

Chapter 4

EGUE(2)- $SU(4)$: Group Theoretical

Results

4.1 Introduction

Spin-isospin $SU(4)$ supermultiplet scheme for nuclei was introduced by Wigner [Wi-37] and there is good evidence for the goodness of this symmetry in some parts of the nuclear chart [Pa-78, Va-95, Na-01, Va-05, Ko-07a, Va-99, Va-07] and also more recently there is new interest in $SU(4)$ symmetry for heavy $N \sim Z$ nuclei [Va-95, Na-01, Va-05, Ko-07a]. Therefore, it is important to define and study EGE's generated by random two-body interactions with $SU(4)$ symmetry [EGUE(2)- $SU(4)$]. Given m fermions (nucleons) in Ω number of sp orbitals with spin and isospin degrees of freedom, for $SU(4)$ scalar Hamiltonians, the symmetry algebra is $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ and all the states within an $SU(4)$ [but not $U(\Omega)$] irrep will be degenerate in energy. In the past, applying Wigner-Racah algebra of the embedding algebra $U(2\Omega) \supset U(\Omega) \otimes SU(2)$ some analytical results are derived for EGUE(2)- \mathbf{s} ; see Appendix C for some details. Going beyond the spin ensemble (discussed in Chapters 2, 3 and Appendix C), our purpose in the present chapter is to define EGUE(2)- $SU(4)$, develop analytical formulation for solving the ensemble and derive analytical formulas, for the lower order moments of the one-point (density of eigenvalues) and two-point (defining level fluctuations) functions, for some simple class of $SU(4)$ irreps. In addition, analytical formulation developed in the chapter allows one to consider all these, numerically, for any $SU(4)$ [or $U(\Omega)$] irrep. Using these, studied are: ensemble averaged spec-

tral variances, expectation values of the quadratic Casimir invariant of $SU(4)$ algebra, four periodicity in the gs energies and lower order cross-correlations in energy centroids and spectral variances generated by this ensemble. Before proceeding further, let us mention that a preliminary report of some of the results in this chapter is given in [Ma-09a] and all the details are published in the long paper [Ma-10b].

4.2 Preliminaries of $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ Algebra

Although all the results in this section are well-known [Pa-78, He-69, He-74a], we will discuss these here for completeness and also for introducing various quantities and notations used in the reminder of the chapter^a.

4.2.1 Generators of $U(\Omega)$ and $SU(4)$ algebras

Let us begin with m fermions distributed in 4Ω number of sp states. Then the spectrum generating algebra is $U(4\Omega)$. Associating two quantum numbers i (i -space) and α (α -space) to each sp state, the sp states are denoted by $|i, \alpha\rangle$, where $i = 1, 2, \dots, \Omega$ and $\alpha = 1, 2, 3, 4$. In nuclear applications, the i -space corresponds to the orbital space and the α -space corresponds to the spin(**s**)-isospin(**t**) space, then $|\alpha\rangle = |m_{\mathbf{s}}, m_{\mathbf{t}}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle, |-\frac{1}{2}, \frac{1}{2}\rangle$ and $|-\frac{1}{2}, -\frac{1}{2}\rangle$, respectively. From now on in this section we will present results both in single state representation defined by $|i, \alpha\rangle$ states and also in the spin-isospin representation defined by $|i; \mathbf{s} = \frac{1}{2}, m_{\mathbf{s}}; \mathbf{t} = \frac{1}{2}, m_{\mathbf{t}}\rangle$ states. For the EGUE(2)- $SU(4)$ ensemble, the former will suffice. However the later (spin-isospin) representation is useful for understanding the physical relevance of the ensemble. In the single state representation, the $(4\Omega)^2$ number of operators $C_{i\alpha; j\beta}$ generate $U(4\Omega)$ algebra and with respect to this algebra, all the m fermion states transform as the irrep $\{1^m\}$. In terms of the creation operators $a_{i,\alpha}^\dagger$ and the annihilation operators $a_{j,\beta}$, the generators $C_{i\alpha; j\beta}$ and their commutation relations are,

$$C_{i\alpha; j\beta} = a_{i,\alpha}^\dagger a_{j,\beta}; \quad [C_{i\alpha; j\beta}, C_{k\alpha'; l\beta'}] = C_{i\alpha; l\beta'} \delta_{jk} \delta_{\beta\alpha'} - C_{k\alpha'; j\beta} \delta_{li} \delta_{\beta'\alpha}. \quad (4.2.1)$$

It is possible to define commuting unitary transformations in the i -space and α -space separately and then we have $U(\Omega)$ and $U(4)$ algebras describing unitary trans-

^aWe use different notations in this chapter for mathematical ease

formations in the two respective spaces. With this we have the direct product group-subgroup structure $U(4\Omega) \supset U(\Omega) \otimes U(4)$. We can easily write the generators A_{ij} and $B_{\alpha\beta}$ for the $U(\Omega)$ and $U(4)$ algebras, respectively, using the fact that the generators of $U(\Omega)$ are scalars in α -space and similarly the $U(4)$ generators in the i -space,

$$A_{ij} = \sum_{\alpha=1}^4 C_{i\alpha;j\alpha}, \quad B_{\alpha\beta} = \sum_{i=1}^{\Omega} C_{i\alpha;i\beta}. \quad (4.2.2)$$

Their commutation relations can be derived using Eq. (4.2.1) by summing over the appropriate indices,

$$\begin{aligned} [A_{ij}, A_{kl}] &= A_{il}\delta_{jk} - A_{kj}\delta_{li}, \\ [B_{\alpha\beta}, B_{\alpha'\beta'}] &= B_{\alpha\beta'}\delta_{\beta\alpha'} - B_{\alpha'\beta}\delta_{\alpha\beta'}. \end{aligned} \quad (4.2.3)$$

Also the A 's commute with the B 's. Instead of $U(4)$, it is possible to consider $SU(4)$ by making the generators B 's traceless [see Eq. (4.2.11) ahead].

In the orbital \times spin-isospin realization of the $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ algebra, $SU(4)$ corresponds to the Wigner's supermultiplet algebra [Wi-37]. In this physically relevant spin-isospin representation, the $SU(4)$ generators can be written in terms of the one-body operators $\mathcal{A}_{ij;\mu_s,\mu_t}^{s,t}$ where,

$$\begin{aligned} \mathcal{A}_{ij;\mu_s,\mu_t}^{s,t} &= \left(a_i^\dagger \bar{a}_j \right)_{\mu_s,\mu_t}^{s,t} \\ &= \sum_{m_s(m'_s), m_t(m'_t)} \left\langle \frac{1}{2} m_s \frac{1}{2} m'_s \mid s \mu_s \right\rangle \left\langle \frac{1}{2} m_t \frac{1}{2} m'_t \mid t \mu_t \right\rangle a_{i;\frac{1}{2},m_s;\frac{1}{2},m_t}^\dagger \bar{a}_{j;\frac{1}{2},m'_s;\frac{1}{2},m'_t}. \end{aligned} \quad (4.2.4)$$

Note that $\bar{a}_{j;\frac{1}{2},\mu_s;\frac{1}{2},\mu_t} = (-1)^{1+\mu_s+\mu_t} a_{j;\frac{1}{2},-\mu_s;\frac{1}{2},-\mu_t}$. The operators $\mathcal{A}_{ij;\mu_s,\mu_t}^{s,t}$ generate $U(4\Omega)$ algebra. Similarly, the operators $\mathcal{A}_{ij}^{0,0}$ (Ω^2 in number) and $\sum_i \mathcal{A}_{ii;\mu_s,\mu_t}^{s,t}$ (16 in number) generate the $U(\Omega)$ and $U(4)$ algebras, respectively. The 16 generators of $U(4)$ can be written in terms of the number operator \hat{n} , the three spin generators S_μ^1 , the three isospin generators T_μ^1 and the nine components $(\sigma\tau)_{\mu,\mu'}^{1,1}$ of the Gamow-Teller operator $\sigma\tau$. Dropping the number operator, we obtain the $SU(4)$ algebra. Given a one-body operator \mathcal{O} , it can be expressed in terms of the creation and annihilation

operators,

$$\mathcal{O} = \sum_{i,j,m_s,m_t,m'_s,m'_t} \left\langle i; \frac{1}{2}, m_s; \frac{1}{2}, m_t \mid \mathcal{O} \mid j; \frac{1}{2}, m'_s; \frac{1}{2}, m'_t \right\rangle a_{i;\frac{1}{2},m_s;\frac{1}{2},m_t}^\dagger a_{j;\frac{1}{2},m'_s;\frac{1}{2},m'_t}. \quad (4.2.5)$$

Starting with Eq. (4.2.5), applying the angular-momentum algebra [Ed-74] and using Eq. (4.2.4), will give [Ko-06b]

$$\hat{n} = 2 \sum_i \mathcal{A}_{ii;0,0}^{0,0}, \quad S_\mu^1 = \sum_i \mathcal{A}_{ii;\mu,0}^{1,0}, \quad (4.2.6)$$

$$T_\mu^1 = \sum_i \mathcal{A}_{ii;0,\mu}^{0,1}, \quad (\sigma\tau)_{\mu,\mu'}^{1,1} = \sum_i \mathcal{A}_{ii;\mu,\mu'}^{1,1}.$$

Commutation relations for the $SU(4)$ generators in the spin-isospin (sometimes called spherical) representation are,

$$[S_\mu^1, S_{\mu'}^1] = -\sqrt{2} \langle 1 \mu 1 \mu' \mid 1 \mu + \mu' \rangle S_{\mu+\mu'}^1,$$

$$[T_\mu^1, T_{\mu'}^1] = -\sqrt{2} \langle 1 \mu 1 \mu' \mid 1 \mu + \mu' \rangle T_{\mu+\mu'}^1,$$

$$[S_\mu^1, (\sigma\tau)_{\mu',\mu''}^{1,1}] = -\sqrt{2} \langle 1 \mu 1 \mu' \mid 1 \mu + \mu' \rangle (\sigma\tau)_{\mu+\mu',\mu''}^{1,1}, \quad (4.2.7)$$

$$[T_\mu^1, (\sigma\tau)_{\mu',\mu''}^{1,1}] = -\sqrt{2} \langle 1 \mu 1 \mu'' \mid 1 \mu + \mu'' \rangle (\sigma\tau)_{\mu',\mu+\mu''}^{1,1},$$

$$\begin{aligned} [(\sigma\tau)_{\mu_1,\mu_2}^{1,1}, (\sigma\tau)_{\mu_3,\mu_4}^{1,1}] &= \sqrt{2} (-1)^{\mu_1+1} \langle 1 \mu_2 1 \mu_4 \mid 1 \mu_2 + \mu_4 \rangle \delta_{\mu_1,-\mu_3} T_{\mu_2+\mu_4}^1 \\ &+ \sqrt{2} (-1)^{\mu_2+1} \langle 1 \mu_1 1 \mu_3 \mid 1 \mu_1 + \mu_3 \rangle \delta_{\mu_2,-\mu_4} S_{\mu_1+\mu_3}^1. \end{aligned}$$

Now we will consider the quadratic Casimir invariants (C_2) of $U(\Omega)$ and $SU(4)$ and their physical interpretation. However we will not consider here the cubic (C_3) and quartic (C_4) invariants of $SU(4)$ although they are needed for some purposes as discussed ahead; see for example [Pa-78] for C_3 and C_4 operators.

4.2.2 Quadratic Casimir operators of $U(\Omega)$ and $SU(4)$ and the Majorana operator

In the $|i, \alpha\rangle$ representation it is easy to write down the quadratic Casimir invariant of $U(4)$,

$$C_2[U(4)] = \sum_{\alpha, \beta} B_{\alpha, \beta} B_{\beta, \alpha} = 4\hat{n} + \sum_{i, j, \alpha, \beta} a_{i, \alpha}^\dagger a_{j, \beta}^\dagger a_{j, \alpha} a_{i, \beta}. \quad (4.2.8)$$

The operator $C_2[U(4)]$ commutes with the generators $B_{\alpha, \beta}$ or equivalently with \hat{n} , S_μ^1 , T_μ^1 and $(\sigma\tau)_{\mu, \mu}^{1, 1}$. Just as $C_2[U(4)]$, the quadratic Casimir invariant of $U(\Omega)$ is,

$$C_2[U(\Omega)] = \sum_{i, j} A_{ij} A_{ji} = \hat{n} \Omega - \sum_{i, j, \alpha, \beta} a_{i, \alpha}^\dagger a_{j, \beta}^\dagger a_{j, \alpha} a_{i, \beta}. \quad (4.2.9)$$

Combining Eqs. (4.2.8) and (4.2.9) we have

$$C_2[U(\Omega)] + C_2[U(4)] = \hat{n}(\Omega + 4). \quad (4.2.10)$$

It is also easy to see that the $C_2[SU(4)]$ can be written in terms of $C_2[U(4)]$ and $C_2[U(\Omega)]$,

$$\begin{aligned} C_2[SU(4)] &= \sum_{\alpha, \beta} B'_{\alpha, \beta} B'_{\beta, \alpha}; \quad B'_{\alpha, \beta} = B_{\alpha, \beta} - \frac{\text{Tr}(B)}{4} \delta_{\alpha, \beta}, \quad \text{Tr}(B) = \sum_{\alpha} B_{\alpha, \alpha} \\ &= C_2[U(4)] - \frac{\hat{n}^2}{4} \\ &= - \left[C_2[U(\Omega)] - \hat{n}(\Omega + 4) + \frac{\hat{n}^2}{4} \right]. \end{aligned} \quad (4.2.11)$$

In the angular-momentum coupled representation, $C_2[SU(4)] = S^2 + T^2 + (\sigma\tau) \cdot (\sigma\tau)$. In order to obtain a physical interpretation for $C_2[SU(4)]$, we will consider the space exchange or the Majorana operator \widehat{M} , with strength κ , that exchanges the spatial coordinates of the particles (the index i) and leaves the index α (equivalently spin-isospin quantum numbers) unchanged. Then [Pa-78],

$$\widehat{M}|i, \alpha; j, \beta\rangle = \kappa |j, \alpha; i, \beta\rangle. \quad (4.2.12)$$

As $|i, \alpha; j, \beta\rangle = a_{i,\alpha}^\dagger a_{j,\beta}^\dagger |0\rangle$, Eq. (4.2.12) gives, with κ a constant,

$$\begin{aligned}\widehat{M} &= \frac{\kappa}{2} \left[\sum_{i,j,\alpha,\beta} (a_{j,\alpha}^\dagger a_{i,\beta}^\dagger) (a_{i,\alpha}^\dagger a_{j,\beta}^\dagger)^\dagger \right] \\ &= \frac{\kappa}{2} \left[\sum_{i,j} \left(\sum_{\alpha} a_{j,\alpha}^\dagger a_{i,\alpha} \right) \left(\sum_{\beta} a_{i,\beta}^\dagger a_{j,\beta} \right) - \Omega \sum_{j,\alpha} a_{j,\alpha}^\dagger a_{j,\alpha} \right] \\ &= \frac{\kappa}{2} \{C_2[U(\Omega)] - \Omega \hat{n}\}.\end{aligned}\quad (4.2.13)$$

Eqs. (4.2.11) and (4.2.13) allow us to write the \widehat{M} operator in terms of $C_2[SU(4)]$.

