CHAPTER V :

SOME RESULTS ON THE MINIMAL COMPLETE CLASS OF LINEAR ESTIMATORS

5.0 SUMMARY

Let $\mathcal{C}$ denote the class of all homogeneous, linear, unbiased estimators of the mean of a finite population, which do not take into account the order or the number of repetitions of a unit. In this chapter we discuss a sufficient condition under which $\mathcal{C}$ is not minimal complete. We also give in this chapter a complete description of the minimal complete subclass of $\mathcal{C}$ in the special case of taking a simple random sample of size 2 from a population of size 3.

5.1 INTRODUCTION

Consider the problem of estimating the mean of a finite population. As noted in Chapter I it is known that the class L of unbiased linear estimators contains a best estimator in the sense of minimum variance, if and only if, the design (S,P) is a unicluster. Roy-Chakravarti [17] have proved that the subclass $\mathcal{C}$ of L consisting of those estimators which do not take into consideration either the order or the number of repetitions of a unit is complete in L. Godambe and Joshi [4] and Dharmadhikari [1] gave examples of inadmissible estimators in $\mathcal{C}$. Thus, in general, $\mathcal{C}$ is not minimal complete. In the next section we obtain a sufficient condition on the design under which $\mathcal{C}$ is not minimal complete. Also examples are given in the next section to show that this sufficient condition is not necessary and to show that $\mathcal{C}$ may be minimal complete even if the design is non-unicluster. In the last section we give a complete description of the minimal complete class for the artificial special case when one takes a simple random sample of size 2 without replacement from a population of size 3. In this special case, the set $\mathcal{C}$ can be indexed by points $\alpha \in R^3$. We show that both the admissible and inadmissible estimators in $\mathcal{C}$ lead to α-sets of infinite Lebesgue measure. Further, in a certain sense, the inadmissible estimators vastly out-number the admissible ones. Thus while, the concept of admissibility does not lead to a unique choice, it does weed-out a large- sub-class of estimators.

5.2 A SUFFICIENT CONDITION UNDER WHICH THE ROY-CHAKRAVARTI CLASS IS NOT MINIMAL COMPLETE

For a design (S,P), a homogeneous linear estimator of the population mean has the form given in (1.2.6),

i.e. $$t(s,\underset{\sim}{Y}) = \sum_{i \in s} b(s,i)\, Y_i . \qquad \ldots(5.2.1)$$

The conditions for $t(s,\underset{\sim}{Y})$ of (5.2.1) to be unbiased for $\overline{Y}$, the population mean, are

$$\sum_{s \ni i} b(s,i)\, p(s) = N^{-1}, \quad i=1,2,\ \ldots,\ N. \qquad \ldots(5.2.2)$$

Note that in our set-up all samples which consist of the same set of distinct units are treated as equivalent and hence L and $\mathcal{C}$, mentioned in section 5.1, coincide. We now prove the theorem.

<u>Theorem</u> 5.2.1 : <u>If</u> $\sum_{s \in S} n(s) > \frac{N(N+1)}{2}$, <u>then</u> L <u>is not minimal complete</u>.

<u>Proof</u> : Let T denote the Horvitz-Thompson estimator. That is,

$$T(s,\underset{\sim}{Y}) = \sum_{i \in s} \left[Y_i / N\, \pi_i\right], \text{ where } \pi_i = \sum_{s \ni i} p(s).$$

We want to construct a linear estimator $T_1(s,\underset{\sim}{Y}) = \sum_{i \in s} c(s,i) Y_i$ such that T_1 is unbiased for the zero function and

$cov\ (T_1, T) \equiv 0$ for all values of $\underset{\sim}{Y}$.

