CHAPTER - III

FRACTIONAL FACTORIAL DESIGNS OF THE TYPE 3^n

3.1 INTRODUCTION

The method of construction of fractional factorial design of the type 3^n is given in section 1.11. It is said that fractional replication is not as satisfactory in 3^n design as in 2^m design. In this chapter, fractional replicate of a 3^n experiment with blocks is discussed.

3.2 FRACTIONAL REPLICATE OF A 3^n EXPERIMENT

Theorem 3.1 : Let the q linearly independent forms

$$M_\theta \equiv b_{\theta 1} Y_1 + b_{\theta 2} Y_2 + \cdots + b_{\theta n} Y_n$$
$$(\theta = 1,2,\ldots, q)$$

generate a class of arrays $\emptyset_2$ each of strength 2 in EG(n,3). If the fraction of the 3^n design consisting of the assemblies belonging to the arrays $T_1, T_2, \ldots, T_k$ of $\emptyset_2$ estimates the main effects and the two-factor interactions, then the corresponding incomplete block design with k blocks, each block containing assemblies of some array $T_v (v=1,2,\ldots,k)$

estimates the same effects orthogonally to the (k-1) contrasts between the k block effects (assuming higher factor interactions to be absent).

Proof : The proof is similar to that given in case of 2^m design in Theorem, 2.6.

The (k-1) linearly independent contrasts between the k block totals are linear functions of interactions involving 3 or more factors corresponding to the linear forms

$$\mu_1 M_1 + \mu_2 M_2 + \dots + \mu_q M_q,$$
$$\mu_\theta = 0,1,2 \; (\theta = 1,2, \dots, q),$$
$$(\mu_1, \mu_2, \dots, \mu_q) \neq (0,0,\dots,0)$$

each of weight ≥ 3 and the contrasts between the k block effects.

The theorem holds good if arrays of strength 2 are replaced by arrays of strength 3 in EG(n,3).

Examples

(1) Fractional replicate of a 3^4 design.

(2) Fractional replicate of a 3^5 design.

(3) Fractional replicate of a 3^6 design.

Example (1)

$4/9^{th}$ Fraction of 3^4 design in 4 blocks, 9 assemblies in each

Design

Block-1	Block-2	Block-3	Block-4
$\{T_1\}$	$\{T_2\}$	$\{T_3\}$	$\{T_4\}$

where the arrays of strength 2 in EG(4,3) are as given below ;

$$\begin{aligned} Z_1 + Z_2 + Z_3 &= 0 \\ Z_2 + 2Z_3 + Z_4 &= 0 \end{aligned} \qquad \ldots(3.2.1)$$

in GF(3). The identity relationship for the fraction of 3^4 design is

$$I = \beta_1\beta_2\beta_3 = \beta_2\beta_3^2\beta_4 = \beta_1\beta_2^2\beta_4 = \beta_1\beta_3^2\beta_4^2$$

The sets of effects aliased are

$$\begin{aligned} &\{\beta_1,\ \beta_2\beta_3,\ \beta_3\beta_4,\ \beta_2\beta_4^2\}, \\ &\{\beta_2,\ \beta_1\beta_3,\ \beta_1\beta_4,\ \beta_3\beta_4^2\}, \\ &\{\beta_3,\ \beta_1\beta_2,\ \beta_2\beta_4,\ \beta_1\beta_4^2\}, \\ &\{\beta_4,\ \beta_2\beta_3^2,\ \beta_1\beta_2^2,\ \beta_1\beta_3^2\}, \end{aligned} \qquad \ldots(3.2.2)$$

The 9 assemblies satisfying the equation (3.2.1) will estimate (1) the grand average, (2) two linearly independent functions of effects for each alias set in (3.2.2).

Since the number of effects in each alias set is 4, each effect carrying 2 d.f., let us consider the estimation of effects (3.2.2) from the assemblies of the four arrays in EG(4,3) given below :

Array	T_1	T_2	T_3	T_4
$Z_1 + Z_2 + Z_3 =$	0	0	0	1
$Z_2 + 2Z_3 + Z_4 =$	0	1	2	0

...(3.2.3)

For each set of effects in (3.2.2), the corresponding coefficients in the expected value of the response column vector $\underline{y}$ of assemblies belonging to the arrays T_1, T_2, T_3, T_4 in (3.2.3) are given below :

It should be noted that (1) each coefficient corresponding to three assemblies, (2) the estimates of any two effects coming from different alias sets are orthogonal.

