CHAPTER - IV

FRACTIONAL FACTORIAL DESIGNS

OF THE TYPE $2^m \times 3^n$

4.1 INTRODUCTION

In earlier chapters, the procedure for estimating of main effects and the two-factor interactions of various fractional factorial designs of the type 2^m and 3^n is given. In a $2^m x 3^n$ factorial experiment when m or n or both m and n are large, a large number of assemblies create a problem. Suppose one is interested only in main effects and the two-factor interactions, then all the assemblies are not needed. Providing a reasonable margin for estimating error, methods of construction are given so that the normal equations estimating the effects break up into independent sets and render the solution easy, i.e. to construct a fraction (including the whole) of a $2^m \times 3^n$ design is to take a fraction of the 2^m complete factorial and a fraction of the 3^n complete factorial, and to adjoin every treatment combination in the fraction of the 2^m to every treatment combination in the fraction of the 3^n. To make these points clear, consider a fraction $2^2 \times 3^2$ consisting of the assemblies (X_1, X_2, Z_1, Z_2) obtained by taking the symbolic direct product of assemblies of an array given by $X_1 + X_2 = 0$ in EG(m,2) and those of an array given by $Z_1 + Z_2 = 0$ in EG(n,3).

Writing

$$S_1 = \begin{bmatrix} (0, & 0) \\ (1, & 1) \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} (0, & 0) \\ (1, & 2) \\ (2, & 1) \end{bmatrix}$$

The required assemblies are given by

$$S_1 \otimes T_1 = \begin{bmatrix} (0, & 0, & 0, & 0) \\ (0, & 0, & 1, & 2) \\ (0, & 0, & 2, & 1) \\ (1, & 1, & 0, & 0) \\ (1, & 1, & 1, & 2) \\ (1, & 1, & 2, & 1) \end{bmatrix}$$

Let $\underline{\beta}' = [\mu, A_1, A_2, A_1 A_2, L(\beta_1), Q(\beta_1), L(\beta_2), Q(\beta_2), L(\beta_1\beta_2), Q(\beta_1\beta_2), L(\beta_1\beta_2^2), Q(\beta_1\beta_2^2), L(A_1\beta_1), Q(A_1\beta_1), L(A_1\beta_2), Q(A_1\beta_2), L(A_2\beta_1), Q(A_2\beta_1), L(A_2\beta_2), Q(A_2\beta_2)]$.

Then

$$E\begin{bmatrix} Y(0,\ 0,\ 0,\ 0) \\ Y(0,\ 0,\ 1,\ 2) \\ Y(0,\ 0,\ 2,\ 1) \\ Y(1,\ 1,\ 0,\ 0) \\ Y(1,\ 1,\ 1,\ 2) \\ Y(1,\ 1,\ 2,\ 1) \end{bmatrix} =$$

$$\left[\begin{array}{c|ccc|cccccccc|ccccccccc}
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & 2 & -1 & -1 & 0 & 2 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & -1 & -1 & 0 & 2 & -1 & -1 & 0 & 2 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & +1 & -1 & +1 & -1 & +1 & -2 & 1 \\
1 & 1 & 1 & -1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & -2 & 1 & 1 & 0 & -2 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & -1 & 1 & 1 & 0 & -2
\end{array}\right] \cdot \beta$$

The identity relationship for the fraction and the other aliased sets of effects are

$$I = A_1A_2 = \beta_1\beta_2$$

$$\{A_1A_2\},\ \{\beta_1\beta_2,\ \ \beta_1\beta_2^2\},\ \{A_1\beta_1\ ,\ A_2\beta_1\},\ \{A_1\beta_2\ ,\ A_2\beta_2\},$$

assuming higher factor interactions to be absent.

Thus the six estimable functions of effects are, one each for the first two and two each for the remaining two sets of the effects given above.