Since $T_1(s, \underset{\sim}{Y})$ is unbiased for the zero function

$$\sum_{s \ni i} c(s,i)\ p(s) = 0 \qquad i=1,2,\ \ldots, N. \qquad \ldots(5.2.3)$$

The condition $Cov\ (T_1, T) \equiv 0$ gives

$$(N\ \pi_i)^{-1} \sum_{s \ni i} c(s,i)\ p(s) = 0 \quad i=1,2,\ \ldots,\ N \quad \ldots(5.2.4)$$

and

$$(N\ \pi_i)^{-1} \sum_{s \supset \{i,j\}} c(s,j)\ p(s) + (N\ \pi_j)^{-1} \sum_{s \supset \{i,j\}} c(s,i).p(s) = 0$$

$$i,j = 1,2,\ \ldots\ N \text{ and } i \neq j. \qquad \ldots(5.2.5)$$

Conditions (5.2.3) and (5.2.4) are equivalent. Hence the total number of linear equations in (5.2.3) and (5.2.5) are

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}.$$

Therefore the maximum number of linearly independent equations is $\frac{N(N+1)}{2}$. The number of unknown constants $c(s,i)$ equals $\sum_{s \in S} n(s)$.

Thus we have a non-zero solution as soon as

$$\sum_{s \in S} n(s) > \frac{N(N+1)}{2}. \qquad \ldots(5.2.6)$$

If (5.2.6) holds, then it follows from the lemma in Dharmadhikari [1] that T dominates $T + T_1$ and hence L is not minimal complete. The steps (5.2.4), (5.2.5) and (5.2.6) in the proof of theorem 5.2.1 are contained in section (5.3) of Ramkrishnan[18].

<u>Example</u> : Consider a situation where one takes simple random samples of size 2 without replacement from a population of size 3. Here $\frac{N(N+1)}{2} = 6 = \sum_{s \in S} n(s)$. Thus the condition (5.2.6) of the theorem 5.2.1 just fails. However, as shown by Dharmadhikari[1] , L does contain inadmissible estimators. Hence condition (5.2.6) is not necessary.

<u>Example</u> : Let N = 2 and suppose that the only samples of positive probability (each equal to 1/2) are $s_1 = \{1\}$ and $s_2 = \{1,2\}$. Then $T \in L$ is of the form

$$T(s_1,\underset{\sim}{Y}) = \alpha Y_1 ; \; T(s_2,\underset{\sim}{Y}) = (1-\alpha)\, Y_1+Y_2 ; \; \alpha \in R.$$

For any $\alpha \in R$, the resulting estimator is admissible, because it is the only estimator which has zero variance at all points (Y_1,Y_2) on the line $(2\alpha - 1)\, Y_1 = Y_2$. Thus L is minimal complete while $\sum_{s \in S} n(s) = 3 = \frac{N(N+1)}{2}$.

It is clear from the two examples above that (5.2.6) cannot be improved in general.

5.3 IDENTIFICATION OF THE MINIMAL COMPLETE CLASS IN A SPECIAL CASE

Consider the extreme special case of a simple random sample of size 2 drawn without replacement from a population of size 3. Here we have three samples $s_1 = \{1,2\}$, $s_2 = \{2,3\}$ and $s_3 = \{3,1\}$ each having the same probability 1/3. Here L can be indexed by points in R^3. The estimator T_α corresponding to $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in R^3$ is defined by

$$T_\alpha (s_i, \underset{\sim}{Y}) = \alpha_i Y_i + (1-\alpha_{i+1}) Y_{i+1} , \quad i=1,2,3,$$

where i+1 is interpreted as 1 when i=3. This convention regarding the subscript (i+1) will be followed throughout this section without mention. We now prove the following lemma :

Lemma 5.3.1 : Every $T_\alpha \in L$ reaches zero variance on some line l in R^3 passing through origin and not coincident with any coordinate axis. Conversely, given any line l which passes through the origin and which does not coincide with any coordinate axis, there is a uniquely determined class $\{T(c), c \in R\}$ of estimators in L such that every T(c) reaches zero variance on l.

Proof : (a) Let $T_\alpha \varepsilon L$. For T_α to have zero variance at

$\underset{\sim}{Y} = (Y_1, Y_2, Y_3)$, we must have

$$T_\alpha(s_1, \underset{\sim}{Y}) = T_\alpha(s_2, \underset{\sim}{Y}) = T_\alpha(s_3, \underset{\sim}{Y}) = \bar{Y}.$$