In the table given 3.1, the coefficient vectors within each set are in order of the assemblies, but between the sets they are in different order of the assemblies. Two coefficients vectors belonging to different sets are orthogonal i.e. their inner product is zero.

Table 3.1

Array \ Effect	Set-1: β_1 L	β_1 Q	$\beta_2\beta_3$ L	$\beta_2\beta_3$ Q	$\beta_3\beta_4$ L	$\beta_3\beta_4$ Q	$\beta_2\beta_4^2$ L	$\beta_2\beta_4^2$ Q	Set-2: β_2 L	β_2 Q	$\beta_1\beta_3$ L	$\beta_1\beta_3$ Q	$\beta_1\beta_4$ L	$\beta_1\beta_4$ Q	$\beta_3\beta_4^2$ L	$\beta_3\beta_4^2$ Q
T_1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
	0	2	1	1	0	-2	0	-2	0	-2	1	1	0	-2	0	-2
	1	1	0	-2	1	1	1	1	1	1	0	-2	1	1	1	1
T_2	-1	1	-1	1	0	-2	1	-2	-1	1	-1	1	0	-2	1	1
	0	-2	1	1	1	1	-1	1	0	-2	1	1	1	1	-1	1
	1	1	0	-2	-1	1	0	1	1	1	0	-2	-1	1	0	-2
T_3	-1	1	-1	1	1	1	0	-2	-1	1	-1	1	1	1	0	-2
	0	-2	1	1	-1	1	1	1	0	-2	1	1	-1	1	1	1
	1	1	0	-2	0	-2	-1	1	1	1	0	-2	0	-2	-1	1
T_4	-1	1	0	-2	1	1	1	1	-1	1	0	-2	0	-2	-1	1
	0	-2	-1	1	-1	1	-1	1	0	-2	-1	1	1	1	0	-2
	1	1	1	1	0	-2	0	-2	1	1	1	1	-1	1	1	1

Table 3.1

Array \ Effect	Set-3: β_3 L	Set-3: β_3 Q	Set-3: $\beta_1\beta_2$ L	Set-3: $\beta_1\beta_2$ Q	Set-3: $\beta_2\beta_4$ L	Set-3: $\beta_2\beta_4$ Q	Set-3: $\beta_1\beta_4^2$ L	Set-3: $\beta_1\beta_4^2$ Q	Set-4: β_4 L	Set-4: β_4 Q	Set-4: $\beta_2\beta_3^2$ L	Set-4: $\beta_2\beta_3^2$ Q	Set-4: $\beta_1\beta_2^2$ L	Set-4: $\beta_1\beta_2^2$ Q	Set-4: $\beta_1\beta_3^2$ L	Set-4: $\beta_1\beta_3^2$ Q
T_1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
	0	-2	1	1	0	-2	0	-2	0	-2	1	1	1	1	0	-2
	1	1	0	-2	1	1	1	1	1	1	0	-2	0	-2	1	1
T_2	-1	1	-1	1	0	-2	1	1	-1	1	0	-2	0	-2	1	1
	0	-2	1	1	1	1	-1	1	0	-2	-1	1	-1	1	-1	1
	1	1	0	-2	-1	1	0	-2	1	1	1	1	1	1	0	-2
T_3	-1	1	-1	1	1	1	0	-2	-1	1	1	1	1	1	0	-2
	0	-2	1	1	-1	1	1	1	0	-2	0	-2	0	-2	1	1
	1	1	0	-2	0	-2	-1	1	1	1	-1	1	-1	1	-1	1
T_4	-1	1	0	-2	-1	1	0	-2	-1	1	-1	1	0	-2	0	-2
	0	-2	-1	1	0	-2	1	1	0	-2	1	1	-1	1	1	1
	1	1	1	1	1	1	-1	1	1	1	1	-2	1	1	-1	1