Then, we have

$$\widehat{M} = \kappa \left\{ 2\hat{n} \left(1 + \frac{\hat{n}}{16} \right) - \frac{1}{2} C_2[SU(4)] \right\}.\quad (4.2.14)$$

Using Eq. (4.2.14) one can identify the $SU(4)$ [or $U(4)$] irrep for gs, assuming that the Hamiltonian is represented by the Majorana operator. Towards this end, now we will consider the $SU(4)$ and $U(\Omega)$ irreps and the reduction of the $SU(4)$ irreps to (S, T) .

4.2.3 $SU(4)$ and $U(\Omega)$ irreps and identification of the ground state $U(\Omega)$ or $SU(4)$ irreps

With m fermions in 4Ω sp states, we can decompose the basis space with dimension $\binom{4\Omega}{m}$ into irreps of $U(4)$ [or $SU(4)$] and $U(\Omega)$ and further the $U(4)$ irreps into (S, T) . Firstly, the $U(4)$ irreps are represented by the Young tableaux (see Fig. 4.2) or the partitions $\{F\}$,

$$\{F\} = \{F_1, F_2, F_3, F_4\}, \quad F_1 \geq F_2 \geq F_3 \geq F_4 \geq 0, \quad m = \sum_{i=1}^4 F_i.\quad (4.2.15)$$

Note that F_α are the eigenvalues of $B_{\alpha\alpha}$ defined in Eq. (4.2.2). As the total m -particle wavefunctions are antisymmetric, the $U(\Omega)$ irreps $\{f\}$ are uniquely defined by $\{F\}$ and $\{f\} = \{\tilde{F}\}$ (alternatively $\{F\} = \{\tilde{f}\}$) which is obtained by changing rows to columns in the Young tableaux $\{F\}$; see for example [Pa-78, Wy-70, Ha-62]. Due to this symmetry constraint, $F_j \leq \Omega$, $j = 1, 2, 3, 4$ and $f_i \leq 4$, $i = 1, 2, \dots, \Omega$. Given the $U(4)$ irrep $\{F\}$, the corresponding $SU(4)$ irrep $\{F'\}$, which is three rowed Young tableaux, can be defined

by

$$\{F'\} = \{F'_1, F'_2, F'_3\} = \{F_1 - F_4, F_2 - F_4, F_3 - F_4\}. \quad (4.2.16)$$

The $\{F\} \rightarrow (S, T)$ reductions can be obtained using group theoretical methods [Wy-70, Ha-62]. Alternatively a physically intuitive procedure, easy to implement on a machine, is as follows. First, the $\{F\} \rightarrow (S, T)$ reductions for a symmetric $U(4)$ irrep $\{F\} = \{F_1, 0, 0, 0\}$ can be obtained by distributing $m = F_1$ identical bosons in the four spin-isospin orbitals labeled by $|m_s m_t\rangle$. From these distributions, the S_z and T_z eigenvalues $M_S = \sum_i m_i (m_s)_i$ and $M_T = \sum_i m_i (m_t)_i$ and the corresponding degeneracies $d(m : M_S, M_T)$ follow easily. Here m_i are the number of bosons in the i th orbit with $m_s = (m_s)_i$ and $m_t = (m_t)_i$. Let us denote the number of times (S, T) appears in a given $\{F\}$ by $D(\{F\} : S, T)$. It is easy to see that $D(\{m, 0, 0, 0\} : S, T)$ is given by the double difference,

$$\begin{aligned} D(\{m, 0, 0, 0\} : S, T) &= d(m : M_S = S, M_T = T) - d(m : M_S = S, M_T = T + 1) \\ &\quad - d(m : M_S = S + 1, M_T = T) + d(m : M_S = S + 1, M_T = T + 1). \end{aligned} \quad (4.2.17)$$

Carrying out this exercise on a machine for many m values, we obtain the following (well known in literature) general result,

$$\{m, 0, 0, 0\} \rightarrow (S, T) = \left(\frac{m}{2}, \frac{m}{2}\right), \left(\frac{m}{2} - 1, \frac{m}{2} - 1\right), \dots, (0, 0) \text{ or } \left(\frac{1}{2}, \frac{1}{2}\right). \quad (4.2.18)$$

It is important to note that here $D(\{m, 0, 0, 0\} : S, T) = 1$ for all allowed (S, T) values (i.e., multiplicity is unity). The reductions for a general $U(4)$ irrep $\{F\} = \{F_1, F_2, F_3, F_4\}$ follow by writing $\{F\}$ as a determinant involving only totally symmetric irreps with the multiplication of the elements in the determinant replaced by outer products. Then we have [Wy-70, Ja-81]

$$\{F\} = |\mathcal{F}_{ij}|, \quad \mathcal{F}_{ij} = \{F_i + j - i, 0, 0, 0\}; \quad \{0\} = 1, \quad \{-x, 0, 0, 0\} = 0. \quad (4.2.19)$$

Substituting the dimensions for symmetric irreps in the above determinant gives the

dimension formula for $U(4)$ irreps,

$$d_4(\{F\}) = |d_{ij}|, \quad d_{ij} = \binom{F_i + j - i + 3}{3}. \quad (4.2.20)$$

Also the corresponding (S, T) values and their multiplicities can be obtained by substituting the (S, T) values for \mathcal{F}_{ij} in the determinant in Eq. (4.2.19) and evaluating the determinant by applying angular-momentum coupling rules. Note that $d_4(\{F\}) = \sum_{S,T} (2S+1)(2T+1)D(\{F\} : S, T)$. In carrying out the algebra we can exploit the equivalence between $SU(4)$ and $U(4)$ irreps and employ just 3 rowed $U(4)$ irreps. This procedure is used in constructing Tables 4.1 and 4.2. For a realistic system such as the atomic nucleus, given the Ω value and the number of valence nucleons m , we can enumerate all the allowed $U(4)$ or $SU(4)$ irreps using Eqs. (4.2.15) and (4.2.16). Table 4.1 gives all the possible $U(4)$ irreps for $\Omega = 10$ and $m = 0 - 6$ along with their spin-isospin structure.

Assuming that the Majorana operator is the Hamiltonian with κ in Eq. (4.2.14) negative, we can identify the $SU(4)$ irreps labeling gs as follows. Using the formulas for the eigenvalues of $C_2[U(4)]$ and $C_2[U(\Omega)]$,

$$\langle C_2[U(4)] \rangle^{\{F\}} = \sum_{i=1}^4 F_i(F_i + 5 - 2i), \quad \langle C_2[U(\Omega)] \rangle^{\{f\}} = \sum_{i=1}^{\Omega} f_i(f_i + \Omega + 1 - 2i), \quad (4.2.21)$$

and applying Eq. (4.2.14), we can order the $SU(4)$ irreps. For physical systems, generally, the $U(\Omega)$ (spatial) irrep for the ground states should be the most symmetric one. The symmetric irrep, as seen from Eq. (4.2.21), will have the largest eigenvalue for $C_2[U(\Omega)]$. From Eqs. (4.2.13) and (4.2.14), then it follows that the $SU(4)$ irrep for gs should be the one with the lowest eigenvalue for $C_2[SU(4)]$ and these eigenvalues can be obtained by combining Eq. (4.2.11) with Eq. (4.2.21). Now, for a given (m, T_z) with $T = |T_z|$ and $T_z = (N-Z)/2$ for a nucleus with N neutrons and Z protons, enumerating $\{F\} \rightarrow (S, T)$ reductions, we can determine the $U(4)$ irreps labeling gs, by applying Eq. (4.2.14) with κ negative. In Table 4.2, $U(4)$ and $U(\Omega)$ irreps for gs are listed for $\Omega = 10$ and $m = 4 - 11$ for all T_z values. As it is well known and also seen from Table 4.2, for the Majorana operator or equivalently for the $SU(4)$ invariant Hamiltonians, for $N=Z$ even-even ($m = 4r$), $N=Z$ odd-odd ($m = 4r + 2$) and $N=Z \pm 1$ ($m = 4r \pm 1$) odd-A nuclei,

Table 4.1: $m \rightarrow \{F\} \rightarrow (S, T)$ reductions for $\Omega = 10$ and $m = 0 - 6$. In the table, r in $(S, T)^r$ gives the multiplicity of the irrep (S, T) .

m	$\{F_1, F_2, F_3, F_4\}$	(S, T)
0	{0, 0, 0, 0}	(0, 0)
1	{1, 0, 0, 0}	$(\frac{1}{2}, \frac{1}{2})$
2	{1, 1, 0, 0}	(1, 0), (0, 1)
	{2, 0, 0, 0}	(1, 1), (0, 0)
3	{1, 1, 1, 0}	$(\frac{1}{2}, \frac{1}{2})$
	{2, 1, 0, 0}	$(\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
	{3, 0, 0, 0}	$(\frac{3}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
4	{1, 1, 1, 1}	(0, 0)
	{2, 1, 1, 0}	(1, 1), (1, 0), (0, 1)
	{2, 2, 0, 0}	(2, 0), (1, 1), (0, 2), (0, 0)
	{3, 1, 0, 0}	(2, 1), (1, 2), (1, 1), (1, 0), (0, 1)
	{4, 0, 0, 0}	(2, 2), (1, 1), (0, 0)
5	{2, 1, 1, 1}	$(\frac{1}{2}, \frac{1}{2})$
	{2, 2, 1, 0}	$(\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
	{3, 1, 1, 0}	$(\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
	{3, 2, 0, 0}	$(\frac{5}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{2}), (\frac{1}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
	{4, 1, 0, 0}	$(\frac{5}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{5}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{2}), (\frac{1}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
	{5, 0, 0, 0}	$(\frac{5}{2}, \frac{5}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{1}{2}, \frac{1}{2})$
6	{2, 2, 1, 1}	(1, 0), (0, 1)
	{2, 2, 2, 0}	(1, 1), (0, 0)
	{3, 1, 1, 1}	(1, 1), (0, 0)
	{3, 2, 1, 0}	(2, 1), (2, 0), (1, 2), (1, 1) ² , (1, 0), (0, 2), (0, 1)
	{3, 3, 0, 0}	(3, 0), (2, 1), (1, 2), (1, 0), (0, 3), (0, 1)
	{4, 1, 1, 0}	(2, 2), (2, 1), (1, 2), (1, 1), (1, 0), (0, 1)
	{4, 2, 0, 0}	(3, 1), (2, 2), (2, 1), (2, 0), (1, 3), (1, 2), (1, 1) ² , (0, 2), (0, 0)
	{5, 1, 0, 0}	(3, 2), (2, 3), (2, 2), (2, 1), (1, 2), (1, 1), (1, 0), (0, 1)
	{6, 0, 0, 0}	(3, 3), (2, 2), (1, 1), (0, 0)

the $U(\Omega)$ irreps for the gs, with lowest T , are $\{4^r\}$, $\{4^r, 2\}$, $\{4^r, 1\}$ and $\{4^r, 3\}$ with spin-isospin structure (see Table 4.1) being $(0, 0)$, $(1, 0) \oplus (0, 1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$, respectively. For convenience, we introduce the notation $\{f_m^{(p)}\}$ where

$$\{f_m^{(p)}\} = \{4^r, p\}; \quad m = 4r + p \text{ and } p = \text{mod}(m, 4) \quad (4.2.22)$$

and this is used in the remainder of the chapter. We shall see ahead that for the special $U(\Omega)$ irreps in Eq. (4.2.22), analytical formulas are much simpler than for a general $U(\Omega)$ irrep.