The equations $T_\alpha(s_1, \underset{\sim}{Y}) = T_\alpha(s_3, \underset{\sim}{Y})$, $T_\alpha(s_2, \underset{\sim}{Y}) = T_\alpha(s_1, \underset{\sim}{Y})$

and $T_\alpha(s_2, \underset{\sim}{Y}) = T_\alpha(s_3, \underset{\sim}{Y})$

give

$$(2\alpha_1 - 1)Y_1 + (1-\alpha_2)\,Y_2 - \alpha_3 Y_3 = 0,$$

$$-\alpha_1 Y_1 + (2\alpha_2 - 1)\,Y_2 + (1-\alpha_3)Y_3 = 0,$$

$$(1-\alpha_1)Y_1 - \alpha_2 Y_2 + (2\alpha_3 - 1)Y_3 = 0, \qquad \ldots(5.3.1)$$

respectively.

The equations (5.3.1) are linearly dependent. Therefore there is a non-zero solution and the solution space has dimension 1 or 2. If the solution space has dimension 2, clearly it will contain a line l which passes through the origin and does not coincide with any coordinate axis. If the solution space has dimension 1, then it coincides with a line l passing through the origin. But then l cannot be a coordinate axis. For if $(0, 0, Y_3)$ is a solution of (5.3.1) with $Y_3 \neq 0$, then we get the contradictory results

$$\alpha_3 = 0, \alpha_3 = 1 \text{ and } \alpha_3 = 1/2.$$

This proves the first part of the lemma. Conversely, let l be the line determined by the origin and a point $\underset{\sim}{Y}$ with $Y_1 \neq 0$, and $Y_2 \neq 0$. Then an estimator T_α attains zero variance on l if, and only if, it attains zero variance at $\underset{\sim}{Y}$. Thus for having zero variance on l we rewrite (5.3.1) as follows :

$$
\begin{aligned}
2Y_1\alpha_1 - Y_2\alpha_2 - Y_3\alpha_3 &= Y_1 - Y_2, \\
-Y_1\alpha_1 + 2Y_2\alpha_2 - Y_3\alpha_3 &= Y_2 - Y_3, \\
-Y_1\alpha_1 - Y_2\alpha_2 + 2Y_3\alpha_3 &= Y_3 - Y_1. \qquad \ldots(5.3.2)
\end{aligned}
$$

The non-homogeneous system (5.3.2) is consistent. A solution of homogeneous system corresponding to (5.3.2) is $Y_i\alpha_i = c$, i = 1,2,3 where $c \in R$. A particular solution of (5.3.2) obtained by using the constraint

$$Y_1\alpha_1 + Y_2\alpha_2 + Y_3\alpha_3 = 0, \text{ is}$$

$$Y_i\alpha_i = (1/3)(Y_i - Y_{i+1}), \quad i=1,2,3.$$

Hence, if $Y_3 \neq 0$, a solution of (5.3.2) is

$$Y_i\alpha_i = c + \frac{1}{3}(Y_i - Y_{i+1}), \quad i=1,2,3 \qquad \ldots(5.3.3)$$

where $c \in R$ is arbitrary.

If $Y_3 = 0$, then α_3 is arbitrary and α_1, α_2 are uniquely determined. In any case, we get a family of estimators indexed by the real numbers. This proves the lemma.

If an estimator T reaches zero variance on a line l, then any estimator T' which dominates T must also have zero variance on l. Thus, to determine admissibility, one can consider subclasses of estimators reaching zero variance on different lines. The above lemma enables us to split L into two types of subclasses.

(i) Subclass I consists of those estimators which reach zero variance at points $\underset{\sim}{Y}$ such that exactly one $Y_i=0$.

(ii) Subclass II consists of those estimators which reach zero variance at points $\underset{\sim}{Y}$ such that no Y_i vanishes.

We will now consider each subclass separately and identify the admissible estimators.

Theorem 5.3.1 : (i) An estimator T_α attains zero variance at some point $(0, a_2, a_3)$ with $a_2 \neq 0$ and $a_3 \neq 0$ if, and only if, (α_2, α_3) lies on the rectangular hyperbola

$$(3\alpha_2 - 2)(3\alpha_3 - 1) + 1 = 0. \qquad \ldots(5.3.4)$$

(ii) <u>Let</u> (α_2, α_3) <u>satisfy</u> (5.3.4). <u>If</u> $(\alpha_2, \alpha_3) = (1,0)$, <u>then</u> T_α <u>is admissible if, and only if,</u> $\alpha_1 = 1/2$. <u>If</u> $(\alpha_2, \alpha_3) \neq (1,0)$, <u>then</u> T_α <u>is admissible for every</u> $\alpha_1 \in R$.

