate a class of arrays Ω_2 in EG(m,2) each of strength 2. There are 2^p arrays in this class. Let $S_1, S_2, \ldots, S_{p+1}$ be (p+1) arrays which correspond to the linear forms equated to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

column-wise. Then the fractional replicate of the 2^m experiment consisting of the assemblies belonging to these (p+1) arrays will estimate the main effects and the two-factor interactions (assuming interactions involving three or more factors to be negligible).

<u>Proof</u> : Denoting the $(l+1)^{th}$ column vector by

$$\underline{\delta}_{\ell} = \begin{bmatrix} \delta_{l1} \\ \delta_{l2} \\ \cdots \\ \cdots \\ \delta_{lp} \end{bmatrix}$$

where $\delta_{lr} = 0, (l \neq r, \ \ell = 0, 1, 2, \ldots, p; \ r = 1, 2, \ldots, p)$ and $\delta_{11} = \delta_{22} = \ldots = \delta_{pp} = 1$, the array S_{l+1} $(l=0,1,2,\ldots,p)$ correspond to the linear forms $L_r (r=1,2,\ldots,p)$ equated to $\underline{\delta}_l$.

Let $S = \bigcup_{l=0}^{p} S_{l+1}$ be the required fractional replicate consisting of $(p+1)\ 2^{m-p}$ assemblies of the 2^m design.

Since each is an array of strength 2, the fraction S obviously estimates all the main effects. Consider the two-factor interaction T(L). There may be three cases :

Case (1) : T(L) is not aliased with any main effect or two-factor interaction.

3 In this case from corrllary (2.4.1) it follows that T(L) is estimable.

Case (2) : T(L) is aliased with a main effect A_{i_1}.

Suppose $T(L) = A_{i_2} A_{i_3}$. Then there exists

$(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1p})$ such that

$$L + \lambda_{11}L_1 + \lambda_{12}L_2 + \cdots + \lambda_{1p}L_p = X_{i_1}$$

so $$\lambda_{11}L_1 + \lambda_{12}L_2 + \cdots + \lambda_{1p}L_p = X_{i_1} + X_{i_2} + X_{i_3}$$

Suppose the interactions $A_{i_4} A_{i_5}, A_{i_6} A_{i_7}, \ldots, A_{i_{2k}} A_{i_{2k+1}}$ are also aliased with A_{i_1}. Then from lemma (2.1..), it follows that all the indices $i_2, i_3, \ldots, i_{2k+1}$ are distinct and $k \leqslant p$.

Suppose

$$\lambda_{21}L_1 + \lambda_{22}L_2 + \cdots + \lambda_{2p}L_p = X_{i_1} + X_{i_4} + X_{i_5}$$

$$\lambda_{31}L_1 + \lambda_{32}L_2 + \cdots + \lambda_{3p}L_p = X_{i_1} + X_{i_6} + X_{i_7}$$

. . . .

. . . .

$$\lambda_{k1}L_1 + \lambda_{k2}L_2 + \cdots + \lambda_{kp}L_p = X_{i_1} + X_{i_2} + X_{i_{2k+1}}.$$

Then from lemma (2.1..), the linear forms

$X_{i_1} + X_{i_2} + X_{i_3},\ X_{i_1} + X_{i_4} + X_{i_5}, \ldots, X_{i_1} + X_{i_{2k}} + X_{i_{2k+1}}$

are mutually independent. So the matrix.

$$\Lambda(k \times p) = ((\lambda_{jr}))\ ;\ j = 1,2,\ldots,k;\ r=1,2,\ldots,p$$

is of rank k. Without less of generality we shall assume that the (k x k) principal minor Λ_1 is non-singular.

Suppose T_{l+1} denotes the estimate of A_{i_1} based on the fraction of assemblies given by the array S_{l+1} $(l=0,1,2,\ldots,p)$.

Let $\underline{\lambda}_j' = (\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{jp})$. We know that $\underline{\delta}_l' = (\delta_{l1}, \delta_{l2}, \ldots, \delta_{lp})$, then we get

$$\underline{\lambda}_j' \cdot \underline{\delta}_l' = \lambda_{jl} \quad (j=1,2,\ldots,k;\ l=0,1,\ldots,k)$$

where λ_{jl} will be either 0 or 1.