From the table 3.1 the normal equations for each set of effects can be worked out as below :

Let $\underline{\rho}_1' = [L(\beta_1), Q(\beta_1), L(\beta_2\beta_3), Q(\beta_2\beta_3), L(\beta_3,\beta_4), Q(\beta_3\beta_4), L(\beta_2\beta_4^2), Q(\beta_2\beta_4^2)]$,

$\underline{\rho}_2' = [L(\beta_2), Q(\beta_2), L(\beta_1\beta_3), Q(\beta_1\beta_3), L(\beta_1\beta_4), Q(\beta_1\beta_4), L(\beta_3\beta_4^2), Q(\beta_3\beta_4^2)]$,

$\underline{\rho}_3' = [L(\beta_3), Q(\beta_3), L(\beta_1\beta_2), Q(\beta_1\beta_2), L(\beta_2\beta_4), Q(\beta_2\beta_4), L(\beta_1\beta_4^2), Q(\beta_1\beta_4^2)]$,

$\underline{\rho}_4' = [L(\beta_4), Q(\beta_4), L(\beta_2\beta_3^2), Q(\beta_2\beta_3^2), L(\beta_1\beta_2^2), Q(\beta_1\beta_2^2), L(\beta_1\beta_3^2), Q(\beta_1\beta_3^2)]$,

where $\underline{\rho}_1'$, $\underline{\rho}_2'$, $\underline{\rho}_3'$, $\underline{\rho}_4'$ denote the row vectors of effects in the other three sets as indicated in the table.

Also let

$$Y[L(\beta_1\beta_2)] = [(\{Z_1+Z_2=2\} \cap \{T\}) - (\{Z_1+Z_2=0\} \cap \{T\})] ,$$

$$Y[Q(\beta_1\beta_2)] = [(\{Z_1+Z_2=0\} \cap \{T\}) - 2(\{Z_1+Z_2=1\} \cap \{T\}) + (\{Z_1+Z_2=2\} \cap \{T\})] ,$$

where $\{T\}$ denotes the set of assemblies of the fraction defined in (3.2.3) and the other notations have the same meaning as given in section 1.10.

Then

$$\left[\begin{array}{c|c} P_j & Q_j \\ \hline Q_j' & R_j \end{array}\right] \cdot \hat{\underline{\rho}}_j = Y(\underline{\rho}_j) \qquad \ldots(3.2.4)$$

$j = 1,2,3,4$

where $P_1=P_2=P_3 = \left[\begin{array}{cc|cc} 24 & 0 & 12 & -18 \\ 0 & 72 & -18 & -36 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right]$,

$$P_4 = \left[\begin{array}{cc|cc} 24 & 0 & 3 & -9 \\ 0 & 72 & -9 & -9 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right].$$

$R_1 = \left[\begin{array}{cc|cc} 24 & 0 & 6 & 0 \\ 0 & 72 & 0 & 18 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right]$, $R_2 = \left[\begin{array}{cc|cc} 24 & 0 & -3 & -9 \\ 0 & 72 & 9 & -9 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right]$,

$R_3 = \left[\begin{array}{cc|cc} 24 & 0 & -3 & 9 \\ 0 & 72 & -9 & -9 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right]$, $R_4 = \left[\begin{array}{cc|cc} 24 & 0 & 3 & -27 \\ 0 & 72 & -27 & 9 \\ \hline & & 24 & 0 \\ \text{sym.} & & 0 & 72 \end{array}\right]$.

$$Q_1 = \left[\begin{array}{cc|cc} -3 & -9 & -3 & -9 \\ 9 & -9 & 9 & -9 \\ \hline 3 & -9 & 3 & -9 \\ -9 & -9 & -9 & -9 \end{array}\right], \quad Q_2 = \left[\begin{array}{cc|cc} -3 & 9 & 6 & 0 \\ -9 & -9 & 0 & 18 \\ \hline -6 & 0 & 3 & 9 \\ 0 & 18 & 9 & -9 \end{array}\right],$$

$$Q_3 = \left[\begin{array}{cc|cc} 6 & 0 & -3 & 9 \\ 0 & 18 & -9 & -9 \\ \hline 3 & 9 & -6 & 0 \\ 9 & -9 & 0 & 18 \end{array}\right], \quad Q_4 = \left[\begin{array}{cc|cc} 3 & 9 & -3 & 9 \\ 9 & -9 & -9 & -9 \\ \hline 15 & 9 & 12 & -18 \\ -9 & 45 & -18 & -36 \end{array}\right].$$