<u>Proof</u> : Suppose that the estimator T_α attains zero variance at $(0, a_2, a_3)$ with $a_2 \neq 0$ and $a_3 \neq 0$. Then (5.3.1) gives

$$\begin{aligned} (1-\alpha_2)\, a_2 - \alpha_3 a_3 &= 0, \\ -\alpha_2 a_2 + (2\alpha_3 - 1)\, a_3 &= 0. \end{aligned} \qquad \ldots(5.3.5)$$

From (5.3.5) we get

$$\frac{a_2}{a_3} = \frac{\alpha_3}{(1-\alpha_2)} = \frac{(2\alpha_3 - 1)}{\alpha_2} \quad . \qquad \ldots(5.3.6)$$

Hence $3\alpha_2\alpha_3 - 2\alpha_3 - \alpha_2 + 1 = 0$

which by simple algebra reduces to (5.3.4). Conversely, if (α_2, α_3) satisfies (5.3.4) then a_2/a_3 can be determined uniquely by (5.3.6) so that (5.3.5) holds and hence T_α will have zero variance at $(0, a_2, a_3)$. This proves part (i).

Now let (α_2, α_3) satisfy (5.3.4). For varying α_1, the variance of T_α at a point $\underset{\sim}{Y}$ with $Y_1 \neq 0$ is

$$\text{Var}\,(T_\alpha) = \frac{1}{3} \sum_{i=1}^{3} \left[\alpha_i Y_i + (1-\alpha_{i+1})\, Y_{i+1}\right]^2 - \bar{Y}^2. \qquad \ldots(5.3.7)$$

Thus Var (T_α) is minimum when

$$\alpha_1 = \frac{Y_1 + (\alpha_2 - 1) Y_2 + \alpha_3 Y_3}{2Y_1} \qquad \ldots(5.3.8)$$

If $(\alpha_2, \alpha_3) \neq (1,0)$, then the right side of (5.3.8) can be made to assume any real value by a suitable choice of $\underset{\sim}{Y}$. Therefore, for any $\alpha_1 \in R$, the estimator T_α is the unique estimator which attains minimum variance at a suitable point $\underset{\sim}{Y}$ amongst all estimators which attain zero variance at another suitable point $(0, a_2, a_3)$. Thus every such T_α is admissible. On the other hand, if $(\alpha_2, \alpha_3) = (1,0)$, then right side of (5.3.8) reduces to 1/2. Hence $\alpha = (1/2, 1, 0)$ leads to an admissible estimator. For $\alpha_1 \neq 1/2$, Dharmadhikari[1] has shown that the estimator corresponding to $(\alpha_1, 1, 0)$ is inadmissible. This proves (ii) and completes the proof of the theorem.

<u>Theorem</u> 5.3.2 : <u>Let</u> $\underset{\sim}{a} = (a_1, a_2, a_3)$, <u>where</u> $a_i \neq 0$, $i=1,2,3$.

Let $\bar{a} = (a_1 + a_2 + a_3)/3$. <u>An estimator</u> T_α <u>attains zero variance at</u> $\underset{\sim}{a}$ <u>if, and only if,</u>

$$a_i \alpha_i = c + \delta_i, \; i = 1,2,3 \qquad \ldots(5.3.9)$$

<u>where</u> $c \in R$ <u>is arbitrary and</u> $\delta_i = (a_i - a_{i+1})/3$.

<u>An estimator satisfying</u> (5.3.9) <u>is admissible if, and only if,</u>

$$|c - \tfrac{1}{2}\bar{a}| \leq h^*, \text{ where } h^* = \sqrt{\left[(\delta_1^2 + \delta_2^2 + \delta_3^2)/6\right]} .$$

<u>Proof</u> : The first assertion has already been noted in the proof of Lemma 5.3.1. Let e_c denote the estimator T_α where α satisfies (5.3.9). Then

$$\alpha_i\, a_i + (1-\alpha_{i+1})\, a_{i+1} = \bar{a}, \quad i=1,2,3, \qquad \ldots(5.3.10)$$

as $T_\alpha\ (s_i, \underset{\sim}{a}) = \bar{a}$ for $i = 1,2,3$.