The normal equations estimating effects are

$$\begin{bmatrix} 6 & -6 & -6 & 6 \\ -6 & 6 & 6 & -6 \\ -6 & 6 & 6 & -6 \\ 6 & -6 & -6 & 6 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \widehat{A_1 A_2} \\ L(\widehat{\beta_1 \beta_2}) \\ Q(\widehat{\beta_1 \beta_2}) \end{bmatrix} = \begin{bmatrix} Y(I) \\ Y(A_1 A_2) \\ Y \cdot L(\beta_1 \beta_2) \\ Y \cdot Q(\beta_1 \beta_2) \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} Y(A_1) \\ Y(A_2) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 2 & -6 & 2 & -6 \\ 0 & 12 & -6 & -6 & -6 & -6 \\ 2 & -6 & 4 & 0 & 4 & 0 \\ -6 & -6 & 0 & 12 & 0 & 12 \\ 2 & -6 & 4 & 0 & 4 & 0 \\ -6 & 6 & 0 & 12 & 0 & 12 \end{bmatrix} \begin{bmatrix} L(\hat{\beta}_1) \\ Q(\hat{\beta}_1) \\ L(\hat{\beta}_2) \\ Q(\hat{\beta}_2) \\ L(\widehat{\beta_1 \beta_2^2}) \\ Q(\widehat{\beta_1 \beta_2^2}) \end{bmatrix} = \begin{bmatrix} Y \cdot L(\beta_1) \\ Y \cdot Q(\beta_1) \\ Y \cdot L(\beta_2) \\ Y \cdot Q(\beta_2) \\ Y \cdot L(\beta_1 \beta_2^2) \\ Y \cdot Q(\beta_1 \beta_2^2) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 2 & -6 & 4 & 0 & 2 & -6 \\ 0 & 12 & -6 & -6 & 0 & 12 & -6 & -6 \\ 2 & -6 & 4 & 0 & 2 & -6 & 4 & 0 \\ -6 & -6 & 0 & 12 & -6 & -6 & 0 & 12 \\ 4 & 0 & 2 & -6 & 4 & 0 & 2 & -6 \\ 0 & -12 & -6 & -6 & 0 & 12 & -6 & -6 \\ 2 & -6 & 4 & 0 & 2 & -6 & 4 & 0 \\ -6 & -6 & 0 & 12 & -6 & -6 & 0 & 12 \end{bmatrix} \begin{bmatrix} L(\widehat{A_1 \beta_1}) \\ Q(\widehat{A_1 \beta_1}) \\ L(\widehat{A_1 \beta_2}) \\ Q(\widehat{A_1 \beta_2}) \\ L(\widehat{A_2 \beta_1}) \\ Q(\widehat{A_2 \beta_1}) \\ L(\widehat{A_2 \beta_2}) \\ Q(\widehat{A_2 \beta_2}) \end{bmatrix} = \begin{bmatrix} Y \cdot L(A_1 \beta_1) \\ Y \cdot Q(A_1 \beta_1) \\ Y \cdot L(A_1 \beta_2) \\ Y \cdot Q(A_1 \beta_2) \\ Y \cdot L(A_2 \beta_1) \\ Y \cdot Q(A_2 \beta_1) \\ Y \cdot L(A_2 \beta_2) \\ Y \cdot Q(A_2 \beta_2) \end{bmatrix}$$

which imply the estimability of

$$I-A_1A_2 - L(\beta_1\beta_2) + Q(\beta_1\beta_2),$$

$$A_1+A_2,$$

$$4L(\beta_1) + 2L(\beta_2) - 6Q(\beta_2) + 2L(\beta_1\beta_2^2) - 6Q(\beta_1\beta_2^2),$$

$$12Q(\beta_1) - 6L(\beta_2) - 6Q(\beta_2) - 6L(\beta_1\beta_2^2) - 6Q(\beta_1\beta_2^2),$$

$$4L(A_1\beta_1)+2L(A_1\beta_2)-6Q(A_1\beta_2)+4L(A_2\beta_1)+2L(A_2\beta_2)-6Q(A_2\beta_2),$$

$$12Q(A_1\beta_1)-6L(A_1\beta_2)- 6Q(A_1\beta_2)+12Q(A_2\beta_1)-6L(A_2\beta_2)-6Q(A_2\beta_2).$$

This is due to Bose and Connor [10] .