From (3.2.4) follows

$$\hat{\underline{\rho}}_j = \left[\begin{array}{c|c} p_j & q_j \\ \hline q_j' & r_j \end{array}\right] \cdot Y(\underline{\rho}_j), \qquad j = 1,2,3,4 \qquad \ldots(3.2.5)$$

where

$$p_1 = p_2 = p_3 = \frac{1}{108}\left[\begin{array}{cc|cc} 12 & 0 & -6 & 4 \\ 0 & 4 & 4 & 2 \\ \hline & & 12 & 0 \\ \multicolumn{2}{c|}{\text{sym.}} & 0 & 4 \end{array}\right], \quad p_4 = \frac{1}{324}\left[\begin{array}{cc|cc} 18 & 0 & \multicolumn{2}{c}{0} \\ 0 & 6 & & \\ \hline & & 18 & 0 \\ \multicolumn{2}{c|}{\text{sym.}} & 0 & 6 \end{array}\right],$$

$$r_1 = r_2 = r_3 = \frac{1}{108}\left[\begin{array}{cc|cc} 6 & 0 & \multicolumn{2}{c}{0} \\ 0 & 2 & & \\ \hline & & 6 & 0 \\ \multicolumn{2}{c|}{\text{sym.}} & 0 & 2 \end{array}\right], \quad r_4 = \frac{1}{324}\left[\begin{array}{cc|cc} 48 & 0 & 3 & 21 \\ 0 & 16 & 21 & -1 \\ \hline & & 48 & 0 \\ \multicolumn{2}{c|}{\text{sym.}} & 0 & 16 \end{array}\right],$$

$$q_1 = \frac{1}{108}\left[\begin{array}{cc|cc} 3 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ \hline -3 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right], \quad q_2 = \frac{1}{108}\left[\begin{array}{cc|cc} 0 & -2 & -3 & 1 \\ 2 & 0 & -1 & -1 \\ \hline 3 & 1 & 0 & -2 \\ 1 & -1 & -2 & 0 \end{array}\right],$$

$$q_3 = \frac{1}{108}\left[\begin{array}{cc|cc} -3 & 1 & 0 & -2 \\ -1 & -1 & 2 & 0 \\ \hline 0 & -2 & 3 & 1 \\ -2 & 0 & 1 & -1 \end{array}\right], \quad q_4 = \frac{1}{324}\left[\begin{array}{cc|cc} -9 & -3 & 0 & -6 \\ -3 & 3 & 6 & 0 \\ \hline -9 & -9 & -18 & 0 \\ 9 & -3 & 0 & 6 \end{array}\right].$$

Hence, all the 33 main effects and the two factor interactions (including the grand average) of the 3^4 design are estimable from 36 assemblies of the fraction consisting of the arrays T_1, T_2, T_3, T_4. as given in (3.2.3).

Using the transformation given in (1.11.3), the above effects which are estimated in Geometric set can be transformed into corresponding effects in Product Set.

Since the block contrasts together with the effects of the 3^4 design account for all 35 d.f., nothing is left to estimate the error. Hence, error has to be obtained from previous knowledge or by having one more replication.

Example (2)

$1/3^{rd}$ Fractional Factorial of a 3^5 Design in 3 Blocks, 27 Assemblies Each.

Design

Block-1	Block-2	Block-3
$\{T_1\}$	$\{T_2\}$	$\{T_3\}$

where the arrays of strength 2 in EG(5,3) are as given below :

Array	T_1	T_2	T_3
$Z_1 + Z_2 + Z_3 =$	0	1	2
$Z_3 + Z_4 + Z_5 =$	0	2	1

In this design, the estimates of effects are orthogonal.