Table 4.2: $U(4)$ and $U(\Omega)$ irreps F_m and f_m , respectively, with the smallest value for $\langle C_2[SU(4)] \rangle^{f_m}$ for a given (m, T_z) value in the $(2p1f)$ -shell [$\Omega = 10$]. For the results in the Table, isospin $T = |T_z|$.

m	$ T_z $	$F_m = \tilde{f}_m$	f_m	m	$ T_z $	$F_m = \tilde{f}_m$	f_m
4	0	{1, 1, 1, 1}	{4}	9	1	{3, 2, 2, 2}	{4, 4, 1}
	1	{2, 1, 1, 0}	{3, 1}			{3, 3, 2, 1}	{4, 3, 2}
	2	{2, 2, 0, 0}	{2, 2}			{4, 3, 1, 1}	{4, 2, 2, 1}
5	1	{2, 1, 1, 1}	{4, 1}	10	2	{4, 4, 1, 0}	{3, 2, 2, 2}
		{2, 2, 1, 0}	{3, 2}			{5, 4, 0, 0}	{2, 2, 2, 2, 1}
		{3, 2, 0, 0}	{2, 2, 1}			{3, 3, 2, 2}	{4, 4, 2}
6	0	{2, 2, 1, 1}	{4, 2}	11	3	{3, 3, 2, 2}	{4, 4, 2}
	1	{2, 2, 1, 1}	{4, 2}			{4, 3, 2, 1}	{4, 3, 2, 1}
	2	{3, 2, 1, 0}	{3, 2, 1}			{4, 4, 1, 1}	{4, 2, 2, 2}
	3	{3, 3, 0, 0}	{2, 2, 2}			{5, 4, 1, 0}	{3, 2, 2, 2, 1}
7	1	{2, 2, 2, 1}	{4, 3}	10	4	{5, 5, 0, 0}	{2, 2, 2, 2, 2}
		{3, 2, 1, 1}	{4, 2, 1}			{3, 3, 3, 2}	{4, 4, 3}
		{3, 3, 1, 0}	{3, 2, 2}			{4, 3, 2, 2}	{4, 4, 2, 1}
8	2	{4, 3, 0, 0}	{2, 2, 2, 1}	11	5	{4, 4, 2, 1}	{4, 3, 2, 2}
		{2, 2, 2, 2}	{4, 4}			{5, 4, 1, 1}	{4, 2, 2, 2, 1}
		{3, 2, 2, 1}	{4, 3, 1}			{5, 5, 1, 0}	{3, 2, 2, 2, 2}
		{3, 3, 1, 1}	{4, 2, 2}			{6, 5, 0, 0}	{2, 2, 2, 2, 2, 1}
3	1	{4, 3, 1, 0}	{3, 2, 2, 1}	10	6		
		{4, 4, 0, 0}	{2, 2, 2, 2}				

Having described some of the essential properties of the $U(\Omega) \otimes SU(4)$ algebra, now we will introduce the EGUE(2)- $SU(4)$ random matrix ensemble and analyze in some detail its properties. From now on we denote the irreps $\{f\}$ and $\{F\}$ as f and F , respectively when there is no confusion.

4.3 Definition and Basic Properties of EGUE(2)- $SU(4)$

4.3.1 Definition of EGUE(2)- $SU(4)$

Let us begin with the normalized two-particle states $|f_2 \nu_2; F_2 \beta_2\rangle$ where the $U(4)$ irreps $F_2 = \{1^2\}$ and $\{2\}$ and the corresponding $U(\Omega)$ irreps f_2 are $\{2\}$ (symmetric) and $\{1^2\}$ (antisymmetric), respectively. Similarly ν_2 are the additional quantum numbers that belong to f_2 and β_2 belong to F_2 . As f_2 uniquely defines F_2 , from now on we will drop F_2 unless they are explicitly needed and also we will use the $f_2 \leftrightarrow F_2$ equivalence

whenever needed. With $A^\dagger(f_2 v_2 \beta_2)$ and $A(f_2 v_2 \beta_2)$ denoting creation and annihilation operators for the normalized two-particle states, a general two-body Hamiltonian operator \hat{H} that is $SU(4)$ scalar can be written as

$$\hat{H} = \hat{H}_{\{2\}} + \hat{H}_{\{1^2\}} = \sum_{f_2, v_2^i, v_2^f, \beta_2; f_2 = \{2\}, \{1^2\}} H_{f_2 v_2^i v_2^f}(2) A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2). \quad (4.3.1)$$

In Eq. (4.3.1), the two-body matrix elements $H_{f_2 v_2^i v_2^f}(2) = \langle f_2 v_2^f \beta_2 | \hat{H} | f_2 v_2^i \beta_2 \rangle$ are independent of the β_2 's. The uniform summation over β_2 in Eq. (4.3.1) ensures that \hat{H} is $SU(4)$ scalar and therefore it will not connect states with different f_2 's. However, \hat{H} is not a $SU(4)$ invariant operator. Just as the two-particle states, we can denote the m -particle states by $|f_m v_m^f \beta_m^F\rangle; F_m = \tilde{f}_m$. Action of \hat{H} on these states generates states that are degenerate with respect to β_m^F but not v_m^f . Therefore for a given f_m , there will be $d_\Omega(f_m)$ number of levels each with $d_4(\tilde{f}_m)$ number of degenerate states. Formula for the dimension $d_\Omega(f_m)$ is [Wy-70],

$$d_\Omega(f_m) = \prod_{i < j=1}^{\Omega} \frac{f_i - f_j + j - i}{j - i}, \quad (4.3.2)$$

where, $f_m = \{f_1, f_2, \dots\}$. Equation (4.3.2) also gives $d_4(F_m)$ with the product ranging from $i = 1$ to 4 and replacing f_i by F_i . As \hat{H} is a $SU(4)$ scalar, the m -particle H matrix will be a direct sum of matrices with each of them labeled by the f_m 's with dimension $d_\Omega(f_m)$. Thus

$$H(m) = \sum_{f_m} H_{f_m}(m) \oplus. \quad (4.3.3)$$

Figure 4.1 shows an example for Eq. (4.3.3). As seen from Eq. (4.3.1), the H matrix in two-particle spaces is a direct sum of two matrices $H_{f_2}(2)$, one in the $f_2 = \{2\}$ space and the other in $\{1^2\}$ space. Similarly, for the 6 particle example shown in Fig. 4.1, there are 9 f_m 's and therefore the H matrix is a direct sum of 9 matrices. It should be noted that the matrix elements of $H_{f_m}(m)$ matrices receive contributions from both $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$.

Embedded random matrix ensemble EGUE(2)- $SU(4)$ for a m fermion systems with a fixed f_m , i.e., $\{H_{f_m}(m)\}$, is generated by the ensemble of H operators given in Eq. (4.3.1) with $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$ matrices replaced by independent GUE ensem-

$\{4,2\}, 5$ 19305	0	0	0	0	0	0	0	0
0	$\{4,1^2\}, 9$ 17160	0	0	0	0	0	0	0
0	0	$\{3^2\}, 9$ 9075	0	0	0	0	0	0
0	0	0	$\{3,2,1\}, 15$ 21120	0	0	0	0	0
0	0	0	0	$\{3,1^3\}, 21$ 9240	0	0	0	0
0	0	0	0	0	$\{2^3\}, 21$ 4950	0	0	0
0	0	0	0	0	0	$\{2^2,1^2\}, 25$ 6930	0	0
0	0	0	0	0	0	0	$\{2,1^4\}, 33$ 2310	0
0	0	0	0	0	0	0	0	$\{1^6\}, 45$ 210

Figure 4.1: Direct sum matrix structure for a $SU(4)$ scalar Hamiltonian. The example in the figure is for $m = 6$ particles in $\Omega = 10$ sp orbitals. The $U(\Omega)$ irreps and the corresponding eigenvalues for the quadratic Casimir invariant of $SU(4)$ along with the dimensions for the diagonal blocks are shown in the figure. For example, for the block that corresponds to the irrep $f_m = \{3, 2, 1\}$, we have $\langle C_2[SU(4)] \rangle^{f_m} = 15$ and $d_\Omega = 21120$. As shown in the figure (with all the off-diagonal blocks having all matrix elements zero), $H(m) = \sum_{f_m} H_{f_m}(m) \oplus$ and for each diagonal block, we have a EGUE(2)- $SU(4)$ matrix ensemble labeled by (m, f_m) .

bles of random matrices,

$$\{H(2)\} = \{H_{\{2\}}(2)\}_{\text{GUE}} \oplus \{H_{\{1^2\}}(2)\}_{\text{GUE}}. \quad (4.3.4)$$

Random variables defining the real and imaginary parts of the matrix elements of $H_{f_2}(2)$ are independent Gaussian variables with zero center and variance given by (with bar representing ensemble average),

$$\overline{H_{f_2} v_2^1 v_2^2(2) H_{f_2'} v_2^3 v_2^4(2)} = \delta_{f_2 f_2'} \delta_{v_2^1 v_2^4} \delta_{v_2^2 v_2^3} (\lambda_{f_2})^2. \quad (4.3.5)$$

Also, the independence of the $\{H_{\{2\}}(2)\}$ and $\{H_{\{1^2\}}(2)\}$ GUE ensembles imply,

$$\begin{aligned} & \overline{\left[H_{\{2\}v_2^1v_2^2}(2) \right]^P \left[H_{\{1^2\}v_2^3v_2^4}(2) \right]^Q} = 0 \text{ for } P \text{ or } Q \text{ odd,} \\ & = \left\{ \overline{\left[H_{\{2\}v_2^1v_2^2}(2) \right]^P} \right\} \left\{ \overline{\left[H_{\{1^2\}v_2^3v_2^4}(2) \right]^Q} \right\} \text{ for } P \text{ and } Q \text{ even.} \end{aligned} \quad (4.3.6)$$

Action of \widehat{H} defined by Eq. (4.3.1) on m -particle basis states with a fixed f_m , along with Eqs. (4.3.4)-(4.3.6) generates EGUE(2)- $SU(4)$ ensemble $\{H_{f_m}(m)\}$; it is labeled by the $U(\Omega)$ irrep f_m with matrix dimension $d_\Omega(f_m)$.

4.3.2 Matrix structure

For a better understanding of the size of the EGUE(2)- $SU(4)$ matrices and the number of independent matrix elements they contain, let us consider the example of 8 fermions in $N = 24$ sp states. For spinless fermion systems, we have EGUE(2) with a two-particle GUE of dimension 276 and the number of independent variables [denoted by $i_2(0)$] is 76176. These generate the m fermion EGUE(2) ensemble with H matrices of dimension $d(8) = 735471$. For fermions with spin symmetry, we have EGUE(2)-s with $\Omega = 12$. This ensemble is generated by independent GUE's in two-particle spin $s = 0$ and $s = 1$ spaces with dimensions 78 and 66, respectively. Then the number of independent variables [denoted by $i_2(2)$] for this system is 10440. The H matrix dimensions for EGUE(2)-s ensembles for the 8 particle system with spins $S = 0, 1, 2, 3$ and 4 are $d(8, S) = 70785, 113256, 51480, 9009$, and 495, respectively. Going further, with $SU(4)$ symmetry we have EGUE(2)- $SU(4)$ ensembles with $\Omega = 6$. These ensembles are generated by two independent GUE's in $f_2 = \{2\}$ and $\{1^2\}$ spaces with dimensions 21 and 15 respectively. Then the number of independent variables [denoted by $i_2(4)$] for this system is 666. The H matrix dimensions for EGUE(2)- $SU(4)$ ensembles for the 8 particle system with $f_8 = \{2^2, 1^4\}, \{2^3, 1^2\}, \{2^4\}, \{3, 1^5\}, \{3, 2, 1^3\}, \{3, 2^2, 1\}, \{3^2, 1^2\}, \{3^2, 2\}, \{4, 1^4\}, \{4, 2, 1^2\}, \{4, 2^2\}, \{4, 3, 1\}$, and $\{4^2\}$ are 15, 105, 105, 21, 384, 1050, 1176, 1470, 315, 2430, 2520, 4410, and 1764, respectively. Thus i_2 will be considerably reduced as the symmetry increases (with fixed N), i.e., $i_2(4) \ll i_2(2) \ll i_2(0)$. Similarly the H matrix dimensions decrease as we go from EGUE(2) to EGUE(2)-s to EGUE(2)- $SU(4)$. For further insight, let us consider the fraction of independent ma-

trix elements $\mathcal{I}(m, f_m)$, for $m \gg 2$ for the EGUE(2)- $SU(4)$ ensemble, defined as the ratio of $i_2(4)$ to the total number (without counting the hermitian conjugates) of matrix elements,

$$\mathcal{I}(m, f_m) = \frac{i_2(4)}{[d_\Omega(f_m)]^2}. \quad (4.3.7)$$

Similarly, for EGUE(2) and EGUE(2)-s ensembles, we can define the fraction of independent matrix elements as $\mathcal{I}(m) = i_2(0)/[d(m)]^2$ and $\mathcal{I}(m, S) = i_2(2)/[d(m, S)]^2$, respectively. In our above example, for EGUE(2), EGUE(2)-s with $S = 0$ and EGUE(2)- $SU(4)$ with $f_8 = \{4^2\}$, we have $\mathcal{I} = 1.4 \times 10^{-7}$, 2×10^{-6} , and 2×10^{-4} , respectively. Therefore the H matrices with more symmetry are characterized by relatively large fraction of independent matrix elements.