Thus the 6 assemblies of 2^2x3^2 factorial design, taken as above estimate the six linear functions of effects.

4.2 CONSTRUCTION

By combining the arrays $S_1, S_2, \ldots, S_t$, each of the same strength in EG(m,2) with the arrays $T_1, T_2, \ldots, T_t$ of some suitable strength in EG(n,3) by the method of symbolic direct product, designs are constructed. The resulting fractionally factorial design may be denoted as

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ S_t & \otimes & T_t \end{bmatrix}$$

where columns stand for the factors and rows represent the assemblies.

The assemblies may be further divided into blocks by regarding one factor or two factor of the 2^m factorial or the 3^n factorial or one factor of each factorial as block factors.

Let $\underline{A}_i$ denote the column vector of coefficients corresponding to the main effect A_i of the 2^m design and $\underline{B}_j^{\mu}$, the column vector of coefficients corresponding to the main effect B_j^{μ} (μ=1,2) of the 3^n design. B_j^1 will refer to the linear component and B_j^2, to the quadratic component of the main effect of B_j. For convenience B_j^1 will be written as B_j(j=1,2,...,n).

For the two factor interaction $A_iB_j^{\mu}$, the column vector of coefficients in C is obtained by multiplying the corresponding coefficients in $\underline{A}_i$ and $\underline{B}_j^{\mu}$ and similarly for any other pure or mixed two factor interaction. [We consider the model $\underline{y} = C\underline{\mu} + \underline{\varepsilon}$, the column vectors of C correspond to the effect in $\underline{\mu}$. They will also be referred to as the column vector of the coefficients].

In what follows $(\underline{A}_1\underline{A}_2 \ldots \underline{A}_r)$ will mean the r-product of the column vectors. $\underline{A}_1\underline{A}_2 \ldots \underline{A}_r$ for $r \geqslant 2$ and we shall say that the effect is orthogonally estimable if the inner product of its column vector and any other column in C corresponding to any other effect in $\underline{\mu}$ is zero.

The estimates of two effects will be called orthogonal if the inner product of the corresponding column vectors in C is zero. If not, they will be said to be correlated. For example $(\underline{A}_1\underline{A}_2\ \underline{B}_1\underline{B}_2^2) = 0$ will imply that the estimates of (1) A_1A_2 and $B_1B_2^2$ or (2) A_1B_1 and $A_2B_2^2$ or (3) $A_1B_2^2$ and A_2B_1 are orthogonal. If $(\underline{A}_1 \cdot \underline{A}_2 \cdot \underline{B}_1 \cdot \underline{B}_2^2) \neq 0$, they are all correlated.

The estimate of $\underline{\mu}$ is obtained from the normal equations as

$$\hat{\underline{\mu}} = (C'C)^{-1}\ C'\underline{y}$$

provided C'C is non-singular.

The methods of construction in this thesis are based on choosing the assemblies in such a way that the matrix C'C can be arranged into a block diagonal matrix, thus, dividing the effects into sets and estimating them separately.

As soon as $(C'C)^{-1}$ is available, the effects are estimable and the variance-covariance matrix of their estimates can be obtained. We shall say that the two set of effects are estimated by the same matrix, if the matrices in the normal equations estimating them are the same.

It can be seen that if $\underline{y}$ corresponds to the assemblies of an array of strength 4 in EG(m,2)

$$(\underline{I}\cdot\underline{A}_{i_1}) = 0,$$
$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}) = 0,$$
$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{A}_{i_3}) = 0,$$
$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{A}_{i_3}\cdot\underline{A}_{i_4}) = 0$$

for $i_1 \neq i_2 \neq i_3 \neq i_4 \neq 1, 2, \ldots, m$ which imply that all main effects and the two factor interactions (including the grand average I) of the 2^m design are estimable orthogonally. Similar remarks hold for an orthogonal array of strength 4 in EG(m,3).