Example (3)

$2/9^{th}$ Fractional Factorial of 3^6 Design in 6 Blocks, 27 Assemblies Each.

This design is obtained from array of strength 2 in EG(6,3), assuming higher factor interactions to be absent.

Consider an array of strength 2 in EG(6,3) defined by the equations

2 $$2Z_1 + Z_3 + 2Z_6 = 0,$$
$$2Z_2 + Z_4 + 2Z_6 = 0, \qquad \ldots(3.2.6)$$
$$2Z_3 + Z_4 + Z_5 = 0,$$
in GF(3).

The identity relationship for the fraction of the 3^6 design given by (3.2.6) is

$$I = \beta_1^2\beta_3\beta_6^2 = \beta_2^2\beta_4\beta_6^2 = \beta_1^2\beta_4\beta_5 = \beta_1\beta_2\beta_5\beta_6^2$$
$$= \beta_3\beta_4\beta_5^2\beta_6^2 = \beta_1^2\beta_2\beta_3\beta_4^2 = \beta_2^2\beta_3\beta_5^2\beta_6$$
$$= \beta_1\beta_2\beta_3\beta_5 = B_2^2B_3^2B_4^2B_5$$

omitting interactions involving five or more factors.

Restricting to any of the main effects and the two factor interactions, the sets of effects aliased are

$$\{\beta_1, \beta_3\beta_6^2, \beta_4\beta_5\},$$
$$\{\beta_2, \beta_4\beta_6^2\},$$
$$\{\beta_3, \beta_1\beta_6\},$$
$$\{\beta_4, \beta_2\beta_6, \beta_1\beta_5^2\},$$
$$\{\beta_5, \beta_1\beta_4^2\},$$
$$\{\beta_6, \beta_1\beta_3^2, \beta_2\beta_4^2\}, \qquad \ldots(3.2.7)$$
$$\{\beta_1\beta_2, \beta_3\beta_5\},$$
$$\{\beta_1\beta_3, \beta_2\beta_5, \beta_1\beta_6^2, \beta_3\beta_6\},$$

$\{\beta_1\beta_4,\ \beta_4^2\beta_5,\ \beta_2\beta_3,\ \beta_1\beta_5\},$

$\{\beta_2\beta_4,\ \beta_4\beta_6,\ \beta_3\beta_5^2,\ \beta_2\beta_6^2\},$

$\{\beta_3\beta_4,\ \beta_2\beta_5^2\},$

$\{\beta_5\beta_6\ ,\ \beta_2\beta_3^2\},$

$\{\beta_5\beta_6,\ \beta_1\beta_2^2,\ \beta_3\beta_4^2\}.$

Let us consider, for simplicity in calculation, the estimation of the effects (3.2.7) from the assemblies of the six arrays in EG(6,3) given below :

Array	T_1	T_2	T_3	T_4	T_5	T_6	
$2Z_1 + Z_3 + 2Z_6 =$	0	1	2	0	1	2	
$2Z_2 + Z_4 + 2Z_6 =$	0	2	1	1	2	1	...(3.2.8)
$2Z_1 + Z_4 + Z_5 =$	1	0	1	1	2	2	

While estimating the effects from the assemblies of the six arrays in (3.2.8), the aliasing disappears, and the effects (3.2.7) become estimable either orthogonally or in correlated sets.

The sets of correlated effects for the fraction defined by (3.2.8) are

$$\begin{array}{ccc} \text{Set-1} & \text{Set-2} & \text{Set-3} \\ \beta_2\beta_5,\ \beta_1\beta_6^2\ , & \beta_5\beta_6,\ \beta_1\beta_2^2\ , & \beta_2\beta_6,\ \beta_1\beta_5^2\ , \\ \text{set-4} & \text{set-5} & \text{Set-6} \\ \beta_4\beta_5^2\ ,\ \beta_2\beta_3\ , & \beta_3\beta_5^2,\ \beta_2\beta_4\ , & \beta_2\beta_5^2\ ,\ \beta_3\beta_4 \end{array} \qquad \ldots(3.2.9)$$

derived from the linear forms

$$Z_1 + 2Z_2 + 2Z_5 + 2Z_6$$

$$2Z_2 + 2Z_3 + 2Z_4 + Z_5 \quad ,$$

which are equated to

$$\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

Columnwise in the six arrays defined by (3.2.8)