We now compute Var (e_c) at a point $\underset{\sim}{Y}$.

For convenience we write $b_i = Y_i/a_i$.

All summations below are for $i = 1,2,3$.

$$3\left[\mathrm{Var}(e_c) + \bar{Y}^2\right] = \sum\left[\alpha_i Y_i + (1-\alpha_{i+1})\, Y_{i+1}\right]^2$$

$$= \sum\left[\alpha_i a_i b_i + (1-\alpha_{i+1})\, a_{i+1}\, b_{i+1}\right]^2$$

$$= \sum\left[\alpha_i a_i b_i + (\bar{a} - \alpha_i a_i)\, b_{i+1}\right]^2 \text{ using } (5.3.10)$$

$$= \sum\left[(c+\delta_i) b_i + (\bar{a} - c - \delta_i)\, b_{i+1}\right]^2 \text{ using } (5.3.9)$$

$$= \sum\left[c(b_i - b_{i+1}) + \delta_i(b_i - b_{i+1}) + \bar{a} b_{i+1}\right]^2$$

$$= c^2 \sum (b_i - b_{i+1})^2 + 2c \sum \delta_i (b_i - b_{i+1})^2$$

$$+ 2c\,\bar{a} \sum (b_i - b_{i+1})\, b_{i+1} + Q \qquad \ldots(5.3.11)$$

where Q denotes terms which do not involve c.

Now

$$\sum(b_i - b_{i+1})\, b_{i+1} = \sum(b_i\, b_{i+1} - b_{i+1}^2)$$

$$= b_1b_2 - b_2^2 + b_2b_3 - b_3^2 + b_3b_1 - b_1^2$$

$$= -\tfrac{1}{2}\,(2b_1^2 + 2b_2^2 + 2b_3^2 - 2b_1b_2 - 2b_2b_3 - 2b_3b_1)$$

$$= -\tfrac{1}{2}\left[(b_1-b_2)^2 + (b_2-b_3)^2 + (b_3-b_1)^2\right]$$

$$= -\tfrac{1}{2}\sum(b_i-b_{i+1})^2.$$

Therefore (5.3.11) gives

$$3\left[\text{Var}\,(e_c) + \bar{Y}^2\right] = c^2\sum(b_i-b_{i+1})^2 + c\sum(2\,\delta_i-\bar{a})(b_i-b_{i+1})^2+Q.$$

It follows that

$$3\left[\text{Var}\,(e_c) - \text{Var}\,(e_d)\right] = (c-d)\sum(c+d-\bar{a}+2\,\delta_i)(b_i-b_{i+1})^2. \qquad \ldots(5.3.12)$$

Write $h = c+d-\bar{a}$, $Z_i = (b_i-b_{i+1})$, then $\sum Z_i=0$.

Hence (5.3.12) becomes

$$3\left[\text{Var}\,(e_c) - \text{Var}\,(e_d)\right] = (c-d)\sum(h+2\,\delta_i)\,Z_i^2.$$

Then, using the results $\sum\delta_i = 0$ and $\sum Z_i=0$ after some simplification, we get

$$3\left[\text{Var}(e_c) - \text{Var}(e_d)\right] = 2(c-d)\left[(h-\delta_2)\, Z_1^2 + (h-\delta_1)\right.$$

$$\left. \cdot Z_2^2 + (h+2\delta_3) Z_1 Z_2\right]. \qquad \ldots(5.3.13)$$

The quadratic form on right side of (5.3.13) is definite if, and only if,

$(h-\delta_2) > 0$, $(h-\delta_1) > 0$ and $(h-\delta_2)(h-\delta_1) - (\frac{h}{2}+\delta_3)^2 > 0$.

Or

$(h-\delta_2) < 0$, $(h-\delta_1) < 0$ and $(h-\delta_2)(h-\delta_1) - (\frac{h}{2}+\delta_3)^2 > 0$.