The main effects and the two factor interactions will be sometimes referred to as the effect and the mixed two-factor interactions as the mixed effects.

An array of strength 4 in EG(m,2) or EG(n,3) can always be replaced by the corresponding complete factorial when m, n are small.

Theorem 4.2

Suppose the p linearly independent forms

$L_r = a_{r1} X_1 + a_{r2} X_2 + \ldots + a_{rm} X_m (r=1,2,\ldots,p)$ generate a class Ω_2 of arrays, each of strength 2 in EG(m,2). There are 2^p arrays in this class. Let $S_1, S_2, \ldots, S_{p+1}$ be (p+1) of these arrays which correspond to

$$\begin{array}{lllll} L_1 = 0 & 1 & 0 & \ldots & 0 \\ L_2 = 0 & 0 & 1 & \ldots & 0 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ L_p = 0 & 0 & 0 & \ldots & 1 \end{array}$$

Also let q linearly independent forms

$$M_\theta \equiv b_{\theta 1} Y_1 + b_\theta Y_2 + \ldots + b_{\theta n} Y_n . (\theta = 1,2,\ldots,q)$$

generate a class of $\emptyset_1$ of arrays, each of strentgh 1 in EG(n,3). There are 3^q arrays in this class. Let $T_1, T_2, \ldots, T_k$ be k arrays of $\emptyset_1$, which jointly form an array of strength 4. There are three cases

(1) $p+1 < k$ (2) $p+1 > k$ (3) $p+1 = k$.

Case (1) : If $p+1 < k$, take $[k-(p+1)]$ additional arrays of Ω_2, say $S_{p+2}, S_{p+3}, \ldots, S_k$ (not necessarily all different) and form a design

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ S_k & \otimes & T_k \end{bmatrix}$$

Case (2) : If $p+1 > k$, take $(p+1-k)$ additional arrays of $\emptyset_1$, say T_{k+1}, T_{k+2}, ..., T_{p+1} (not necessarily all different) and form a design

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ S_{p+1} & \otimes & T_{p+1} \end{bmatrix}$$

Case (3) : If $p+1 = k$, no adjustment is necessary and the design is

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ S_k & \otimes & T_k \end{bmatrix}$$

Then in the fractional replicates of the $2^m \times 3^n$ design so constructed (a) all main effects and the two-factor interactions are estimable (assuming interactions involving three or more factors to be negligible) (b) of the three sets of effects (i) the main effects and the two-factor interactions of the 2^m design (ii) the main effects and the two factor interactions of the 3^n design (iii) the mixed two-factor interactions, the estimates of any two effects belonging to two different sets are uncorrelated.

Proof : (1) Since the fractional replicates are obtained by combining the arrays of strength 2 in EG(m,2) with the arrays of strength 1 in EG(n,3) by symbolic direct product, it follows that every assembly of the fraction of the 2^m factorial will occur an equal number of times with each level of every factor of the 3^n factorial and every assembly of the fraction of the 3^n factorial will occur an equal number of times with each level of every factor or with each combination of levels of every pair of factors of the 2^m factorial. Hence, we have

$$(\underline{A}_{i1}\underline{A}_{i2}\underline{A}_{i3}\ \underline{B}_{j1}^{\mu_1}) = 0,$$

$$(\underline{A}_{i_1}\underline{B}_{j_1}^{\mu_1}\underline{B}_{j_2}^{\mu_2}\ \underline{B}_{j_3}^{\mu_3}) = 0,$$

$$(\underline{A}_{i_1}\ \underline{B}_{j_1}^{\mu_1}\ \underline{B}_{j_1}^{\mu_2}\ \underline{B}_{j_1}^{\mu_3}) = 0,$$

$$(\underline{A}_{i_1}\ \underline{A}_{i_2}\ \underline{B}_{j_1}^{\mu_1}\ \underline{B}_{j_2}^{\mu_2}) = 0, \qquad \ldots(4.2.1)$$