Due to the two-body selection rules, many of the m -particle matrix elements of the EGUE(2)'s will be zero. In order to understand the sparse nature of the EGUE matrices we employ the sparsity index \mathbf{S} with \mathbf{S}^{-1} defined as the ratio of number of m -particle states that are directly coupled by the two-body interaction to the m -particle matrix dimension. The number of many-particle states that are coupled by the two-body interaction, i.e., the connectivity factor $K(m, f_m)$, is given by the spectral variances; see Chapter 2 and [Ja-97]. Therefore, for the EGUE(2)- $SU(4)$ ensemble,

$$\mathbf{S}^{-1}(m, f_m) = \frac{K(m, f_m)}{d_\Omega(f_m)}. \quad (4.3.8)$$

Similarly, $\mathbf{S}^{-1}(m) = K(m)/d(m)$ for EGUE(2) and $\mathbf{S}^{-1}(m, S) = K(m, S)/d(m, S)$ for EGUE(2)-s. Formulas for the $K(m)$ and $K(m, S)$ are given in ([Fl-96], [Ko-05]) and ([Ko-07], Chapter 2), respectively. For EGUE(2)- $SU(4)$, given the two-particle variances to be $\lambda_{f_2}^2 = \lambda^2$, the variances $\langle \widehat{H}^2 \rangle^{m, f_m}$ in m -particle space are $\sigma^2(m, f_m) = \lambda^2 K(m, f_m)$ with $K(m, f_m)$ propagating the two-particle variances to m -particle spaces. Results in Table 4.6 ahead give formulas for the variance propagator $K(m, f_m)$ for the $U(\Omega)$ irreps $f_m^{(p)}$. For example, $K(m = 4r, f_m = \{4^r\}) = r(\Omega - r + 4) \{2r(2\Omega - 2r + 9) - \Omega - 8\}$, and $K(m = 4r + 1, f_m = \{4^r, 1\}) = r(\Omega - r + 4) \{4r(2\Omega - 2r + 7) + 2\Omega - 15\} / 2$. For the 8 particle example (with $N = 24$) considered before, the connectivity factors K are 4284, 1440, and 864, respectively for EGUE(2), EGUE(2)-s with $S = 0$ and EGUE(2)- $SU(4)$ with $f_8 = \{4^2\}$. These give $\mathbf{S}^{-1} = 5.8 \times 10^{-3}$, 0.02, and 0.49, respectively for these

ensembles. Therefore as symmetry increases, in general, the many-particle EGUE matrices will become more dense. Consequences of this will be discussed further in Section 4.7.

4.3.3 Matrix construction

Before proceeding to the analytical formulation, we will briefly outline a method for numerical construction of EGUE(2)- $SU(4)$ ensemble for a given (Ω, m, f_m) . Consider m fermions in Ω number of sp orbitals each four-fold degenerate. Then in the spin-isospin representation, the sp states are denoted by $|i; \frac{1}{2}, m_s; \frac{1}{2}, m_t\rangle$ as discussed before, where $i = 1, 2, \dots, \Omega$. We arrange the sp states in such a way that the first Ω states have $(m_s, m_t) = (\frac{1}{2}, \frac{1}{2})$, $\Omega + 1$ to 2Ω sp states have $(m_s, m_t) = (\frac{1}{2}, -\frac{1}{2})$, $2\Omega + 1$ to 3Ω sp states have $(m_s, m_t) = (-\frac{1}{2}, \frac{1}{2})$ and $3\Omega + 1$ to 4Ω sp states have $(m_s, m_t) = (-\frac{1}{2}, -\frac{1}{2})$. In this single state representation we denote the sp states as $|k_r\rangle$, $r = 1, 2, \dots, 4\Omega$. Now distributing in all possible ways the m fermions in these 4Ω sp states will generate the m -particle configurations $\mathbf{m} = [m(k_1), m(k_2), \dots, m(k_{4\Omega})]$, with $m(k_r) = 0$ or 1 and $\sum_{r=1}^{4\Omega} m(k_r) = m$. The corresponding (M_S, M_T) values are $M_S = [\sum_{r=1}^{\Omega} m(k_{r1}) + \sum_{r=2}^{2\Omega} m(k_{r2}) - \sum_{r=3}^{3\Omega} m(k_{r3}) - \sum_{r=4}^{4\Omega} m(k_{r4})]/2$ and $M_T = [\sum_{r=1}^{\Omega} m(k_{r1}) - \sum_{r=2}^{2\Omega} m(k_{r2}) + \sum_{r=3}^{3\Omega} m(k_{r3}) - \sum_{r=4}^{4\Omega} m(k_{r4})]/2$. The m -particle H matrix in the basis defined by \mathbf{m} 's with $(M_S^{min}, M_T^{min}) = (0, 0)$ will contain states with all (S, T) values for even m and similarly with $(M_S^{min}, M_T^{min}) = (\frac{1}{2}, \frac{1}{2})$ for odd m . The dimension of this basis space, called $\mathcal{D}(M_S^{min}, M_T^{min})$, is $\sum_{f_m} d_{\Omega}(f_m) \sum_{S, T} D(\tilde{f}_m : S, T)$. In the (st) coupled representation the two-particle matrix elements of \hat{H} are

$$\langle (i, j)s, m_s, t, m_t | \hat{H} | (k, l)s', m_{s'}, t', m_{t'} \rangle .$$

As the $SU(4)$ irreps $\{2\} \rightarrow (st) = (11) \oplus (00)$ and $\{1^2\} \rightarrow (10) \oplus (01)$, it is easy to put these matrix elements in one to one correspondence with the two-body matrix elements $H_{f_2 v_2^i v_2^f}(2)$ in Eq. (4.3.1). Applying angular-momentum algebra, it is then possible to transform these matrix elements into two-body matrix elements $\langle k_c k_d | \hat{H} | k_a k_b \rangle$ in the single state representation. Then the construction of the m -particle H matrix in the \mathbf{m} -basis with (M_S^{min}, M_T^{min}) defined above reduces to the problem of EGUE(2) for spinless fermion systems. The construction of EGUE(2) for spinless fermion sys-

tems on a machine is straightforward. For instance, the dimensions of the matrices $\mathcal{D}(M_S^{min} = 0, M_T^{min} = 0)$ for $m = 6, 8$ and 12 , with $\Omega = 6$, are 17000, 79875, and 263844, respectively. On the other hand, the total m -particle matrix dimensions are $d(6) = 134596$, $d(8) = 735471$, and $d(12) = 2704156$. Therefore, the \mathbf{m} -basis formulation reduces the matrix dimensions considerably.

After constructing this matrix, it is possible to generate the H matrix defined over a fixed f_m space, for some special f_m 's easily, using the $C_2[SU(4)]$ operator as the projection operator; eigenvalues of $C_2[SU(4)]$ will in general have degeneracies with respect to f_m . Some of the special irreps that can be identified uniquely by $C_2[SU(4)]$ are the following: (a) for $m = 4r$, the irreps $\{4^r\}$, $\{4^{r-1}, 3, 1\}$ and $\{4^{r-1}, 2^2\}$ with eigenvalues 0, 8, and 12, respectively; (b) for $m = 4r + 2$, the irreps $\{4^r, 2\}$ and $\{4^{r-1}, 3, 2, 1\}$ with eigenvalues 5, and 15, respectively; (c) for $m = 4r + 1$, the irreps $\{4^r, 1\}$, $\{4^{r-1}, 3, 2\}$ and $\{4^{r-1}, 3, 1^2\}$ with eigenvalues 3, 9, and 13, respectively; and (d) for $m = 4r + 3$, the irreps $\{4^r, 3\}$, $\{4^r, 2, 1\}$, and $\{4^{r-1}, 3^2, 1\}$ with eigenvalues 3, 9, and 13, respectively. For convenience, we denote these special irreps by f_m^s . It should be noted that $f_m^{(p)}$ belong to f_m^s . For the $C_2[SU(4)]$ operator, the m -particle matrix in the \mathbf{m} -basis can be constructed by identifying the two-particle matrix elements, in single state representation, using Eqs. (4.2.8)-(4.2.11). Diagonalizing this matrix gives a direct sum of unitary matrices and the unitary matrix that corresponds to a given f_m^s can be identified from the eigenvalues of $C_2[SU(4)]$. Applying the unitary transformation defined by this unitary matrix, the m -particle H matrix with $(M_S, M_T) = (0, 0)$ for even m [$(M_S, M_T) = (\frac{1}{2}, \frac{1}{2})$ for odd m] can be transformed to the basis with good f_m^s . This method can be successfully implemented on a machine for the irreps f_m^s . Results in Section 4.2 are sufficient for constructing EGUE(2)- $SU(4)$ for these irreps. It is important to note that the $C_2[SU(4)]$ alone will not suffice to identify the matrices corresponding to all the f_m 's. To distinguish them, we need to construct the m -particle matrices for the cubic and quartic Casimir invariants of $SU(4)$ algebra and these are more complicated. Numerical investigations of EGUE(2)- $SU(4)$ by matrix construction are impractical as the dimensions $\mathcal{D}(M_S^{min}, M_T^{min})$ are prohibitively large (even for $\Omega = 6$ and $m = 6$, $\mathcal{D} = 17000$). Therefore our focus in this chapter is in developing analytical formulation for solving the EGUE(2)- $SU(4)$ ensemble (Secs. 4.4 and 4.5 and Appendix F) and using this we have carried out some numerical investigations (Secs. 4.6 and 4.7).

4.4 $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ Wigner-Racah Algebra for Solving EGUE(2)- $SU(4)$

Analytical solutions for EGUE(2)- $SU(4)$ follow, as discussed before for EGUE(k) and EGUE(2)- s (see Sec. 1.2.3 and Appendix C), from the tensorial decomposition of the \hat{H} operator [equivalently $A^\dagger A$ in Eq. (4.3.1)] with respect to $U(\Omega) \otimes SU(4)$. As \hat{H} is a $SU(4)$ scalar, it transforms as the irrep $\{0\}$ with respect to the $SU(4)$ algebra. However with respect to $SU(\Omega)$, the tensorial characters, in the Young tableaux notation, for $f_2 = \{2\}$ are $F_\nu = \{0\}, \{21^{\Omega-2}\}$ and $\{42^{\Omega-2}\}$ with $\nu = 0, 1,$ and $2,$ respectively. Note that F_ν follow from the Kronecker product of the $U(\Omega)$ irreps $\{2\}$ and $\{2^{\Omega-1}\}$ as A^\dagger and A transform as these irreps. Similarly for $f_2 = \{1^2\}, F_\nu = \{0\}, \{21^{\Omega-2}\}$ and $\{2^2 1^{\Omega-4}\}$ with $\nu = 0, 1, 2,$ respectively. Then we can introduce unitary tensors B 's,

$$B(f_2 F_\nu \omega_\nu) = \sum_{v_2^i, v_2^f, \beta_2} A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2) \langle f_2 v_2^f \bar{f}_2 \bar{v}_2^i | F_\nu \omega_\nu \rangle \langle F_2 \beta_2 \bar{F}_2 \bar{\beta}_2 | \{0\}0 \rangle, \quad (4.4.1)$$

and expand \hat{H} in terms of these tensor operators. In Eq. (4.4.1), $\langle f_2 \bar{f}_2 \bar{v}_2^i \bar{v}_2^f \rangle$ are $SU(\Omega)$ Wigner coefficients and $\langle F_2 \bar{F}_2 \bar{\beta}_2 \beta_2 \rangle$ are $SU(4)$ Wigner coefficients. Some properties of the Wigner coefficients are discussed in Appendix E. Note that in Eq. (4.4.1), irreps \bar{f}_2 are complex conjugate of the irreps f_2 [Bu-81]. For example, for the $U(\Omega)$ irrep $f = \{2^r\}$, the irrep that corresponds to \bar{f} is $\{2^{\Omega-r}\}$. Similarly, $\bar{f} = \{4^{\Omega-r}\}$ for $f = \{4^r\}$, $\bar{f} = \{4^{\Omega-r-2}, 2, 1\}$ for $f = \{4^r, 3, 2\}$ and so on. Using the orthonormal properties of the Wigner coefficients appearing in Eq. (4.4.1) and the action of operators A and A^\dagger on the vacuum and two-particle states respectively, it can be proved that the tensors B 's are orthonormal with respect to the traces over fixed f_2 spaces,

$$\langle \langle B(f_2 F_\nu \omega_\nu) B(f_2' F_\nu' \omega_\nu') \rangle \rangle^{f_2} = \delta_{f_2 f_2'} \delta_{F_\nu F_\nu'} \delta_{\omega_\nu \omega_\nu'}. \quad (4.4.2)$$

Expanding \hat{H} in terms of B 's will give the expansion coefficients W 's,

$$\hat{H} = \sum_{f_2, F_\nu, \omega_\nu} W(f_2 F_\nu \omega_\nu) B(f_2 F_\nu \omega_\nu), \quad (4.4.3)$$

and they can be written in terms of the $H(2)$ matrix elements using Eq. (4.4.2),

$$\begin{aligned} W(f_2 F_\nu \omega_\nu) &= \langle \langle \widehat{H} B(f_2 F_\nu \omega_\nu) \rangle \rangle^{f_2} \\ &= \sum_{v_2^i, v_2^f} \sqrt{d_4(\tilde{f}_2)} \langle f_2 v_2^f \overline{f_2 v_2^i} | F_\nu \omega_\nu \rangle H_{f_2 v_2^i v_2^f}(2). \end{aligned} \quad (4.4.4)$$

Now the most significant result is that the W 's are also independent Gaussian variables just as $H(2)$'s with ensemble averaged variances given by,

$$\overline{W(f_2 F_\nu \omega_\nu) W(f_2' F_\nu' \omega_\nu')} = \delta_{f_2 f_2'} \delta_{F_\nu F_\nu'} \delta_{\omega_\nu \omega_\nu'} (\lambda_{f_2})^2 d_4(F_2). \quad (4.4.5)$$

Above result is derived using Eq. (4.3.5) and (4.4.4). As we will see ahead, Eq. (4.4.5) and the (m, f_m) -space matrix elements of H as given by the Wigner-Eckart theorem applied using $SU(\Omega) \otimes SU(4)$ Wigner-Racah algebra, will completely solve EGUE(2)- $SU(4)$.