The weight of the linear form representing a two-factor interaction is 2. Hence we get from theorem 2.4.

$$E \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ - \\ - \\ \hat{T}_{k+1} \end{bmatrix} \begin{matrix} 1 \\ \\ k \end{matrix} = \left[\begin{array}{c|c} 1 & -\underline{J}' \\ \hline \underline{J} & ((c(\lambda_{jl}))) \end{array}\right]_{\substack{1 \quad\quad k}} \begin{bmatrix} A_{i_1} & \\ A_{i_2} & A_{i_3} \\ \cdots & \cdots \\ \cdots & \cdots \\ A_{i_{2k}} & A_{i_{2k+1}} \end{bmatrix} \quad \ldots(2.2.1)$$

The matrix $\Lambda_1 = ((\lambda_{jl}))$; $j = 1,2,\ldots,k$; $l=0,1,2,\ldots,k$ is non-singular. Hence, by Theorem 2.3, the matrix occuring on the right hand side of the above expectation equation is non-singular. Therefore $A_{i_2} A_{i_3}$ is estimable.

Case (3) : A_{i_2} A_{i_3} is not aliased with any main effect, but is aliased with a two-factor interaction.

Using the argument similar to that in case (2), we can easily establish the estimability of A_{i_2} A_{i_3} with the help of theorem 2.2. and theorem 2.4.

This completes the proof.

The following is an example to estimate the main effects and the two-factor interactions of a 2^5 experiment. In the example, $3/4^{th}$ fraction is considered.

Consider the assemblies belonging to three arrays given by the equations

$$X_1 + X_2 + X_3 = 0 \quad 1 \quad 0$$
$$X_1 + X_4 + X_5 = 0 \quad 0 \quad 1$$

in EG(m,2).

The identity relationship for the fraction of the 2^5

6 design defined as above is

$$I = A_1A_2A_3 = A_1A_4A_5 = A_2A_3A_4A_5$$

from which follow the sets of aliased effects

(1) $\{A_1,\ A_2A_3,\ A_4A_5\}$,

(2) $\{A_2,\ A_1A_3\}$, $\{A_3,\ A_1A_2\}$,

$\{A_4,\ A_1A_5\}$, $\{A_5,\ A_1A_4\}$,

(3) $\{A_2A_5,\ A_3A_4\}$, $\{A_2A_4,\ A_3A_5\}$.

<u>Estimation of Effects</u>

(1*) For the effects in (1), the three estimable linear functions corresponding to the three arrays are

$$A_1 - A_2A_3 - A_4A_5,$$

$$A_1 + A_2A_3 - A_4A_5,$$

$$A_1 - A_2A_3 + A_4A_5.$$

The matrix of coefficients

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

is clearly non-singular, which implies that

7 the effects (1) are estimable.

(2*) For any pair of effects in (2) say A_2, A_1A_3, the two estimable functions are $A_1 - A_2A_3$ and $A_1 + A_2A_3$ which are linearly independent and, hence, the effects involved are estimable.

(3*) For any pair of effects in (3), say A_2A_5, A_3A_4, the two estimable functions are

$$A_2A_5 + A_3A_4 \text{ and } A_2A_5 - A_3A_4$$

and, hence, the effects involved are estimable as in (2*).