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{B}_{j_1}^{\mu_1}) = 0,$$

$$(\underline{A}_{i_1}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_2}^{\mu_2}) = 0,$$

$$(\underline{A}_{i_1}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_1}^{\mu_1}) = 0,$$

$$(\underline{A}_{i_1},\ \underline{B}_{j_1}^{\mu_1}) = 0,$$

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_1}\cdot\underline{B}_{j_1}^{\mu_1}) = 0$$

[$\mu_1, \mu_2, \mu_3 = 1,2,;\ i_1 \neq i_2 \neq i_3 = 1,2,\ldots,m;\ j_1 \neq j_2 \neq j_3 = 1,2,\ldots,n$ which imply (b)].

(2) The estimability of the effects of the 2^m design follows from (b) and theorem 2.5 of Chapter II while that of the effects of the 3^n design follows from (b) and the fact that the number of arrays in EG(n,3) jointly form an array of strength 4 and in case (3) even exceeds that number.

3 (3) In case (2), besides (4.2.1) being true,

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_2}^{\mu_2}) = 0 \text{ but } (\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_2}^{\mu_2}) \text{ is not}$$

necessarily zero which implies that the estimates of mixed two-factor interactions will be correlated into m sets

$$\left\{A_{i_1}B_1,\ A_{i_1}B_1^2;\ A_{i_1}B_2;\ A_{i_1}B_2^2;\ \ldots;\ A_{i_1}B_n, A_{i_1}B_n^2\right\},$$

$$(i_1 = 1,2,\ldots,m).$$

Each of the above sets of effects is estimated by the same matrix that estimates the set

$$\left\{B_1,\ B_1^2;\ B_2,\ B_2^2;\ \ldots;\ B_n, B_n^2\right\}$$

which is estimable as in (2). Hence, all mixed two factor interactions are estimable in this case.

In cases (1) and (3), in addition to (4.2.1)

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_1}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_2}^{\mu_2}) = 0$$

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_2}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_1}^{\mu_2}) = 0$$

$$(\underline{A}_{i_1}\cdot\underline{A}_{i_1}\cdot\underline{B}_{j_1}^{\mu_1}\cdot\underline{B}_{j_1}^{\mu_2}) = 0 \qquad (\mu_1 \neq \mu_2)$$

$(\mu_1, \mu_2 = 1,2;\ i_1 \neq i_2 = 1,2,\ldots,m;\ j_1 \neq j_2 = 1,2,\ldots,n)$

Since the arrays $T_1, T_2, \ldots, T_k$ form an array of strength 4.

4 Hence, all mixed two-factor interactions and the effects of the 3^n design are estimated orthogonally, (2) and (3) jointly imply (a). This completes the proof.

It is to be noted that the (p+1) array $S_1, S_2, \ldots, S_{p+1}$ in EG(m,2) as given in theorem 2.5 are sufficient to estimate the main effects and the two-factor interactions of the 2^m design (assuming higher factor interactions to be negligible), but in certain cases, the estimation may be possible by using a lesser number of arrays.

Also the number of assemblies of the fractional replicates given in the above theorem can be further reduced by replacing the k arrays $T_1, T_2, \ldots, T_k$ which together form an array of strength 4 in EG(n,3) by the arrays of $T_1, T_2, \ldots, T_{k'}'$, $(k' < k)$ which jointly form an array, sufficient to estimate the main effects and the two factor interactions of the 3^n factorial.