Analysis of the random matrix ensemble EGUE(2)- $SU(4)$ involves construction of the one-point function $\overline{\rho^{m, f_m}(E)}$, the ensemble averaged density of eigenvalues given by Eq. (2.3.1) with $\Gamma = f_m$ and the two-point and other higher point functions defining fluctuations. The two-point function is given by,

$$S^{m, \Gamma: m', \Gamma'}(E, E') = \overline{\rho^{m, \Gamma}(E) \rho^{m', \Gamma'}(E')} - \left\{ \overline{\rho^{m, \Gamma}(E)} \right\} \left\{ \overline{\rho^{m', \Gamma'}(E')} \right\}, \quad (4.4.6)$$

with $\rho^{m, \Gamma}(E)$ defining fixed- (m, Γ) density of eigenvalues. The two-point function $S^{m, \Gamma: m', \Gamma'}$ generates 'self-correlations' when $m = m'$ and $\Gamma = \Gamma'$ and 'cross-correlations' between states with $m \neq m'$ and/or $\Gamma \neq \Gamma'$. For EGUE(2)- $SU(4)$ ensemble, $\Gamma = f_m$. In Chapter 6, Γ corresponds to the m -particle spin S . Therefore, for EGUE(2)- $SU(4)$, the two-point function $S^{m f_m: m' f_{m'}}$ generates self-correlations when $m = m'$ and $f_m = f_{m'}$ and cross-correlations between states with $m = m'$ and $f_m \neq f_{m'}$ and also between states with $m \neq m'$ and $f_m \neq f_{m'}$. It should be emphasized that with $m = m'$ it is possible to have $f_m \neq f_{m'}$ and this should not be confused as $f_m = f_{m'}$ (confusion may arise if one substitutes the numerical value for $m = m'$). Towards deriving the forms for the one and two-point functions (discussion of higher point functions is beyond the scope of the present thesis), the moment approach is adopted and the lower or-

der moments are analyzed. By definition, all odd moments of $\overline{\rho^{m,f_m}(E)}$ will vanish and therefore the lower order moments of interest are the ensemble averaged spectral variances $\overline{\langle \hat{H}^2 \rangle^{m,f_m}}$ and the fourth moment $\overline{\langle \hat{H}^4 \rangle^{m,f_m}}$ giving the excess parameter $\gamma_2(m, f_m)$ where,

$$\gamma_2(m, f_m) = \left[\overline{\langle \hat{H}^2 \rangle^{m,f_m}} \right]^{-2} \left[\overline{\langle \hat{H}^4 \rangle^{m,f_m}} \right] - 3. \quad (4.4.7)$$

Similarly the two lower order normalized bivariate moments of the two-point function are Σ_{rr} , $r = 1, 2$ give the covariances in energy centroids and spectral variances respectively. The formulas for these are given by,

$$\begin{aligned} \Sigma_{11}(m, \Gamma; m', \Gamma') &= \frac{\overline{\langle H \rangle^{m, \Gamma} \langle H \rangle^{m', \Gamma'}}}{\left[\left\{ \overline{\langle H^2 \rangle^{m, \Gamma}} \right\} \left\{ \overline{\langle H^2 \rangle^{m', \Gamma'}} \right\} \right]^{1/2}}, \\ \Sigma_{22}(m, \Gamma; m', \Gamma') &= \frac{\overline{\langle [H]^2 \rangle^{m, \Gamma} \langle H^2 \rangle^{m', \Gamma'}}}{\left\{ \overline{\langle H^2 \rangle^{m, \Gamma}} \right\} \left\{ \overline{\langle H^2 \rangle^{m', \Gamma'}} \right\}} - 1, \end{aligned} \quad (4.4.8)$$

with $\Gamma = f_m$ and $\Gamma' = f_{m'}$ for EGUE(2)-SU(4). For $m = m'$ and $f_m = f_{m'}$, the Σ_{11} and Σ_{22} give the first two terms in the normal mode decomposition of the level motion in the ensemble [Br-81, Pa-00] and hence they are of importance. Similarly for ($m = m'$, $f_m \neq f_{m'}$) and ($m \neq m'$, $f_m \neq f_{m'}$), the Σ_{11} and Σ_{22} are important as they generate non-zero cross-correlations that are zero if the m -particle H matrices for each f_m are represented by independent GUE's.

In order to derive the analytical results for the moments of the one and two-point functions, the basic quantity that is needed is the ensemble averaged covariance between any two m -particle matrix elements of H , i.e.,

$$\begin{aligned} &\overline{H_{f_m v_m^i v_m^f} H_{f_{m'} v_{m'}^i v_{m'}^f}} \\ &= \overline{\langle f_m F_m v_m^f \beta \mid \hat{H} \mid f_m F_m v_m^i \beta \rangle \langle f_{m'} F_{m'} v_{m'}^f \beta' \mid \hat{H} \mid f_{m'} F_{m'} v_{m'}^i \beta' \rangle}. \end{aligned} \quad (4.4.9)$$

Using the expansion given by Eq. (4.4.3) and applying Eq. (4.4.5) for the ensemble

average of the product of two W 's, $\overline{H\overline{H}}$ reduces to the matrix elements of the unit tensors B 's. Wigner-Eckart theorem in $SU(\Omega)$ and $SU(4)$ spaces will give [He-74a],

$$\begin{aligned}
& \langle f_m F_m v_m^f \beta | B(f_2 F_v \omega_v) | f_m F_m v_m^i \beta \rangle \\
&= \sum_{\rho} \langle f_m || B(f_2 F_v) || f_m \rangle_{\rho} \langle f_m v_m^i F_v \omega_v | f_m v_m^f \rangle_{\rho} \\
&= \frac{1}{\sqrt{d_{\Omega}(f_2) d_4(\tilde{f}_2)}} \sum_{\rho} \langle f_m ||| B(f_2 F_v) ||| f_m \rangle_{\rho} \langle f_m v_m^i F_v \omega_v | f_m v_m^f \rangle_{\rho}; \tag{4.4.10}
\end{aligned}$$

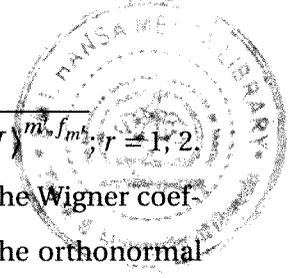
$$\langle f_m ||| B(f_2 F_v) ||| f_m \rangle_{\rho} = \sum_{f_{m-2}} F(m) \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{U(f_m \overline{f_2} f_m f_2; f_{m-2} F_v)_{\rho}}{U(f_m \overline{f_2} f_m f_2; f_{m-2}(0))},$$

where the summation is over the multiplicity index ρ and this arises as $f_m \otimes F_v$ gives in general more than once the irrep f_m . In Eq. (4.4.10), $F(m) = -m(m-1)/2$ and $U(---)$ are the $SU(\Omega)$ Racah coefficients. Similarly, the standard double-barred matrix elements (called reduced matrix elements) are changed into triple-barred matrix elements in Eq. (4.4.10) for convenience. The formula for the dimension $d_{\Omega}(f_m)$ is given by Eq. (4.3.2) and the dimension \mathcal{N}_{f_m} of f_m with respect to the S_m group is [Wy-70],

$$\mathcal{N}_{f_m} = \frac{m! \prod_{i < k=1}^r (\ell_i - \ell_k)}{\ell_1! \ell_2! \dots \ell_r!}; \quad \ell_i = f_i + r - i. \tag{4.4.11}$$

Here, r denotes total number of rows in the Young tableaux for f_m . Correlations generated by EGUE(2)- $SU(4)$ between states with (m, f_m) and $(m', f_{m'})$ follow from the covariances between the m -particle matrix elements of H . Applying Eqs. (4.4.9), (4.4.3), (4.4.5) and (4.4.10) in that order, the final expression for $\overline{H\overline{H}}$ is,

$$\begin{aligned}
& \overline{H_{f_m v_m^i v_m^f} H_{f_{m'} v_{m'}^i v_{m'}^f}} \\
&= \sum_{f_2, F_v, \omega_v} \frac{(\lambda_{f_2})^2}{d_{\Omega}(f_2)} \sum_{\rho, \rho'} \langle f_m ||| B(f_2 F_v) ||| f_m \rangle_{\rho} \langle f_{m'} ||| B(f_2 F_v) ||| f_{m'} \rangle_{\rho'} \tag{4.4.12} \\
& \times \langle f_m v_m^i F_v \omega_v | f_m v_m^f \rangle_{\rho} \langle f_{m'} v_{m'}^i F_v \omega_v | f_{m'} v_{m'}^f \rangle_{\rho'}.
\end{aligned}$$



In the following section, we will consider $\overline{\langle \hat{H}^2 \rangle^{m, f_m}}$ and $\overline{\langle \hat{H}^r \rangle^{m, f_m} \langle \hat{H}^r \rangle^{m', f_{m'}}}$, $r = 1, 2$. It is important to mention here that in evaluating these moments, the Wigner coefficients appearing in Eq. (4.4.12) will eventually disappear due to the orthonormal properties of these coefficients [see Eqs. (E6a) and (E6b)] and therefore the final results for these moments will involve only the $SU(\Omega)$ Racah coefficients given in Eq. (4.4.10). In Appendix F, we will consider $\overline{\langle \hat{H}^4 \rangle^{m, f_m}}$ and the algebra here is more complicated giving additional Racah coefficients than in Eq. (4.4.10).

From now onwards, we drop the “hat” symbol over the H operator when there is no confusion.

4.5 Exact Expressions for Spectral Variances, Lower Order Cross-correlations and Analytical Results for Lowest $U(\Omega)$ Irreps

4.5.1 Covariances in energy centroids $\overline{\langle H \rangle^{m, f_m} \langle H \rangle^{m', f_{m'}}$

Firstly the ensemble averaged energy centroid $\overline{\langle H \rangle^{m, f_m}} = 0$ by the definition of EGUE(2)- $SU(4)$ ensemble. As $\langle H \rangle^{m, f_m}$ is the trace of H (divided by the dimensionality) in (m, f_m) space, only $F_v = \{0\}$ will generate this. Therefore for $\overline{\langle H \rangle \langle H \rangle}$, the Wigner coefficients in Eq. (4.4.12) and the ratio of the U -coefficients in Eq. (4.4.10) will be unity. Then trivially,

$$\begin{aligned} \overline{\langle H \rangle^{m, f_m} \langle H \rangle^{m', f_{m'}}} &= F(m) F(m') \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{f_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \sum_{f_{m'-2}} \frac{\mathcal{N}_{f_{m'-2}}}{\mathcal{N}_{f_{m'}}} \\ &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'}), \end{aligned} \quad (4.5.1)$$

where,

$$P^{f_2}(m, f_m) = F(m) \sum_{f_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}}. \quad (4.5.2)$$

Table 4.3 gives the expression for $P^{f_2}(m, f_m)$ for the irreps $f_m^{(p)}$. It is possible to derive Eq. (4.5.1) using the trace propagation formula for the energy centroids [Pa-78],

$$E_c(m, f_m) = \langle H \rangle^{m, f_m} = a_0 + a_1 m + a_2 m^2 + a_3 \langle C_2[SU(4)] \rangle^{\tilde{f}_m}$$

$$\begin{aligned} \Rightarrow E_c(m, f_m) &= \frac{3m^2 + 12m - 4 \langle C_2[SU(4)] \rangle^{f_m}}{16} \langle H \rangle^{2, \{2\}} \\ &+ \frac{5m^2 - 20m + 4 \langle C_2[SU(4)] \rangle^{f_m}}{16} \langle H \rangle^{2, \{1^2\}}. \end{aligned} \quad (4.5.3)$$

Note that $\langle C_2[SU(4)] \rangle^{f_m} = \langle C_2[U(4)] \rangle^{f_m} - m^2/4$ with $\langle C_2[U(4)] \rangle^{f_m}$ given by Eq. (4.2.21).

We have verified that Eq. (4.5.3) reproduces the results given in Table 4.3.

Table 4.3: $P^{f_2}(m, f_m)$ for $f_m = f_m^{(p)} = \{4^r, p\}$ and $f_2 = \{2\}$ and $\{1^2\}$. See Eq. (4.5.2) for the definition of $P^{f_2}(m, f_m)$.