But this means that the condition

$(h-\delta_2)(h-\delta_1) - (\frac{h}{2}+\delta_3)^2 > 0$ is both necessary and sufficient for the quadratic form on right side of (5.3.13) to be definite.

After some simplification this condition reduces to

$h^2 > (2/3)\,(\delta_1^2 + \delta_2^2 + \delta_3^2)$ or

$|h| > 2h^*$ where $h^* = \sqrt{\frac{1}{6}\left[(\delta_1^2 + \delta_2^2 + \delta_3^2)\right]}$.

Further, it is easy to see that the quadratic form in the bracket on the right side of (5.3.13) is positive definite if $h > 2h^*$ and negative definite if $h < -2h^*$.

Let $c > \frac{\bar{a}}{2} + h^*$ and $d = \frac{\bar{a}}{2} + h^*$ then $h = c+d - \bar{a} > 2h^*$. Therefore the bracket on right side of (5.3.13) is positive definite. But $c-d > 0$. Therefore right side of (5.3.13) is positive definite. Thus e_d dominates e_c and e_c is inadmissible. Similarly e_c is not admissible for $c < (\bar{a}/2) - h^*$. Now let $|c - (\bar{a}/2)| \leq h^*$ and $|d-(\bar{a}/2)| < h^*$. We show that e_d does not dominate e_c. This will show that e_c is admissible, because for d outside this range, e_d is dominated by e_f where $f = (\bar{a}/2)+h^*$ or $f = (\bar{a}/2)-h^*$. We have $|c+d-\bar{a}| \leq 2h^*$. Therefore the quadratic form on the right side of (5.3.13) is indefinite. Hence e_c is not dominated by e_d. Thus e_c is admissible. This proves the theorem.

We will now describe a method by which one can determine whether, for a given $\alpha \in R^3$, T_α is admissible. Calculate the three quantities

$$a_i = (3\alpha_{i+1}-2)\ (3\alpha_{i+2}-1)+1, \quad i=1,2,3. \qquad \ldots(5.3.14)$$

Actually (a_1, a_2, a_3) is just a solution of (5.3.2) for the given α.

<u>Case 1</u>. Let one of the a_i's, say, a_1 be zero. Then (α_2, α_3) lies on the hyperbola (5.3.4). If $(\alpha_2, \alpha_3) \neq (1,0)$ then T_α

is admissible. If $(\alpha_2, \alpha_3) = (1,0)$ then T_α is admissible if, and only if, $\alpha_1 = 1/2$. The Case $a_2=0$ or $a_3=0$ are treated similarly.

<u>Case 2</u>. Suppose all the a_i's are non-zero. Calculate $\delta_i = (a_i - a_{i+1})/3$, i=1,2,3 and $h^* = \sqrt{\frac{1}{6}[(\delta_1^2 + \delta_2^2 + \delta_3^2)]}$.
Because of (5.3.9), the value of $\alpha_i a_i - \delta_i$ is same for all i and we denote this value by c. Then T_α is admissible if, and only if,

$$|c - (\bar{a}/2)| \leq h^* \qquad \ldots(5.3.15)$$

where $\bar{a} = (a_1 + a_2 + a_3)/3$

The expression (5.3.15) can be expressed in simple form

$$a_1 . a_2 . a_3 \leq \frac{3}{4} \bar{a}^2. \qquad \ldots(5.3.16)$$

Put i=1 in (5.3.14) and substitute for α_2, α_3 by $\alpha_i = (c + \delta_i)/a_i$, to get

$$c^2 - c\bar{a} = (a_1 a_2 a_3 - \sum a_i a_{i+1})/9. \qquad \ldots(5.3.17)$$

Condition (5.3.15) is equivalent to

$$c^2 - c\bar{a} \leq h^{*2} - \frac{1}{4} \bar{a}^2. \qquad \ldots(5.3.18)$$

From (5.3.17) and (5.3.18) we get (5.3.16).