<u>Examples</u>

(1) $1/8^{th}$ fractional replicate of $2^7 \times 3^2$

(2) $1/16^{th}$ fractional replicate of $2^7 \times 3^3$ design

(3) $5/48^{th}$ fractional replicate of $2^7 \times 3^2$ design

(4) $1/16^{th}$ fractional replicate of $2^9 \times 3^2$ design.

Example (1)

$1/8^{th}$ fractional replicate of $2^7 x 3^2$ design

(65 effects are estimated from 144 assemblies)

(a) Generating functions and arrays in EG (7,2)

Array	S_1	S_2	S_3	S_4	S_5	S_6
$X_1 + X_4 + X_5 =$	0	1	0	0	0	1
$X_2 + X_3 + X_4 =$	0	0	1	0	0	1
$X_1 + X_2 + X_6 =$	0	0	0	1	0	1
$X_1 + X_3 + X_7 =$	0	0	0	0	1	1

(b) Generating functions and arrays in EG(2,3)

Array	T_1	T_2	T_3
$Z_1 + Z_2 =$	0	1	2

The design consists of assemblies given by the mixed array

$$\begin{bmatrix} S_1 \otimes T_1 \\ S_2 \otimes T_2 \\ S_3 \otimes T_3 \\ S_4 \otimes T_1 \\ S_5 \otimes T_2 \\ S_6 \otimes T_3 \end{bmatrix}$$

Estimation of Effects

The following groups of effects are correlated :

(1) $\{A_1,\ A_4A_5,\ A_2A_6,\ A_3A_7\}$,

(2) $\{A_2,\ A_3A_4,\ A_1A_6,\ A_5A_7\}$,

$\{A_3,\ A_2A_4,\ A_1A_7,\ A_5A_6\}$,

$\{A_4,\ A_2A_3,\ A_1A_5,\ A_6A_7\}$,

(3) $\{A_5,\ A_1A_4,\ A_3A_6,\ A_2A_7\}$,

$\{A_6,\ A_1A_2,\ A_3A_5,\ A_4A_7\}$,

$\{A_7,\ A_1A_3,\ A_4A_6,\ A_2A_5\}$,

(1*) Effects in group (1) are estimated by the matrix

$$\frac{1}{48\text{x}12}\begin{bmatrix} 5 & 1 & 1 & 1 \\ & 5 & -1 & -1 \\ & & 5 & -1 \\ \text{sym.} & & & 5 \end{bmatrix}$$

(2*) Effects in group (2) are estimated by the matrix.

$$\frac{1}{48\text{x}8}\begin{bmatrix} 4 & 2 & 0 & -2 \\ & 5 & -2 & -3 \\ & & 4 & 2 \\ \text{sym.} & & & 5 \end{bmatrix}$$

(3*) Effects in group (3) are estimated by the matrix

$$\frac{1}{48 \times 8} \begin{bmatrix} 5 & 3 & -2 & -2 \\ & 5 & -2 & -2 \\ & & 4 & 0 \\ \text{sym.} & & & 4 \end{bmatrix}$$

the rest of the effects are orthogonally estimated.

Example (2)

1/6th fractional replicate of $2^7 \times 3^3$ design

(89 effects are estimated from 216 assemblies)

(a) Generating functions and arrays in EG(7,2)

Array	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
$X_1 + X_4 + X_5 =$	1	0	0	1	1	0	0	1	0
$X_2 + X_3 + X_4 =$	0	0	0	0	1	1	1	1	0
$X_1 + X_2 + X_6 =$	0	1	0	1	0	1	0	1	0
$X_1 + X_3 + X_7 =$	0	0	1	1	1	0	1	1	0

(b) Generating functions and arrays in EG(3,3)

Array	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
$Z_1 + Z_2 =$	0	0	0	1	1	1	2	2	2
$Z_2 + Z_3 =$	0	1	2	0	1	2	0	1	2

The design then consists of assemblies given by the mixed array

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ S_9 & \otimes & T_9 \end{bmatrix}$$

Estimation of effects

The following group of effects are correlated :

$\{A_1, A_4A_5, A_2A_6, A_3A_7\}$,

$\{A_2, A_3A_4, A_1A_6, A_5A_7\}$,

$\{A_3, A_2A_4, A_1A_7, A_5A_6\}$,

$\{A_4, A_2A_3, A_1A_5, A_6A_7\}$,

$\{A_5, A_1A_4, A_3A_6, A_2A_7\}$,

$\{A_6, A_1A_2, A_3A_5, A_4A_7\}$,

$\{A_7, A_1A_3, A_4A_6, A_2A_5\}$.