$f_m^{(p)}$	$P^{f_2}(m, f_m^{(p)})$	
	$f_2 = \{2\}$	$f_2 = \{1^2\}$
$\{4^r\}$	$-3r(r+1)$	$-5r(r-1)$
$\{4^r, 1\}$	$-\frac{3r}{2}(2r+3)$	$-\frac{5r}{2}(2r-1)$
$\{4^r, 2\}$	$-(3r^2+6r+1)$	$-5r^2$
$\{4^r, 3\}$	$-\frac{3}{2}(r+2)(2r+1)$	$-\frac{5r}{2}(2r+1)$

4.5.2 Spectral variances $\overline{\langle H^2 \rangle^{m, f_m}}$

Writing $\overline{\langle H^2 \rangle^{m, f_m}}$ explicitly in terms of the m -particle H matrix elements,

$$\overline{\langle H^2 \rangle^{m, f_m}} = \frac{1}{d_\Omega(f_m)} \sum_{v_m^1, v_m^2} \overline{H_{f_m v_m^1 v_m^2} H_{f_m v_m^2 v_m^1}}, \quad (4.5.4)$$

and then applying Eqs. (4.4.10) and (4.4.12) and the orthonormal properties of the $SU(\Omega)$ Wigner coefficients (see Appendix E) lead to

$$\begin{aligned} \overline{\langle H^2 \rangle^{m, f_m}} &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{v=0,1,2} \sum_{\rho} \left| \langle f_m ||| B(f_2, \mathbf{F}_v) ||| f_m \rangle_{\rho} \right|^2 \\ &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{v=0,1,2} \mathcal{Q}^v(f_2 : m, f_m). \end{aligned} \quad (4.5.5)$$

The functions $\mathcal{Q}^\nu(f_2 : m, f_m)$ involve $SU(\Omega)$ U -coefficients and the explicit expression is,

$$\begin{aligned} \mathcal{Q}^\nu(f_2 : m, f_m) &= [F(m)]^2 \sum_{f_{m-2}, f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}} \mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m} \mathcal{N}_{f_m}} X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu); \\ X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) &= \\ \sum_{\rho} \frac{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \mathbf{F}_\nu)_\rho U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \mathbf{F}_\nu)_\rho}{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \{0\}) U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \{0\})}. \end{aligned} \quad (4.5.6)$$

Tabulations for X_{UU} (also for Y_{UU} defined ahead) or equivalently $SU(\Omega)$ U -coefficients, though in a complex form, are available in [He-74a]. However to gain insight into the spectral variances and the cross-correlations Σ_{rr} , we derive analytical results by restricting ourselves to the physically relevant (in nuclear structure; see Section 4.2) irreps $f_m^{(p)}$.

Summation over the multiplicity index ρ appearing in Eq. (4.5.6) [also Eq. (4.5.17) ahead] arises naturally in applications to physical problems as all the physically relevant results should be independent of ρ which is a label for equivalent $SU(\Omega)$ irreps. Hecht derived formulas for the sums in X_{UU} (also Y_{UU} defined ahead) in the context of spectral distribution methods in nuclei [He-74a]. Tabulations for $X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ are collected in Table 4.4 and they are given in terms of the so-called axial distances τ_{ij} for the boxes i and j in a given Young tableaux. Given a Young tableaux $\{f_m\}$, the axial distance τ_{ij} between the last box in row i and the last box in row j is $\tau_{ij} = f_i - f_j + j - i$, with f_k being the number of boxes in the row k . The f_{m-2} irreps are obtained by removing the two-particle symmetric ($f_2 = \{2\}$) or anti-symmetric ($f_2 = \{1^2\}$) irreps from f_m . Figure 4.2 shows all the allowed f_{m-2} 's for the irreps $f_m^{(p)}$. In the figure, a and b (or c) denote the last boxes in the rows a and b (or c), respectively, that are to be removed from the Young tableaux $\{4^r, p\}$ to obtain the allowed f_{m-2} irreps for $f_2 = \{2\}$ and $\{1^2\}$. It is seen that unlike for EGUE(2)-s studied in [Ko-07], for the EGUE(2)- $SU(4)$ ensemble we need a much wider variety of X_{UU} 's. Results in Table 4.4 (also Table 4.7) for any f_m are given in terms of the following func-

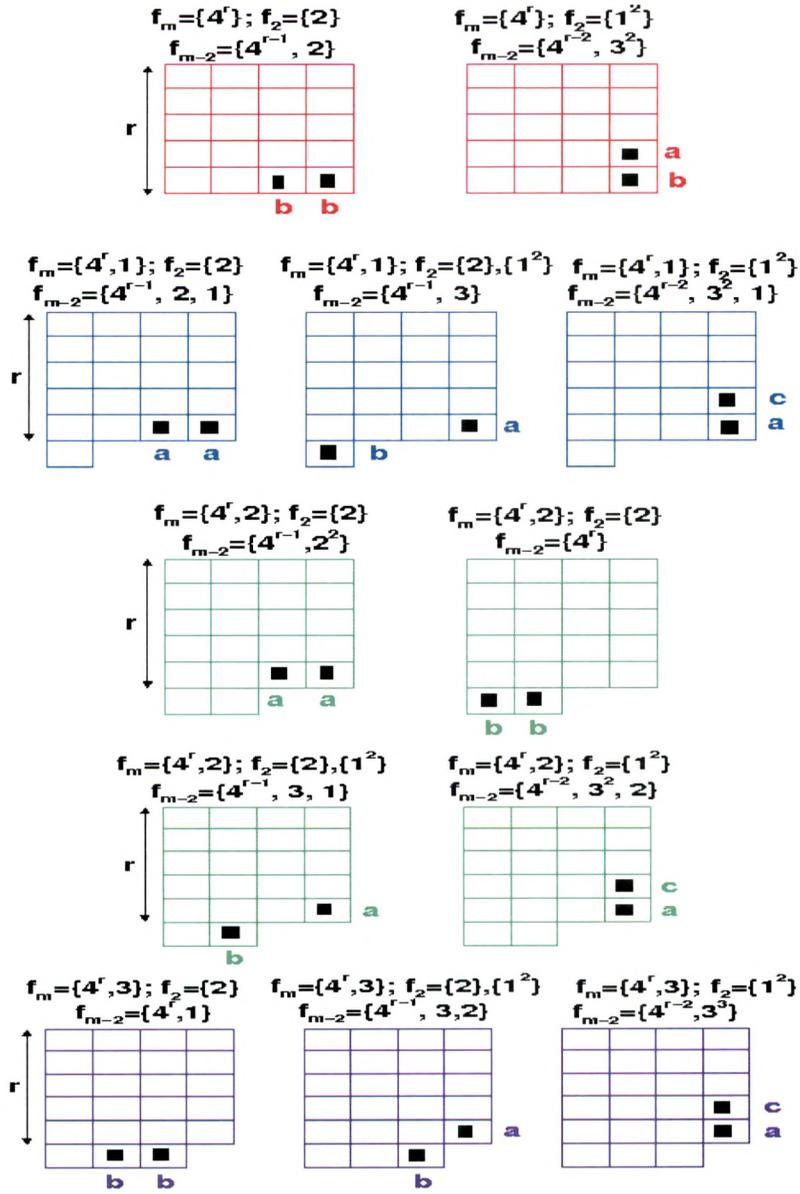


Figure 4.2: Schematic representation of the Young tableaux $f_m = f_m^{(p)} = \{4^r, p\}$ with $p = 0, 1, 2$ and 3 . Shown are the boxes with filled squares denoted by a , b and c whose removal from the irrep f_m generates the irreps f_{m-2} by action of $\overline{f_2}$ where $f_2 = \{2\}$ and $\{1^2\}$.

tions,

$$\Pi_a^{(b)} = \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b} (1 - 1/\tau_{ai}) , \quad (4.5.7a)$$

$$\Pi_b^{(a)} = \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b} (1 - 1/\tau_{bi}) .$$

$$\Pi'_a = \prod_{i=1,2,\dots,\Omega; i \neq a} (1 - 1/\tau_{ai}), \quad (4.5.7b)$$

$$\Pi''_a = \prod_{i=1,2,\dots,\Omega; i \neq a} (1 - 2/\tau_{ai}).$$

$$\Pi_a^{(bc)} = \prod_{i=1,2,\dots,\Omega; i \neq a, i \neq b, i \neq c} (1 - 1/\tau_{ai}); \quad a \neq b \neq c. \quad (4.5.7c)$$

In [He-74a], the functions $\Pi_a^{(b)}$ and $\Pi_b^{(a)}$ are called Π_a and Π_b , respectively and sometimes this (Π_a, Π_b) notation is confusing. Further, we have introduced the functions Π'_a, Π''_a and $\Pi_a^{(bc)}$. These and the notation $\Pi_a^{(b)}$ and $\Pi_b^{(a)}$ simplify considerably the formulas given by Hecht [He-74a] and therefore the results in Table 4.4 (also Table 4.7) are much easier to use in practice. Table 4.5 gives $\tau_{ab}, \Pi_a^{(b)}, \Pi_b^{(a)}, \Pi'_a, \Pi''_a$ and $\Pi_a^{(bc)}$ for the irreps $f_m^{(p)}$ which are required for deriving analytical formulas for the corresponding $X_{UU}(f_2; f_{m-2}, f'_{m-2}; F_\nu)$ and also $Y_{UU}(f_{m-2}, f'_{m-2}; F_\nu)$ defined ahead. Also given in the table are $\mathcal{N}_{f_{m-2}}/\mathcal{N}_{f_m}$ obtained by simplifying Eq. (4.4.11). Combining the results in Tables 4.4 and 4.5 and carrying out simplifications, final formulas for $\overline{\langle H^2 \rangle^{m, f_m}}$ are obtained and they are given in Table 4.6. In principle, the operator generating $\langle H^2 \rangle^{m, f_m}$ for any two or (1+2)-body H , will be a polynomial of maximum body rank 4 in the number operator \hat{n} and the quadratic, cubic and quartic invariants of the $SU(4)$ algebra. The expansion coefficients in the resulting formula will involve $\langle H^2 \rangle^{m, f_m}$ with $m = 0$ to 4 [Pa-73, Pa-72] and they can be calculated by constructing the ensemble, for a fixed Ω , on a computer. Using these inputs, the propagation equation can be used to compute spectral variances for any (m, f_m) . However Eqs. (4.5.5) and (4.5.6) give the ensemble averaged variances directly in terms of $SU(\Omega)$ U -coefficients.

Table 4.4: Formulas for $X_{UU}(f_2; f_{m-2}, f'_{m-2}; F_\nu)$ defined in Eq. (4.5.6). Note that $\{f(ab)\}\{f(ab)\}$ entries satisfy the $a \leftrightarrow b$ symmetry correctly. Similarly the entries $\{f(ab)\}\{f(ac)\}$ are independent of $b \leftrightarrow c$ interchange as required by the X_{UU} function. See text for details.

$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{1^2\}; f_{m-2}, f'_{m-2}; \{2, 1^{\Omega-2}\})$
$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega(\Omega-1)}{2(\Omega-2)} \left\{ \left(1 + \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_b^{(a)}} + \left(1 - \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \right\}$

Table 4.4 – continued

$\{f(ab)\}\{f(ac)\}$	$\frac{\Omega(\Omega-1)}{2(\Omega-2)} \left\{ \frac{1}{\Pi_a^{(bc)}} - \frac{4}{\Omega} \right\}$
$\{f(cd)\}\{f(ab)\}$	$\frac{2(\Omega-1)}{(\Omega-2)}$
$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{1^2\}; f_{m-2}, f'_{m-2}; \{2^2, 1^{\Omega-4}\})$
$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega}{(\Omega-2)} \left\{ 1 + \frac{(\Omega-1)(\Omega-2)}{2\Pi_a^{(b)}\Pi_b^{(a)}} - \frac{(\Omega-1)}{2} \right.$ $\times \left. \left[\left(1 + \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_b^{(a)}} + \left(1 - \frac{1}{\tau_{ab}}\right) \frac{1}{\Pi_a^{(b)}} \right] \right\}$
$\{f(ab)\}\{f(ac)\}$	$\frac{\Omega(\Omega-1)}{2(\Omega-2)} \left\{ \frac{2}{\Omega-1} - \frac{1}{\Pi_a^{(bc)}} \right\}$
$\{f(cd)\}\{f(ab)\}$	$\frac{\Omega}{(\Omega-2)}$
$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{2\}; f_{m-2}, f'_{m-2}; \{2, 1^{\Omega-2}\})$
$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega(\Omega+1)}{2(\Omega+2)} \left\{ \frac{(\tau_{ab}-1)^2}{\tau_{ab}(\tau_{ab}+1)} \frac{1}{\Pi_b^{(a)}} + \frac{(\tau_{ab}+1)^2}{\tau_{ab}(\tau_{ab}-1)} \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \right\}$
$\{f(aa)\}\{f(aa)\}$	$\frac{2\Omega(\Omega+1)}{(\Omega+2)} \left\{ \frac{1}{\Pi'_a} - \frac{1}{\Omega} \right\}$
$\{f(aa)\}\{f(bb)\}$	$\frac{2(\Omega+1)}{(\Omega+2)}$
or	
$\{f(aa)\}\{f(bc)\}$	$\frac{\Omega(\Omega+1)}{(\Omega+2)} \left\{ \frac{(\tau_{ab}+1)}{(\tau_{ab}-1)} \frac{1}{\Pi_a^{(b)}} - \frac{2}{\Omega} \right\}$
$\{f(aa)\}\{f(ab)\}$	
$\{f(ab)\}\{f(ac)\}$	$\frac{\Omega(\Omega+1)}{2(\Omega+2)} \left\{ \frac{(\tau_{ab}+1)(\tau_{ac}+1)}{(\tau_{ab}-1)(\tau_{ac}-1)} \frac{1}{\Pi_a^{(bc)}} - \frac{4}{\Omega} \right\}$
$\{f(ab)\}\{f(cd)\}$	$\frac{2(\Omega+1)}{(\Omega+2)}$
$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{2\}; f_{m-2}, f'_{m-2}; \{4, 2^{\Omega-2}\})$