The coefficient of effects in the expected value of the response column vector $\underline{y}$ of assemblies belonging to the arrays in (3.2.8) are given below :

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Array \ Effect	Set-1				Set-4			
	$\beta_2\beta_5$		$\beta_1\beta_6^2$		$\beta_2\beta_5^2$		$\beta_2\beta_3$	
	L	Q	L	Q	L	Q	L	Q
T_1	-1	1	-1	1	-1	1	-1	1
	0	-2	0	-2	0	-2	1	1
	1	1	1	1	1	1	0	-2
T_2	-1	1	1	1	-1	1	1	1
	0	-2	-1	1	0	-2	0	-2
	1	1	0	-2	1	1	-1	1
T_3	-1	1	-1	1	-1	1	-1	1
	0	-2	0	-2	0	-2	1	1
	1	1	1	1	1	1	0	-2
T_4	-1	1	1	1	-1	1	1	1
	0	-2	-1	1	0	-2	0	-2
	1	1	0	-2	1	1	-1	1
T_5	-1	1	-1	1	-1	1	-1	1
	0	-2	0	-2	0	-2	1	1
	1	1	1	1	1	1	0	-2
T_6	-1	1	1	1	-1	1	1	1
	0	-2	-1	1	0	-2	0	-2
	1	1	0	-2	1	1	-1	1

In the above table :

(1) each coefficient corresponds to 9 assemblies,

(2) the coefficient vectors within the set are in order of the assemblies, but between the sets they are in different order,

(3) any two coefficient vectors coming from two different sets are orthogonal, i.e. their inner prodcut is zero,

(4) for any pair of effects in sets 1, 2, 3 the coefficient vectors are the same, but in different order of assemblies.

Similar remarks holds for the pairs of effects in sets 4, 5, 6.

Let $\underline{\xi}_j$ (j=1,2,3) denote the column vector of effects in sets 1, 2, 3 and $\underline{\eta}_j$ (j=1,2,3) denote the column vector of effects in sets 4, 5, 6.

Then the normal equations are

$$9 \times \left[\begin{array}{cc|cc} 12 & 0 & 3 & -9 \\ 0 & 36 & 9 & 9 \\ \hline & & 12 & 0 \\ \multicolumn{2}{c|}{\text{sym.}} & 0 & 36 \end{array}\right] \cdot \hat{\underline{\xi}}_j = Y(\underline{\xi}_j) \qquad \ldots(3.2.10)$$

and

$$9 \times \left[\begin{array}{cc|cc} 12 & 0 & -3 & -9 \\ 0 & 36 & -9 & 9 \\ \hline \multicolumn{2}{c|}{\text{sym.}} & 12 & 0 \\ & & 0 & 36 \end{array}\right] . \hat{\underline{\eta}}_j = Y(\underline{\eta}_j)$$

where

$$\hat{\underline{\rho}}_j = \left[\begin{array}{c|c} p_j & q_j \\ \hline q'_j & r_j \end{array}\right] . Y(\underline{\rho}_j)$$

and

$$\hat{\underline{\eta}}_j = \left[\begin{array}{c|c} p_j & q_j \\ \hline q'_j & r_j \end{array}\right] . Y(\underline{\eta}_j)$$

$(j = 1,2,3)$

i.e. $Y(\underline{\rho}_j)$, $Y(\underline{\eta}_j)$ have the same meanings as given in example (1), which give

$$\hat{\underline{\rho}}_j = \frac{1}{9 \times 108} \left[\begin{array}{cc|cc} 12 & 0 & -3 & 3 \\ 0 & 4 & -3 & -1 \\ \hline \multicolumn{2}{c|}{\text{sym.}} & 12 & 0 \\ & & 0 & 4 \end{array}\right] . Y(\underline{\rho}_j) \qquad \ldots(3.2.11)$$

$$\hat{\underline{\eta}}_j = \frac{1}{9 \times 108} \left[\begin{array}{cc|cc} 12 & 0 & 3 & 3 \\ 0 & 4 & 3 & -1 \\ \hline \multicolumn{2}{c|}{\text{sym.}} & 12 & 0 \\ & & 0 & 4 \end{array}\right] . Y(\underline{\eta}_j)$$

$(j=1,2,3)$

The rest of the effects are orthogonally estimated. The estimates can be converted into the corresponding ones in Product Set by using the transformation given in (1.11.3).