Each set is estimated by the matrix

$$\frac{1}{96\text{x}24} \begin{bmatrix} 11 & 1 & 1 & 1 \\ & 11 & -1 & -1 \\ & & 11 & -1 \\ \text{sym.} & & & 11 \end{bmatrix}$$

The rest of the effects are orthogonally estimated.

Example (3)

$5/48^{th}$ fractional replicate of $2^7 x 3^2$ design

(65 effects are estimated from 120 assemblies)

(a) Generating functions and arrays in EG(7,2)

Array	S_1	S_2	S_3	S_4	S_5
$X_1 + X_4 + X_5 =$	0	1	0	0	0
$X_2 + X_3 + X_4 =$	0	0	1	0	0
$X_1 + X_2 + X_6 =$	0	0	0	1	0
$X_1 + X_3 + X_7 =$	0	0	0	0	1

(b) Generating functions and array in EG(2,3)

Array	T_1	T_2	T_3
$Z_1 + Z_2 + Z_3 =$	0	1	2

The design then consists of assemblies given by the mixed array

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ S_3 & \otimes & T_3 \\ S_4 & \otimes & T_1 \\ S_5 & \otimes & T_2 \end{bmatrix}$$

Estimation of Effects

The following groups of effects are correlated :

(1) $\{A_1,\ A_4A_5,\ A_2A_6,\ A_3A_7\}$,

(2) $\{A_2,\ A_3A_4,\ A_1A_6,\ A_5A_7\}$,

$\{A_3,\ A_2A_4,\ A_1A_7,\ A_5A_6\}$,

$\{A_4,\ A_2A_3,\ A_1A_5,\ A_6A_7\}$,

(3) $\{A_5,\ A_1A_4,\ A_3A_6,\ A_2A_7\}$,

$\{A_6,\ A_1A_2,\ A_3A_5,\ A_4A_7\}$,

$\{A_7,\ A_1A_3,\ A_4A_6,\ A_2A_5\}$,

(4) $\{A_1B_1,\ A_1B_2\}$,

$\{A_2B_1,\ A_2B_2\}$,

. . . . . .

. . . . . .

$\{A_7B_1,\ A_7B_2\}$,

(5) $\{A_1B_1^2,\ A_1B_2^2\}$,

$\{A_2B_1^2,\ A_2B_2^2\}$,

. . . . . .

$\{A_7B_1^2,\ A_7B_2^2\}$,

(6) $\{I,\ B_1B_2,\ B_1^2B_2^2,\ B_1^2B_2^2\}$,

(7) $\{B_1,\ B_2,\ B_1B_2^2,\ B_1^2B_2\}$.

(1*) Effects in group (1) are estimated by the matrix

$$\frac{1}{128}\begin{bmatrix} 14 & 6 & 6 & 6 \\ & 6 & 2 & 2 \\ & & 6 & 2 \\ \text{sym.} & & & 6 \end{bmatrix}$$

(2*) Effects in group (2) are estimated by the matrix.

$$\frac{1}{64}\begin{bmatrix} 4 & 2 & 2 & 0 \\ & 3 & 0 & -1 \\ & & 4 & 2 \\ \text{sym.} & & & 3 \end{bmatrix}$$

(3*) Effects in group (3) is estimated by the matrix

$$\frac{1}{128}\begin{bmatrix} 7 & 5 & -2 & -2 \\ & 7 & -2 & -2 \\ & & 4 & 0 \\ \text{sym.} & & & 4 \end{bmatrix}$$