Table 4.4 – continued

$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega(\Omega+1)}{2} \left\{ \frac{1}{\Pi_a^{(b)} \Pi_b^{(a)}} + \frac{2}{(\Omega+1)(\Omega+2)} \right.$
	$\left. - \frac{1}{(\Omega+2)} \left[\frac{(\tau_{ab}-1)^2}{\tau_{ab}(\tau_{ab}+1)} \frac{1}{\Pi_b^{(a)}} + \frac{(\tau_{ab}+1)^2}{\tau_{ab}(\tau_{ab}-1)} \frac{1}{\Pi_a^{(b)}} \right] \right\}$
$\{f(aa)\}\{f(aa)\}$	$\frac{\Omega}{(\Omega+2)} \left\{ 1 - 2(\Omega+1) \frac{1}{\Pi'_a} + \frac{(\Omega+1)(\Omega+2)}{2} \frac{1}{\Pi''_a} \right\}$
$\{f(aa)\}\{f(bb)\}$	
or	$\frac{\Omega}{(\Omega+2)}$
$\{f(aa)\}\{f(bc)\}$	
$\{f(aa)\}\{f(ab)\}$	$\frac{\Omega}{(\Omega+2)} \left\{ 1 - \frac{(\Omega+1)(\tau_{ab}+1)}{(\tau_{ab}-1)\Pi_a^{(b)}} \right\}$
$\{f(ab)\}\{f(ac)\}$	$\frac{\Omega}{(\Omega+2)} \left\{ 1 - \frac{(\Omega+1)(\tau_{ab}+1)(\tau_{ac}+1)}{2(\tau_{ab}-1)(\tau_{ac}-1)} \frac{1}{\Pi_a^{(bc)}} \right\}$
$\{f(ab)\}\{f(cd)\}$	$\frac{\Omega}{(\Omega+2)}$

4.5.3 Cross-correlations in energy centroids $\Sigma_{11}(m, f_m; m', f_{m'})$

Analysis of the random matrix ensembles with various symmetries involves construction of the one-point function $\overline{\rho^{m,\Gamma}(E)}$ given by Eq. (2.3.1) and the two-point and other higher point functions defining fluctuations. Covariances in energy centroids $\Sigma_{11}(m, f_m; m', f_{m'})$ follow from Eqs. (4.4.8), (4.5.1) and (4.5.5),

$$\Sigma_{11}(m, f_m; m', f_{m'}) = \frac{\sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'})}{\left[\left\{ \langle H^2 \rangle^{m, f_m} \right\} \left\{ \langle H^2 \rangle^{m', f_{m'}} \right\} \right]^{1/2}}. \quad (4.5.8)$$

For the irreps $f_m^{(p)}$, formulas for the functions $P^{f_2}(m, f_m)$ and the variances $\overline{\langle H^2 \rangle^{m, f_m}}$ are given in Tables 4.3 and 4.6, respectively. Table 4.4 gives X_{UU} required for calculating the covariances for any general f_m . To gain some insight into the structure

Table 4.5: Axial distances τ_{ab} and the functions $\Pi_a^{(b)}$, $\Pi_b^{(a)}$, Π'_a and Π''_a for the irreps $f_m^{(p)}$ shown in Fig. 4.2. For the situations with $f(ac)$, $\tau_{ab} \rightarrow \tau_{ac}$, $\Pi_a^{(b)} \rightarrow \Pi_a^{(c)}$ and $\Pi_b^{(a)} \rightarrow \Pi_b^{(c)}$. Also for $f(bb)$, $\Pi'_a \rightarrow \Pi'_b$ and $\Pi''_a \rightarrow \Pi''_b$. For $f_2 = \{1^2\}$ and for $f_m^{(p)} = \{4^r, p\}$ with $p \neq 0$, we need $\Pi_a^{(bc)}$ (see Fig. 4.2 for a , b and c) and for the examples in the Table, we have $\Pi_a^{(bc)} = 5r/[2(4 + \Omega - r)]$.

$f_m^{(p)}$	f_2	f_{m-2}	τ_{ab}	$\frac{1}{\Pi_a^{(b)}}$	$\frac{1}{\Pi_b^{(a)}}$	$\frac{1}{\Pi'_a}$	$\frac{1}{\Pi''_a}$	$\frac{1}{\Pi_a^{(bc)}}$	$\frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}}$
$\{4^r\}$	$\{2\}$	$\{4^{r-1}, 2\}$	-	-	-	$\frac{4 + \Omega - r}{4r}$	$\frac{(4 + \Omega - r)(3 + \Omega - r)}{6r(r + 1)}$	$\frac{3(r + 1)}{2(4r - 1)}$	
		$\rightarrow f(bb)$							
		$\{4^{r-2}, 3^2\}$	+1	$\frac{5 + \Omega - r}{5(r - 1)}$	$\frac{4 + \Omega - r}{2r}$	-	-	$\frac{5(r - 1)}{2(4r - 1)}$	
$\{4^r, 1\}$	$\{2\}$	$\rightarrow f(ab)$							
		$\{4^{r-1}, 2, 1\}$	-	-	$\frac{4(4 + \Omega - r)}{15r}$	$\frac{(4 + \Omega - r)(3 + \Omega - r)}{5r(r + 1)}$	$\frac{5(r + 1)}{4(4r + 1)}$		
		$\rightarrow f(aa)$							
	$\{1^2\}$	$\{4^{r-1}, 3\}$	+4	$\frac{4 + \Omega - r}{5r}$	$\frac{5(\Omega - r)}{r + 4}$	-	-	$\frac{r + 4}{4(4r + 1)}$	
		$\rightarrow f(ab)$							
		$\{4^{r-1}, 3\}$	+4	$\frac{4 + \Omega - r}{5r}$	$\frac{5(\Omega - r)}{r + 4}$	-	-	$\frac{r + 4}{4(4r + 1)}$	
	$\{1^2\}$	$\rightarrow f(ab)$							
		$\{4^{r-2}, 3^2, 1\}$	-1	$\frac{8(4 + \Omega - r)}{15r}$	$\frac{5(5 + \Omega - r)}{24(r - 1)}$	-	-	$\frac{9(r - 1)}{4(4r + 1)}$	
		$\rightarrow f(ac)$							

Table 4.5 – (continued)

$f_m^{(\varphi)}$	f_2	f_{m-2}	τ_{ab}	$\frac{1}{\Pi_a^{(b)}}$	$\frac{1}{\Pi_b^{(a)}}$	$\frac{1}{\Pi_a'}$	$\frac{1}{\Pi_a''}$	$\frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}}$
$\{4^r, 2\}$	$\{2\}$	$\{4^r\}$	-	-	-	$\frac{3(1+\Omega-r)}{2(r+3)}$	$\frac{6(\Omega-r)(1+\Omega-r)}{(r+3)(r+4)}$	$\frac{(r+3)(r+4)}{6(4r+1)(4r+2)}$
		$\rightarrow f(bb)$						
		$\{4^{r-1}, 2^2\}$	-	-	-	$\frac{3(4+\Omega-r)}{10r}$	$\frac{3(3+\Omega-r)(4+\Omega-r)}{10r(r+1)}$	$\frac{10r(r+1)}{3(4r+1)(4r+2)}$
		$\rightarrow f(aa)$						
		$\{4^{r-1}, 3, 1\}$	+3	$\frac{4+\Omega-r}{5r}$	$\frac{2(1+\Omega-r)}{r+3}$	-	-	$\frac{5r(r+3)}{2(4r+1)(4r+2)}$
$\{1^2\}$	$\{1^2\}$	$\rightarrow f(ab)$						
		$\{4^{r-1}, 3, 1\}$	+3	$\frac{4+\Omega-r}{5r}$	$\frac{2(1+\Omega-r)}{r+3}$	-	-	$\frac{5r(r+3)}{2(4r+1)(4r+2)}$
		$\rightarrow f(ab)$						
		$\{4^{r-2}, 3^2, 2\}$	-1	$\frac{3(4+\Omega-r)}{5r}$	$\frac{2(5+\Omega-r)}{9(r-1)}$	-	-	$\frac{15r(r-1)}{2(4r+1)(4r+2)}$
		$\rightarrow f(ac)$						
$\{4^r, 3\}$	$\{2\}$	$\{4^r, 1\}$	-	-	-	$\frac{2(2+\Omega-r)}{3(r+2)}$	$\frac{(1+\Omega-r)(2+\Omega-r)}{(r+2)(r+3)}$	$\frac{(r+2)(r+3)}{(4r+2)(4r+3)}$
		$\rightarrow f(bb)$						
		$\{4^{r-1}, 3, 2\}$	+2	$\frac{4+\Omega-r}{5r}$	$\frac{(2+\Omega-r)}{r+2}$	-	-	$\frac{5r(r+2)}{(4r+2)(4r+3)}$
		$\rightarrow f(ab)$						
		$\{4^{r-1}, 3, 2\}$	+2	$\frac{4+\Omega-r}{5r}$	$\frac{(2+\Omega-r)}{r+2}$	-	-	$\frac{5r(r+2)}{(4r+2)(4r+3)}$
$\{1^2\}$	$\{1^2\}$	$\rightarrow f(ab)$						
		$\{4^{r-2}, 3^3\}$	-1	$\frac{4(4+\Omega-r)}{5r}$	$\frac{(5+\Omega-r)}{4(r-1)}$	-	-	$\frac{5r(r-1)}{(4r+2)(4r+3)}$
		$\rightarrow f(ac)$						

of $\Sigma_{11}(m, f_m; m', f_{m'})$, we consider the dilute limit defined by $\Omega \rightarrow \infty$, $r \gg 1$ and $r/\Omega \rightarrow 0$. Then the variance formulas in Table 4.6 take a simple form for all $f_m^{(p)}$,

$$\overline{\langle H^2 \rangle^{m, f_m^{(p)}}} = -\frac{\Omega^2}{2} \left[\lambda_{\{2\}}^2 P^{\{2\}}(m, f_m^{(p)}) + \lambda_{\{1^2\}}^2 P^{\{1^2\}}(m, f_m^{(p)}) \right]. \quad (4.5.9)$$

Combining Eqs. (4.5.8) and (4.5.9), we have

$$\Sigma_{11}(m, f_m^{(p)}; m', f_{m'}^{(p)}) \xrightarrow{\Omega \rightarrow \infty, r \gg 1} \frac{4}{\Omega^4} \frac{\sum_{f_2} \lambda_{f_2}^2 P^{f_2}(m, f_m^{(p)}) P^{f_2}(m', f_{m'}^{(p)})}{\left[\left\{ \sum_{f_2} \lambda_{f_2}^2 P^{f_2}(m, f_m^{(p)}) \right\} \left\{ \sum_{f_2} \lambda_{f_2}^2 P^{f_2}(m', f_{m'}^{(p)}) \right\} \right]^{1/2}}. \quad (4.5.10)$$

Thus, $\Sigma_{11}(m, f_m^{(p)}; m', f_{m'}^{(p)})$ will be zero as $\Omega \rightarrow \infty$ and there will be no cross-correlations. However for finite Ω , there will be correlations between energy centroids of different states and some examples are discussed ahead.

Table 4.6: Ensemble averaged spectral variances $\overline{\langle H^2 \rangle^{m, f_m^{(p)}}}$ for various $f_m = f_m^{(p)}$.

$f_m^{(p)}$	$\overline{\langle H^2 \rangle^{m, f_m^{(p)}}}$
$\{4^r\}$	$\frac{r(\Omega - r + 4)}{2} \left[\lambda_{\{2\}}^2 3(r+1)(\Omega - r + 3) + \lambda_{\{1^2\}}^2 5(r-1)(\Omega - r + 5) \right]$
$\{4^r, 1\}$	$\frac{r(\Omega - r + 4)}{4} \left[\lambda_{\{2\}}^2 \{6r(\Omega - r + 1) + 9\Omega + 15\} + \lambda_{\{1^2\}}^2 5\{2r(\Omega - r + 5) - \Omega - 9\} \right]$
$\{4^r, 2\}$	$\lambda_{\{2\}}^2 \frac{1}{2} \left[3r^4 - 6(\Omega + 2)r^3 + (3\Omega^2 + 6\Omega - 5)r^2 + (\Omega + 2)(6\Omega + 17)r + \Omega(\Omega + 1) \right] + \lambda_{\{1^2\}}^2 \frac{5r}{2} (\Omega - r + 4) \{(\Omega + 4)r - r^2 - 3\}$
$\{4^r, 3\}$	$\frac{1}{4} \left[\lambda_{\{2\}}^2 3(r+2)(\Omega - r + 2)(2r\Omega - 2r^2 + 6r + \Omega + 1) + \lambda_{\{1^2\}}^2 5r(\Omega - r + 4)(2r\Omega - 2r^2 + 6r + \Omega - 1) \right]$

4.5.4 Cross-correlations in spectral variances $\Sigma_{22}(m, f_m; m', f_{m'})$

Expression for $\Sigma_{22}(m, f_m; m', f_{m'})$ given by Eq. (4.4.8) involves evaluation of

$$\overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}}.$$

As the two-body H operator defined in Eq. (4.3.1) is a sum of H 's in two-particle spaces defined by $f_2 = \{2\}$ and $\{1^2\}$, we have $H(2) = H_{\{2\}}(2) + H_{\{1^2\}}(2)$. The H_{f_2} 's are independent and the variables defining the matrix elements of H_{f_2} are independent Gaussian variables with zero center and variance given by Eq. (4.3.5). Expanding $\overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}$ and using Eqs. (4.3.5) and (4.3.6), we obtain

$$\begin{aligned} & \overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}} \\ &= \overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m} \langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} + \overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m} \langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} \\ &+ \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} \right\} + \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} \right\} \\ &+ 4 \overline{\langle H_{\{2\}} H_{\{1^2\}} \rangle^{m, f_m} \langle H_{\{1^2\}} H_{\{2\}} \rangle^{m', f_{m'}}}. \end{aligned} \tag{4.5.11}$$