The normal equations estimating effects (1) are

$$\begin{bmatrix} 24 & -8 & -8 \\ & 24 & -8 \\ \text{sym.} & & 24 \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \widehat{A_2A_3} \\ \widehat{A_4A_5} \end{bmatrix} = \begin{bmatrix} Y(A_1) \\ Y(A_2A_3) \\ Y(A_4A_5) \end{bmatrix}$$

which give

$$\begin{bmatrix} \hat{A}_1 \\ \widehat{A_2A_3} \\ \widehat{A_4A_5} \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ \text{sym.} & & 2 \end{bmatrix} \begin{bmatrix} Y(A_1) \\ Y(A_2A_3) \\ Y(A_4A_5) \end{bmatrix}$$

Similarly for any pair of effects in (2), say A_2, A_1A_3

$$\begin{bmatrix} 24 & -8 \\ -8 & 24 \end{bmatrix} \begin{bmatrix} \hat{A}_2 \\ \widehat{A_1A_3} \end{bmatrix} = \begin{bmatrix} Y(A_2) \\ Y(A_1A_3) \end{bmatrix}$$

which give

$$\begin{bmatrix} \hat{A}_2 \\ \widehat{A_1A_3} \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} Y(A_2) \\ Y(A_1A_3) \end{bmatrix}$$

and for any pair of effects in (3) say A_2A_5, A_3A_4

$$\begin{bmatrix} 24 & -8 \\ -8 & 24 \end{bmatrix} \begin{bmatrix} \widehat{A_2A_5} \\ \widehat{A_3A_4} \end{bmatrix} = \begin{bmatrix} Y(A_2A_5) \\ Y(A_3A_4) \end{bmatrix}$$

which give

$$\begin{bmatrix} \widehat{A_2A_5} \\ \widehat{A_3A_4} \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} Y(A_2A_5) \\ Y(A_3A_4) \end{bmatrix}$$

The grand average I is estimated by G/24 where G is the total response of all assemblies.

Thus the 16 effects (main effects, two-factor interactions and the grand average) of the 2^5 design are estimated from 24 assemblies assuming higher factor interaction as negligible.

The notation $\wedge$ over any effect denotes the estimate of that effect.

Using the same method, fractional replicate of 2^6, 2^7, 2^8 etc. designs can be suitably constructed.

2.3 The fractional plans given in this section are more economical. The estimation of treatment effects is more simpler by using orthogonal arrays of strength 2. Also, fractional plans are Group Balanced Fractional Factorial Design (GBFF) of type 2^m. Here each group of main effects and/or two-factor interactions have the same variance.

Construction

The procedure of choosing generators of the designs remains the same.

We choose L_r's such that none of these linear forms is of the weight $\leq$ 2. The solution $(X_1, X_2, \ldots, X_m)$ of the equations

$$L_r = d_r \pmod 2 \qquad \ldots(2.3.1)$$

$$d_r = 0, 1; \quad r = 1, 2, \ldots, p$$

each level of every factor will occur equal number of times. The same is true for each combination of levels of every pair of factors. The solutions to-gether are said to form an orthogonal array of strength 2, Rao [36] .

Right hand side of (2.3.1) is a sub-matrix taken from orthogonal arrays of strength 2 with some column dropped. This result is due to Parikh [31] .

Examples

(1) $3/4^{th}$ fractional raplicate of 2^6 experiment.

(2) $7/16^{th}$ fractional replicate of 2^8 experiment.

(3) $5/32^{nd}$ fractional replicate of 2^9 experiment.

Detailed discussion of these designs is given in the following pages.

Example (1)

$3/4^{th}$ Fractional Replicate of 2^6 Design

The design consists of the treatments satisfying the equations

$$X_1 + X_2 + X_3 = 1,\quad 1,\quad 0,\quad 1,\quad 0,\quad 0$$

$$X_1 + X_4 + X_5 = 0,\quad 1,\quad 1,\quad 0,\quad 1,\quad 0$$

$$X_2 + X4 + X_6 = 0,\quad 0,\quad 1,\quad 1,\quad 0,\quad 1$$

mod 2.