(4*) Effects in group (4) is estimated by the matrix

$$\frac{1}{16x24}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

(5*) Effects in group (5) is estimated by the matrix

$$\frac{1}{48x24}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

(6*) Effects in group (6) is estimated by the matrix

$$\begin{bmatrix} 120 & 16 & -48 & 0 & 0 \\ & 48 & 16 & 16 & 16 \\ & & 432 & 48 & 48 \\ & & & 240 & -48 \\ \text{sym.} & & & & 240 \end{bmatrix}^{-1}$$

i.e.

$$\frac{1}{8x432}\begin{bmatrix} 32 & -12 & 4 & 0 & 0 \\ & 81 & -3 & -6 & -6 \\ & & 9 & -2 & -2 \\ & & & 16 & 4 \\ \text{sym.} & & & & 16 \end{bmatrix}$$

(7*) Effects in group (7) is estimated by the matrix

$$\begin{bmatrix} 80 & 16 & -16 & 16 \\ & 80 & 16 & -16 \\ & & 176 & 16 \\ \text{sym.} & & & 176 \end{bmatrix}^{-1}$$

i.e.

$$\begin{bmatrix} 16 & -4 & 2 & -2 \\ & 16 & -2 & 2 \\ & & 7 & -1 \\ \text{sym.} & & & 7 \end{bmatrix}$$

Example (4)

1/16th Fractional Replicate of $2^9 \times 3^2$ design

(90 effects are estimated from 288 assemblies)

(a) Generating functions and arrays in EG(9,2)

Array	S_1	S_2	S_3
$X_1 + X_2 + X_3 + X_8 =$	1	0	0
$X_1 + X_4 + X_7 + X_9 =$	0	1	0
$X_2 + X_3 + X_6 + X_9 =$	0	0	1
$X_2 + X_3 + X_4 + X_5 =$	0	1	0

(b) Generating functions and arrays in EG(2,3)

Array	T_1	T_2	T_3
$Z_1 + Z_2 =$	0	1	2

The design then consists of assemblies given by the mixed array

$$\begin{bmatrix} S_1 & \otimes & T_1 \\ S_2 & \otimes & T_2 \\ S_3 & \otimes & T_3 \end{bmatrix}$$

Estimation of Effects

The following groups of effects are correlated :

4 (1) $\{A_1A_2,\ A_3A_8\}$,

$\{A_1A_6,\ A_3A_7\}$,

$\{A_1A_9,\ A_4A_7\}$,

$\{A_2A_4,\ A_3A_5\}$,

$\{A_2A_9,\ A_5A_6\}$,

$\{A_5A_7,\ A_8A_9\}$

(2) $\{A_1A_5,\ A_4A_8\}$,

$\{A_2A_7,\ A_6A_8\}$,

$\{A_3A_9,\ A_4A_6\}$,

(3) $\{A_1A_3,\ A_2A_8,\ A_6A_7\}$,

$\{A_1A_7,\ A_3A_6,\ A_4A_9\}$,

$\{A_2A_5,\ A_3A_4,\ A_6A_9\}$,

$\{A_7A_9,\ A_5A_8,\ A_1A_4\}$,

$\{A_2A_3,\ A_1A_8,\ A_4A_5\}$,

$\{A_5A_9,\ A_2A_6,\ A_7A_8\}$,

(1*) Each set of effects in (1) is estimated by the matrix

$$\frac{1}{32\text{x}24}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

(2*) Each set of effects in (2) is estimated by the matrix.

$$\frac{1}{32 \times 24} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(3*) Each set of effects in (3) is estimated by the matrix.

$$\frac{1}{128 \times 3} \begin{bmatrix} 2 & -1 & -1 \\ & 2 & -1 \\ \text{sym.} & & 2 \end{bmatrix}$$