Let A be the set of all $\alpha \in R^3$ such that T_α is admissible and let B be the complementary set of all $\alpha \in R^3$ such that T_α is inadmissible. First observe that the estimators considered in Theorem 5.3.1. correspond to a set of Lebesgue measure zero. Therefore, from the point of view of measure, the essential information is contained in theorem 5.3.2 and hence in the condition (5.3.16). If we substitute for the a_i's from (5.3.14), (5.3.16) becomes a condition involving a sixth degree polynomial in α_1, α_2 and α_3,

$$\{(3\alpha_2-2)\ (3\alpha_3-1)+1\}\ \{(3\alpha_3-2)\ (3\alpha_1-1)+1\}$$

$$\cdot\ \{(3\alpha_1-2)\ (3\alpha_2-1)+1\} \leq \frac{3}{4}\,\bar{a}^2\ . \qquad \ldots(5.3.19)$$

Substitute $d_i = 3\left[\alpha_i-(1/2)\right]$ and write $d = (d_1, d_2, d_3)$. Then the condition (5.3.16) for admissibility takes the form

$f(d) \leq 0$,

where $\quad f(d) = 12(a_1a_2a_3 - \frac{3}{4}\,\bar{a}^2) = J(d)+K(d)+L(d)+M(d)$,

with $\quad J(d) = 12d_1^2\,d_2^2\,d_3^2 + 10d_1d_2d_3(d_1+d_2+d_3)$,

$$K(d) = -4(d_1^2d_2^2 + d_2^2d_3^2 + d_3^2d_1^2) + \frac{9}{4}\,(d_1^2+d_2^2+d_3^2),$$

$$L(d) = 6(d_1-d_2)(d_2-d_3)(d_3-d_1),$$

$$M(d) = -\frac{9}{4}\left[(d_1-d_2)^2 + (d_2-d_3)^2 + (d_3-d_1)^2\right].$$

Note that $M(d) \leq 0$ for all d,

$$L(d) \leq 0 \quad \text{whenever} \quad d_1 > d_2 > d_3. \qquad \ldots(5.3.20)$$

It is easy to verify that

$K(d) \leq 0$ whenever

$$d_2^2 \geqslant 2 \text{ and } d_3^2 \geqslant 2. \qquad \ldots(5.3.21)$$

Now $J(d) = 2d_1d_2d_3 \left[d_3(6d_1d_2+5) + 5(d_1+d_2) \right]$.

Therefore $J(d) \leqslant 0$ whenever

$$-5 < 6d_1d_2 < 0, \quad d_3 < 0 \text{ and } d_1+d_2 < 0. \qquad \ldots(5.3.22)$$

It follows from (5.3.20), (5.3.21) and (5.3.22) that

$f(d) \leq 0$ whenever

$$-\frac{5}{6} < d_1d_2 < 0, \quad d_3 < d_2 < -\sqrt{2}\ . \qquad \ldots(5.3.23)$$

Now the set of $d \in R^3$ for which (5.3.23) holds has infinite Lebesgue measure. This shows that the set of admissible points in the d-space has infinite Lebesgue measure.

Now T_α is inadmissible whenever

$f(d) > 0$ where $f(d)$ is as defined earlier.

Rewrite $f(d)$ as

$$f(d) = 4\, d_1^2 d_2^2 (d_3^2-1) + 4d_2^2 d_3^2 (d_1^2-1) + 4d_3^2 d_1^2 (d_2^2-1)$$
$$+ 10 d_1 d_2 d_3 (d_1+d_2+d_3) + \frac{9}{4}\left[(2d_1-d_2)\, d_2 + (2d_2-d_3) d_3\right.$$
$$\left. + (2d_3-d_1)\, d_1\right] + 6(d_1-d_2)\,(d_2-d_3)(d_3-d_1).$$

One can easily verify that

$$f(d) > 0 \text{ whenever } 1 < d_1 < d_2 < d_3 < 2d_1. \qquad \ldots(5.3.24)$$

Now the set of $d \in R^3$ for which (5.3.24) holds has infinite Lebesgue measure. Thus the set of inadmissible points in the d-space also has infinite Lebesgue measure.

Now for fixed d such that each $d_i \neq 0$ and $k \in R$, $f(kd)$ is a sixth degree polynomial in k. Therefore the set of all values of k for which $f(kd) \leq 0$ is a bounded set. Thus any line through the origin which does not lie in any co-ordinate plane makes only a bounded intercept on the set of admissible points in the d-space. In this sense one can say that inadmissible estimators outnumber the admissible ones.