The identity relationship for the fraction is

$$I = A_1A_2A_3 = A_1A_4A_5 = A_2A_4A_6 = A_3A_5A_6$$

$$= A_2A_3A_4A_5 = A_1A_3A_4A_6 = A_1A_2A_5A_6$$

1 Estimation of Effects

The set of correlated effects are

$$(1)\qquad \{A_1,\ A_2A_3,\ A_4A_5\},$$
$$\{A_2,\ A_1A_3,\ A_4A_6\},$$
$$\{A_3,\ A_1A_2,\ A_5A_6\},$$
$$\{A_4,\ A_1A_5,\ A_2A_6\},$$
$$\{A_5,\ A_1A_4,\ A_3A_6\},$$
$$\{A_6,\ A_2A_4,\ A_3A_5\},$$

$$(2)\qquad \{A_1A_6,\ A_3A_4,\ A_2A_5\}.$$

Each set of effects in (1) is estimated by the matrix

$$\begin{bmatrix} 40 & 0 & 0 \\ & 48 & -16 \\ \text{sym.} & & 48 \end{bmatrix}^{-1} = \frac{1}{384}\begin{bmatrix} 8 & 0 & 0 \\ & 9 & 3 \\ \text{sym.} & & 9 \end{bmatrix}$$

Effects in (2) is estimated by the matrix.

$$\begin{bmatrix} 48 & -16 & -16 \\ & 48 & -16 \\ \text{sym.} & & 48 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 2 & 1 & -1 \\ & 2 & 1 \\ \text{sym.} & & 2 \end{bmatrix}$$

$\hat{\mu} = \frac{G}{48}$, where G is the sum of observed responses.

2 Example (2)

$7/16^{th}$ Fractional Replicate of 2^8 Design

The design consists of the treatments satisfying the equations

$$X_1 + X_3 + \lambda_6 = 0,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0$$
$$X_2 + X_5 + X_7 = 0,\ 0,\ 1,\ 1,\ 0,\ 1,\ 0$$
$$X_4 + X_6 + X_7 = 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 1$$
$$X_3 + X_5 + X_8 = 1,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0$$

mod 2.

The identity relationship for fraction is

$$I = A_1A_3A_6 = A_2A_5A_7 = A_4A_6A_7 = A_3A_5A_8$$
$$= A_1A_3A_4A_7 = A_1A_5A_6A_8 = A_2A_4A_5A_6$$
$$= A_2A_3A_7A_8 = A_1A_2A_4A_8$$

Estimation of Effects

The sets of correlated effects are

(1) $\{A_3,\ A_1A_6,\ A_5A_8\}$,

$\{A_5,\ A_2A_7,\ A_3A_8\}$,

$\{A_6,\ A_1A_3,\ A_4A_7\}$,

$\{A_7,\ A_2A_5,\ A_4A_6\}$,

$\{A_2A_8,\ A_1A_4,\ A_3A_7\}$,

$\{A_1A_8,\ A_5A_6,\ A_2A_4\}$,

3 (2) $\{A_1, A_3A_6\}$,

$\{A_2, A_5A_7\}$,

$\{A_4, A_6A_7\}$,

$\{A_8, A_3A_5\}$,

$\{A_1A_5, A_6A_8\}$,

$\{A_1A_6, A_5A_8\}$,

$\{A_1A_3, A_4A_7\}$,

$\{A_1A_2, A_4A_8\}$,

$\{A_2A_3, A_7A_8\}$.

Each set of effects in (1) is estimated by the matrix

$$\begin{bmatrix} 112 & -16 & -16 \\ & 112 & -16 \\ \text{sym.} & & 112 \end{bmatrix}^{-1} = \frac{1}{664}\begin{bmatrix} 6 & 1 & 1 \\ & 6 & 1 \\ \text{sym.} & & 9 \end{bmatrix}$$

Each set of effects in (2) is estimated by the matrix

$$\begin{bmatrix} 112 & -16 \\ -16 & 112 \end{bmatrix}^{-1} = \frac{1}{768}\begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}$$

$\hat{\mu} = G/112$, where G is the sum of observed responses.

Example (3)

5/32nd Fractional Replicate of 2^9 Design

The design consists of the treatments satisfying the equations.

$$X_1 + X_2 + X_3 = 0,\ 0,\ 1,\ 1,\ 1$$
$$X_4 + X_5 + X_6 = 0,\ 0,\ 1,\ 1,\ 1$$
$$X_7 + X_8 + X_9 = 0,\ 0,\ 1,\ 1,\ 1$$
$$X_2 + X_5 + X_8 = 0,\ 1,\ 1,\ 0,\ 1$$
$$X_3 + X_6 + X_9 = 1,\ 1,\ 1,\ 0,\ 0$$

mod 2.