3.3 FRACTIONAL FACTORIAL OF 3^6 DESIGN FROM ARRAYS OF STRENGTH 3 IN EG (6,3).

In this section, a problem of construction and estimation of main effects and the two-factor interactions of 3^6 design from arrays of strength 3 in EG(6,3) is given.

Consider a fraction of the 3^6 design consisting of the assemblies of an array of strength 3 in EG(6,3) defined by the equations

$$Z_1 + 2Z_2 + 2Z_3 + Z_4 = 0$$
$$Z_2 + 2Z_3 + Z_5 + 2Z_6 = 0 \qquad \ldots(3.3.1)$$

The identity relationship for the fraction defined by (3.3.1) is

$$I = \beta_1\beta_2^2\beta_3^2\beta_4 = \beta_2\beta_3^2\beta_5\beta_6^2 \qquad \ldots(3.3.2)$$

interactions involving five or more factors are not considered.

The aliased sets of effects are

$$\{\beta_1\beta_4,\ \beta_2\beta_3\},$$
$$\{\beta_2\beta_5,\ \beta_3\beta_6\},$$
$$\{\beta_1\beta_2^2,\ \beta_3\beta_4^2\}, \qquad \ldots(3.3.3)$$
$$\{\beta_1\beta_3^2,\ \beta_2\beta_4^2\},$$
$$\{\beta_2\beta_3^2,\ \beta_6\beta_5^2\},$$
$$\{\beta_2\beta_6^2,\ \beta_3\beta_5^2\},$$

Only the main effects and the two-factor interactions are considered.

Consider the estimation of effects from the assemblies of the two arrays given as below :

Array	T_1	T_2
$Z_1+2Z_2+2Z_3+Z_4$ =	0	2
$Z_2+2Z_3+Z_5+2Z_6$ =	0	2

For each pair of effects in (3.3.3) taken in orders the corresponding coefficients in the expected value of the response column vector $\underline{y}$ of assemblies in T_1, T_2 are given below : such coefficients in the table corresponds to 27 assemblies.

Array \ Effect	$\beta_1\beta_4$ L	$\beta_1\beta_4$ Q	$\beta_2\beta_3$ L	$\beta_2\beta_3$ Q
T_1	-1	1	-1	1
	0	-2	0	-2
	1	1	1	1
T_2	-1	1	0	-2
	0	-2	1	1
	1	1	-1	1

The same table holds for every pair of effects in (3.3.3) taken in order. Denoting by $\underline{\rho}_j$ (j=1,2,...,6), the column vector of effects in the j^{th} pair, the normal equations for estimating $\underline{\rho}_j$ are

$$27 \left[\begin{array}{cc|cc} 4 & 0 & 1 & 3 \\ 0 & 12 & -3 & 3 \\ \hline \multicolumn{2}{c|}{\text{sym.}} & 4 & 0 \\ & & 0 & 12 \end{array}\right] \hat{\underline{\rho}}_j = Y\,(\underline{\rho}_j) \qquad \ldots(3.3.4)$$

$$j = 1,2,\ldots,6$$

From (3.3.4)

$$\hat{\underline{\rho}}_j = \frac{1}{36\text{x}27} \left[\begin{array}{cc|cc} 12 & 0 & -3 & -3 \\ 0 & 4 & 3 & -1 \\ \hline \multicolumn{2}{c|}{\text{sym.}} & 12 & 0 \\ & & 0 & 4 \end{array}\right] .Y(\underline{\rho}_j) \qquad \ldots(3.3.5)$$

$$j = 1,2,\ldots,6$$

1 The estimates in (3.3.5) may be converted back to the corresponding estimates in the Product Set by using the transformation given in (1.11.3).

The effects not occuring in (3.3.5) are orthogonally estimated.