The identity relationship for fraction is

$$I = A_1A_2A_3 = A_4A_5A_6 = A_7A_8A_9 = A_2A_5A_8$$
$$= A_3A_6A_9 = A_1A_4A_7 = A_1A_3A_5A_8 = A_1A_2A_6A_9$$
$$= A_2A_4A_6A_8 = A_3A_4A_5A_9 = A_2A_5A_7A_9$$
$$= A_3A_6A_7A_8 = A_1A_4A_8A_9 = A_1A_5A_6A_7$$
$$= A_2A_3A_4A_7$$ (omitting 5 and more factor interactions).

Estimation of Effects

The sets of correlated effects are

(1) $\{A_1, A_2A_3, A_4A_7\}$,

$\{A_2, A_1A_3, A_5A_8\}$,

$\{A_4, A_5A_6, A_1A_7\}$,

$\{A_5, A_4A_6, A_2A_8\}$,

$\{A_7, A_8A_9, A_1A_4\}$,

$\{A_8, A_7A_9, A_2A_3\}$,

(2) $\{A_3, A_1A_2, A_6A_9\}$,

$\{A_6, A_4A_5, A_3A_9\}$,

$\{A_9, A_7A_8, A_3A_6\}$,

(3) $\{A_1A_9, A_2A_6, A_4A_8\}$,

$\{A_2A_9, A_1A_6, A_5A_7\}$,

$\{A_3A_5, A_4A_9, A_1A_8\}$,

$\{A_3A_8, A_6A_7, A_1A_5\}$,

$\{A_5A_9, A_3A_4, A_2A_7\}$,

$\{A_6A_8, A_3A_7, A_2A_4\}$.

Each set of effects in (1) is estimated by the matrix

$$\begin{bmatrix} 80 & 16 & 16 \\ & 80 & 16 \\ \text{sym.} & & 80 \end{bmatrix}^{-1} = \frac{1}{448} \begin{bmatrix} 6 & -1 & -1 \\ & 6 & -1 \\ \text{sym.} & & 6 \end{bmatrix}$$

Each set of effects in (2) is estimated by the matrix

$$\begin{bmatrix} 80 & 16 & 16 \\ & 80 & -48 \\ \text{sym.} & & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 2 & -1 & -1 \\ & 3 & 2 \\ \text{sym.} & & 3 \end{bmatrix}$$

Each set of effects in (3) is estimated by the matrix.

$$\begin{bmatrix} 80 & -48 & 16 \\ & 80 & 16 \\ \text{sym.} & & 80 \end{bmatrix}^{-1} = \frac{1}{128}\begin{bmatrix} 3 & 2 & -1 \\ & 3 & -1 \\ \text{sym.} & & 2 \end{bmatrix}$$

$\hat{\mu} = G/80$, where G is the sum of observed responses.

2.4 FRACTIONAL REPLICATE OF A 2^m DESIGN WITH BLOCK

Theorem 2.6 : If the assemblies belonging to each of the (p+1) arrays, say $S_1 S_2, \ldots, S_{p+1}$ of strength 2 in EG(m,2) defined by the p linearly independent forms.

$$L_r \equiv a_{r1} X_1 + a_{r2} X_2 + \ldots + a_{rm} X_m$$

(r=1,2,...,p), equated to

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

7 column-wise in EG(2) are assigned to a block, then the resulting fractional design of the 2^m experiment in (p+1) blocks, all main effects and two-factor interactions are estimable with their estimate correlated in sets, but orthogonal to p block contrasts.

<u>Proof</u> : Let (S_U) denote the sum of responses of the assemblies in the array S_U (U=1,2,...,p+1). These will be then the (p+1) block totals in some order.

The p linearly independent contrasts between the (p+1) block totals represent linear function of interactions corresponding to the linear forms.

$$\lambda_1 L_1 + \lambda_2 L_2 + \dots + \lambda_p L_p \qquad \text{.,. (2.4.1)}$$

$$\lambda_r = 0,1, \; ; (r=1,2,\dots,p);$$

$$(\lambda_1, \lambda_2, \dots, \lambda_p) \neq (0,0,\dots,0)$$

each of weight $\geqslant 3$, and the contrasts between the block effects. This implies in otherwords that the interactions corresponding to (2.4.1) are mixed up (or confounded) with the contrasts between the block effects.

Next the p linear forms L_r(r=1,2,...,p) in GF(2) partition, the effects of the factorial experiment in alias sets, the estimates of any two effects belonging to

8 different alias sets being orthogonal.

From these it follows that the estimates of the main effects and two-factor interactions are orthogonal to the estimates of the interactions corresponding to linear form (2.4.1) since they belong to alias sets different from the one consisting of the effects belonging to (2.4.1) which in turn implies that they are also orthogonal to the p block contrasts. They are estimable.

For example, 3/4th Fractional Replicate of 2^5 Design in 3 block of 8 Assemblies Each.

The design consists of the treatment satisfying the equations.

Array	S_1	S_2	S_3
$X_1 + X_2 + X_3 =$	0	0	1
$X_3 + X_4 + X_5 =$	0	1	0

Construction of Block Design

Block-1	Block-2	Block-3
S_1	S_2	S_3
(0 0 0 0 0)	(0 0 0 0 1)	(1 0 0 0 0)
(0 0 0 1 1)	(0 0 0 1 0)	(1 0 0 1 1)
(1 1 0 0 0)	(1 1 0 0 1)	(0 1 0 0 0)
(1 1 0 1 1)	(1 1 0 1 0)	(0 1 0 1 1)
(1 0 1 0 1)	(1 0 1 0 0)	(0 0 1 0 1)
(1 0 1 1 0)	(1 0 1 1 1)	(0 0 1 1 0)
(0 1 1 0 1)	(0 1 1 0 0)	(1 1 1 0 1)
(0 1 1 1 0)	(0 1 1 1 1)	(1 1 1 1 0)

where $\{S_U\}$, U = 1,2,3 mean the set of assemblies belonging to the array S_U.

The identity relationship for the fraction is

$$I = A_1A_2A_3 = A_3A_4A_5 = A_1A_2A_4A_5$$

from which follow the sets of aliased effects.

$$\{A_1,\ A_2A_3,\ A_1A_3A_4A_5,\ A_2A_4A_5\},$$
$$\{A_2,\ A_1A_3,\ A_2A_3A_4A_5,\ A_1A_4A_5\},$$
$$\{A_3,\ A_1A_2,\ A_4A_5,\ A_1A_2A_3A_4A_5\},$$
$$\{A_4,\ A_1A_2A_3A_4,\ A_3A_5,\ A_1A_2A_5\},$$
$$\{A_5,\ A_1A_2A_3A_5,\ A_3A_4,\ A_1A_2A_4\},$$
$$\{A_1A_5,\ A_2A_3A_5,\ A_1A_3A_4,\ A_2A_4\}.$$

0 The two block contrasts are

$$(S_2) - (S_1) \text{ and } (S_3) - (S_1)$$

with $E\left[(S_2)-(S_1)\right] = 16\, A_3A_4A_5 - 16\, A_1A_2A_4A_5 + 8(b_2-b_1)$

$E\left[(S_3)-(S_1)\right] = 16\, A_1A_2A_3 - 16\, A_1A_2A_4A_5 + 8(b_3-b_1)$

where b_1, b_2, b_3 denote the block effects.

Thus the interactions $A_1A_2A_3$, $A_3A_4A_5$, $A_1A_2A_4A_5$ are confounded with the contrasts (b_2-b_1), (b_3-b_1) between the three block effects.

Since the estimates of any two effects belonging to two-different alias sets are orthogonal, it follows that the estimates of $A_1A_2A_3$, $A_3A_4A_5$, $A_1A_2A_4A_5$ are orthogonal to the estimates of the rest of the effects in other sets which implies that the contrasts (b_2-b_1), (b_3-b_1) are orthogonal to the estimates of all main effects and the two-factor interactions which are estimable.

Fractional replicates of 2^6, 2^7, etc. can be constructed.

2.5 CONSTRUCTION OF $1/2^{p-1}$ FRACTION OF 2^{3p} FACTORIAL DESIGN

Consider linear forms L_1, L_2, ..., L_p given by

1 $$L_1 = X_1 + X_2 + X_3$$

$$L_2 = X_4 + X_5 + X_6$$

$$\cdots \qquad \ldots(2.5.1)$$

$$\cdots$$

$$L_p = X_{3p-2} + X_{3p-1} + X_{3p}$$

Obviously the solutions of the p linear equations $L_1 = 0$, $L_2 = 0$, ..., $L_p = 0$ (mod 2) given an orthogonal array of strength 2, according to Rao [36]. From this fraction, we can obtain estimates of 2^{2p} linear functions of the main effects and interactions of the 2^{3p} design. Assuming interactions including three factors and more to be absent, this implies that the linear forms which are estimable are functions of the main effects and the two-factor interactions, which means that the two-factor interactions are aliased with main effects. They are obtained from the identity relationship

$$I = A_1A_2A_3 = A_4A_5A_6 = \cdots = A_{3p-2}A_{3p-1}A_{3p},$$

omitting higher factor interactions.

The aliased groups of effects are

$$(A_j,\ A_{j+1},\ A_{j+2})$$

$$(A_{j+1},\ A_j,\ A_{j+2}) \qquad \ldots(2.5.2)$$

$$(A_{j+2},\ A_j,\ A_{j+1})$$

where $j = 1, 4, 7, \ldots, (3p-2)$

To make these aliased effects estimable, let us take one more fraction by the solutions of the equations

$$L_1 = 1,\ L_2 = 1, = \ldots = L_p = 1 \pmod 2.$$

The two fractions together give a fractional design from which the estimates of the main effects and the two-factor interactions come out orthogonal.

Examples

(1) 2^6 Fractional Factorial Design with 32 runs.

(2) 2^9 Fractional Factorial Design with 128 runs.

Detailed discussion of these designs is given in the following pages.

Example (1)

Fractional Replicate of a 2^6 Design with 32 runs

The design consists of the treatments satisfying the equations

$$X_1 + X_2 + X_3 = 0,1$$
$$X_4 + X_5 + X_6 = 0,1$$

mod 2.

3 The identity relationship for each fraction is

$$I = A_1A_2A_3 = A_4A_5A_6$$

from which follows the sets of correlated effects are

$$\{A_1,\ A_2A_3\},$$
$$\{A_2,\ A_1A_3\},$$
$$\{A_3,\ A_1A_2\},$$
$$\{A_4,\ A_5A_6\},$$
$$\{A_5,\ A_4A_6\},$$
$$\{A_6,\ A_4A_5\}.$$

All these effects and the remaining two-factor interactions are orthogonally estimated.

Example (2)

Fractional Replicate of a 2^9 Design with 128 runs

The design consists of the treatments satisfying the equations

$$X_1 + X_2 + X_3 = 0,\ 1$$
$$X_4 + X_5 + X_6 = 0,\ 1$$
$$X_7 + X_8 + X_9 = 0,\ 1$$
$$\bmod 2.$$

4 The identity relationship for each fraction is

$$I = A_1A_2A_3 = A_4A_5A_6 = A_7A_8A_9$$

from which follows the sets of correlated effects are

$$\{A_1,\ A_2A_3\},$$
$$\{A_2,\ A_1A_3\},$$
$$\{A_3,\ A_1A_2\},$$
$$\{A_4,\ A_5A_6\},$$
$$\{A_5,\ A_4A_6\},$$
$$\{A_6,\ A_4A_5\},$$
$$\{A_7,\ A_8A_9\},$$
$$\{A_8,\ A_7A_9\},$$
$$\{A_9,\ A_7A_8\}.$$

All these effects and the remaining two-factor interactions are orthogonally estimated.

2.6 REMARKS

The fractional designs given in section 2.5 can be further assigned to two blocks, each block containing the treatments in each of the two fractions, obviously the estimability of the treatment effects is not affected. The analysis of variance table is to modify to accommodate one degree of freedom between the two blocks.

2.7 PARTIALLY DUPLICATED FRACTIONAL FACTORIAL DESIGN OF TYPE 2^m

Partially duplicated fractional factorial design which requires fewer runs, including duplicates, was given by Daniel [17] . The duplicated runs provide an unbiased estimate of error variance and more precise estimates of effects. Here block designs are considered for fractional factorial. We assume that the main effects and the two-factor interactions are present and the interactions of higher orders are negligible.

Construction

Using the above mentioned theorems on fractional factorial designs of type 2^m, the investigation on partially duplicated fractional factorial designs is given.

The linear form $L_r (r=1,2,...,p)$ are called the generators of the fractional design and will said to generate the fractional design or the fraction.

6 Let $D = (d_{ru})$,

$r = 1,2,\ldots,p;\ u = 1,2,\ldots,p$

be a non-singular matrix of 0, 1 (mod 2).

Then the combined solutions of the (p+1) set of simultaneous equations

$$\begin{aligned} L_1 &= 0,\ d_{11},\ d_{12},\ \ldots,\ d_{1p} \\ L_2 &= 0,\ d_{21},\ d_{22},\ \ldots,\ d_{2p} \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ L_p &= 0,\ d_{p1},\ d_{p2},\ \ldots,\ d_{pp} \end{aligned} \qquad \ldots(2.7.1)$$

mod 2.

This gives a fractional design from which the main effects and the two-factor interactions are estimable.

Example

2^6 experiment with 40 runs

The design consists of the treatments satisfying the equations

$$\begin{aligned} X_1 + X_2 + X_3 &= 0,\ 0,\ 1,\ 0,\ 1 \\ X_1 + X_4 + X_5 &= 0,\ 1,\ 0,\ 1,\ 0 \\ X_2 + X_4 + X_6 &= 0,\ 0,\ 1,\ 1,\ 0 \end{aligned}$$

mod 2.

The identity relationship for the fraction is

$$I = A_1A_2A_3 = A_1A_4A_5 = A_3A_5A_6$$
$$= A_2A_3A_4A_5 = A_1A_3A_4A_6 = A_1A_2A_5A_6$$
$$= A_2A_4A_6$$

The sets of correlated effects are

(1) $\{A_1, A_2A_3, A_4A_5\}$,

$\{A_6, A_2A_4, A_3A_5\}$,

(2) $\{A_2, A_1A_3, A_4A_6\}$,

$\{A_3, A_1A_2, A_5A_6\}$,

$\{A_4, A_1A_5, A_2A_6\}$,

$\{A_5, A_1A_4, A_3A_6\}$,

(3) $\{A_1A_6, A_3A_4, A_2A_5\}$.

(1*) Each set of effects in (1) is estimated by the matrix

$$\begin{bmatrix} 40 & -8 & -8 \\ & 40 & -24 \\ \text{sym.} & & 40 \end{bmatrix}^{-1} = \frac{1}{64}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 2 \\ \text{sym.} & & 3 \end{bmatrix}.$$

(2*) Each set of effects in (2) is estimated by the matrix

$$\begin{bmatrix} 40 & -8 & -8 \\ & 40 & 8 \\ \text{sym.} & & 40 \end{bmatrix}^{-1} = \frac{1}{228}\begin{bmatrix} 6 & 1 & 1 \\ & 6 & -1 \\ \text{sym.} & & 6 \end{bmatrix}$$

(3*) Effects in (3) is estimated by the matrix

$$\begin{bmatrix} 40 & 8 & 8 \\ & 40 & -24 \\ \text{sym.} & & 40 \end{bmatrix}^{-1} = \frac{1}{64} \begin{bmatrix} 2 & -1 & -1 \\ & 3 & 2 \\ \text{sym.} & & 3 \end{bmatrix}$$

$\hat{\mu} = G/40$, where G is the sum of the observed responses.