Similarly, expanding $\overline{\left\{ \langle H^2 \rangle^{m, f_m} \right\} \left\{ \langle H^2 \rangle^{m', f_{m'}} \right\}}$ gives,

$$\begin{aligned} \left\{ \overline{\langle H^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle H^2 \rangle^{m', f_{m'}}} \right\} &= \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} \right\} \\ &+ \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} \right\} \\ &+ \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} \right\} \\ &+ \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} \right\}. \end{aligned} \tag{4.5.12}$$

Using Eqs. (4.5.11) and (4.5.12) in the expression for Σ_{22} given by Eq. (4.4.8), the numerator simplifies to give

$$\begin{aligned}
& \overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}} - \left\{ \overline{\langle H^2 \rangle^{m, f_m}} \right\} \left\{ \overline{\langle H^2 \rangle^{m', f_{m'}}} \right\} \\
&= \left\{ \overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m} \langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} - \left[\overline{\langle (H_{\{2\}})^2 \rangle^{m, f_m}} \right] \left[\overline{\langle (H_{\{2\}})^2 \rangle^{m', f_{m'}}} \right] \right\} \\
&+ \left\{ \overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m} \langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} - \left[\overline{\langle (H_{\{1^2\}})^2 \rangle^{m, f_m}} \right] \left[\overline{\langle (H_{\{1^2\}})^2 \rangle^{m', f_{m'}}} \right] \right\} \\
&+ 4 \overline{\langle H_{\{2\}} H_{\{1^2\}} \rangle^{m, f_m} \langle H_{\{1^2\}} H_{\{2\}} \rangle^{m', f_{m'}}} \\
&= X_{\{2\}} + X_{\{1^2\}} + 4X_{\{1^2\}\{2\}}.
\end{aligned} \tag{4.5.13}$$

Then, we have

$$\Sigma_{22}(m, f_m; m', f_{m'}) = \frac{X_{\{2\}} + X_{\{1^2\}} + 4X_{\{1^2\}\{2\}}}{\left[\overline{\langle H^2 \rangle^{m, f_m}} \right] \left[\overline{\langle H^2 \rangle^{m', f_{m'}}} \right]}. \tag{4.5.14}$$

To evaluate $X_{\{2\}}$ and $X_{\{1^2\}}$, we use Eq. (4.4.3) and carry out the ensemble averaging over W 's using the fact that W 's are Gaussian random variables with zero center and variance given by Eq. (4.4.5). Then, Eq. (4.4.10) and the sum rules for $SU(\Omega)$ Wigner coefficients [see Eqs. (E6a) and (E6b)] will give,

$$X_{f_2} = \frac{2(\lambda_{f_2})^4}{[d_{\Omega}(f_2)]^2} \sum_{\nu=0,1,2} [d(\mathbf{F}_{\nu})]^{-1} \mathcal{Q}^{\nu}(f_2 : m, f_m) \mathcal{Q}^{\nu}(f_2 : m', f_{m'}). \tag{4.5.15}$$

Similarly, we have

$$\begin{aligned}
X_{\{1^2\}\{2\}} &= \frac{\lambda_{\{2\}}^2 \lambda_{\{1^2\}}^2}{d_{\Omega}(\{2\}) d_{\Omega}(\{1^2\})} \\
&\times \sum_{\nu=0,1} [d(\mathbf{F}_{\nu})]^{-1} R^{\nu}(\{1^2\}\{2\} : m, f_m) R^{\nu}(\{1^2\}\{2\} : m', f_{m'}).
\end{aligned} \tag{4.5.16}$$

Note that $\mathcal{Q}^\nu(f_2 : m, f_m)$ are defined in Eq. (4.5.6). The functions $R^\nu(\{1^2\}\{2\} : m, f_m)$ also involve $SU(\Omega)$ U -coefficients and the explicit expression for R^ν is,

$$R^\nu(\{1^2\}\{2\} : m, f_m) = [F(m)]^2 \sum_{f_{m-2}, f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}} \mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m} \mathcal{N}_{f_m}} Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu);$$

$$Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu) =$$

$$\sum_{\rho} \frac{U(f_m, \{1^{\Omega-2}\}, f_m, \{1^2\}; f_{m-2}, \mathbf{F}_\nu)_\rho U(f_m, \{2^{\Omega-1}\}, f_m, \{2\}; f'_{m-2}, \mathbf{F}_\nu)_\rho}{U(f_m, \{1^{\Omega-2}\}, f_m, \{1^2\}; f_{m-2}, \{0\}) U(f_m, \{2^{\Omega-1}\}, f_m, \{2\}; f'_{m-2}, \{0\})}. \quad (4.5.17)$$

In $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$, f_{m-2} comes from $f_m \otimes \{1^{\Omega-2}\}$ and f'_{m-2} comes from $f_m \otimes \{2^{\Omega-1}\}$. In Eq. (4.5.16), the summation is over $\nu = 0$ and 1 only as $\nu = 2$ parts for $f_2 = \{2\}$ and $\{1^2\}$ are different. Here $d(\mathbf{F}_\nu)$ are dimension of the irrep \mathbf{F}_ν , and we have $d(\{0\}) = 1$, $d(\{21^{\Omega-2}\}) = \Omega^2 - 1$, $d(\{42^{\Omega-2}\}) = \Omega^2(\Omega + 3)(\Omega - 1)/4$, and $d(\{2^2 1^{\Omega-4}\}) = \Omega^2(\Omega - 3)(\Omega + 1)/4$. Tables for $X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ are already discussed before (see Table 4.4). Formulas for $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ are tabulated in Table 4.7 and they also involve τ_{ab} , $\Pi_a^{(b)}$, $\Pi_b^{(a)}$, Π'_a , Π''_a and $\Pi_a^{(bc)}$ introduced before.

Table 4.7: Formulas for $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ defined in Eq. (4.5.17). Note that $\{f(ab)\}\{f(ab)\}$ entries satisfy the $a \leftrightarrow b$ symmetry. See text for details.

$\{f_{m-2}\}\{f'_{m-2}\}$	$Y_{UU}(f_{m-2}, f'_{m-2}; \{2, 1^{\Omega-2}\})$
$\{f(ab)\}\{f(ab)\}$	$-\frac{\Omega}{2} \left[\frac{(\Omega^2 - 1)}{(\Omega^2 - 4)} \right]^{1/2} \left\{ \left(1 + \frac{1}{\tau_{ab}} \right) \frac{1}{\Pi_a^{(b)}} \right.$ $\left. + \left(1 - \frac{1}{\tau_{ab}} \right) \frac{1}{\Pi_b^{(a)}} - \frac{4}{\Omega} \right\}$
$\{f(ab)\}\{f(ac)\}$	$-\frac{\Omega}{2} \left[\frac{(\Omega^2 - 1)}{(\Omega^2 - 4)} \right]^{1/2} \left\{ \left(1 + \frac{1}{\tau_{ac}} \right) \frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega} \right\}$
$\{f(ab)\}\{f(aa)\}$	$-\Omega \left[\frac{(\Omega^2 - 1)}{(\Omega^2 - 4)} \right]^{1/2} \left\{ \frac{1}{\Pi_a^{(b)}} - \frac{2}{\Omega} \right\}$
$\{f(ab)\}\{f(cc)\}$	
or	$2 \left[\frac{(\Omega^2 - 1)}{(\Omega^2 - 4)} \right]^{1/2}$
$\{f(ab)\}\{f(cd)\}$	

Using the results in Tables 4.4 and 4.7 and simplifying Eqs. (4.5.6) and (4.5.17), expressions for $\mathcal{Q}^\nu(f_2 : m, f_m)$ and $R^\nu(\{1^2\}\{2\} : m, f_m)$ are derived for the irreps $f_m^{(p)}$. It is found that, with P^{f_2} defined in Eq. (4.5.2),

$$\begin{aligned}\mathcal{Q}^{\nu=0}(f_2 : m, f_m^{(p)}) &= \left[P^{f_2}(m, f_m^{(p)}) \right]^2, \\ R^{\nu=0}(\{1^2\}\{2\} : m, f_m^{(p)}) &= P^{\{1^2\}}(m, f_m^{(p)}) P^{\{2\}}(m, f_m^{(p)}).\end{aligned}\quad (4.5.18)$$

The final results for $\mathcal{Q}^{\nu=1,2}(f_2 : m, f_m^{(p)})$ and $R^{\nu=1}(\{1^2\}\{2\} : m, f_m^{(p)})$ are given in Tables 4.8 and 4.9, respectively. Formulas in these Tables are verified numerically in many examples by directly programming Tables 4.4 and 4.7. In the dilute limit ($\Omega \rightarrow \infty$, $r \gg 1$, $r/\Omega \rightarrow 0$), the cross term $X_{\{1^2\}\{2\}}$ will be very small compared to the direct terms X_{f_2} . Dominant contribution to X_{f_2} comes from $\mathcal{Q}^{\nu=2}(f_2 : m, f_m^{(p)})$ which has the form $-\Omega^4 P^{f_2}(m, f_m^{(p)})/4$ (while the other terms i.e., $\mathcal{Q}^{\nu=1}(f_2 : m, f_m^{(p)})$ and $R^{\nu=1}(\{1^2\}\{2\} : m, f_m^{(p)})$ have Ω^2 dependence). Then in the dilute limit, for the irreps $f_m^{(p)}$, simplifying the results given in Tables 4.8 and 4.9, the covariances in spectral variances take a simple form,

$$\begin{aligned}\Sigma_{22}(m, f_m^{(p)}; m', f_{m'}^{(p)}) &\xrightarrow{\Omega \rightarrow \infty, r \gg 1, r/\Omega \rightarrow 0} \\ &= \frac{8 \sum_{f_2} \lambda_{f_2}^4 P^{f_2}(m, f_m^{(p)}) P^{f_2}(m', f_{m'}^{(p)})}{\Omega^4 \left\{ \sum_{f_2} \lambda_{f_2}^2 P^{f_2}(m, f_m^{(p)}) \right\} \left\{ \sum_{f_2} \lambda_{f_2}^2 P^{f_2}(m', f_{m'}^{(p)}) \right\}}.\end{aligned}\quad (4.5.19)$$

As $\Omega \rightarrow \infty$, $\Sigma_{22}(m, f_m^{(p)}; m', f_{m'}^{(p)}) \rightarrow 0$ and there will be no correlations. For finite Ω , there will be correlations between states with different or same (m, f_m) and examples for these are discussed ahead.

Table 4.8: $\mathcal{Q}^\nu(f_2 : m, f_m)$ for $f_m = f_m^{(p)}$ and $\nu = 1$ and 2. See Eq. (4.5.6) for the definition of \mathcal{Q}^ν .

$f_m^{(p)}$	f_2	ν	$\mathcal{Q}^\nu(f_2 : m, f_m^{(p)})$
$\{4^r\}$	$\{2\}$	1	$\frac{9r(r+1)^2(\Omega-r)(\Omega+1)(\Omega+4)}{2(\Omega+2)}$
		2	$\frac{3r\Omega(r+1)(\Omega-r+1)(\Omega-r)(\Omega+4)(\Omega+5)}{4(\Omega+2)}$
$\{1^2\}$	$\{1^2\}$	1	$\frac{25r(r-1)^2(\Omega-r)(\Omega-1)(\Omega+4)}{2(\Omega-2)}$

Table 4.8 – continued

$f_m^{(p)}$	f_2	ν	$\mathcal{Q}^\nu(f_2: m, f_m^{(p)})$
		2	$\frac{5r\Omega(r-1)(\Omega+3)(\Omega+4)(\Omega-r)(\Omega-r-1)}{4(\Omega-2)}$
$\{4^r, 1\}$	$\{2\}$	1	$\frac{3r(\Omega+1)}{8(\Omega+2)} [-12(\Omega+4)r^3 + 12(\Omega-3)(\Omega+4)r^2 + (33\Omega^2 + 100\Omega - 108)r + 20\Omega(\Omega+4)]$
		2	$\frac{3r\Omega}{8(\Omega+2)} (\Omega-r+1)(\Omega+4) [-2(\Omega+5)r^2 + 2(\Omega-2)(\Omega+5)r + \Omega(3\Omega+11) - 10]$
	$\{1^2\}$	1	$-\frac{5r(\Omega-1)}{8(\Omega-2)} [20(\Omega+4)r^3 - 20(\Omega^2+5\Omega+4)r^2 + (25\Omega^2 + 132\Omega + 20)r - 12\Omega(\Omega+4)]$
		2	$\frac{5r\Omega(\Omega+4)}{8(\Omega-2)} [2(\Omega+3)r^3 - 2(2\Omega^2+5\Omega-3)r^2 + (2\Omega^3+5\Omega^2+\Omega-6)r - \Omega^3 - 6\Omega^2 + 13\Omega - 6]$
$\{4^r, 2\}$	$\{2\}$	1	$\frac{(\Omega+1)}{4(\Omega+2)} [-8(3r^2+6r+1)^2 + (3r+4)(6r^2+13r+1)\Omega^2 - 2(9r^4-79r^2-88r-2)\Omega]$
		2	$\frac{\Omega}{4(\Omega+2)} [3(\Omega+4)(\Omega+5)r^4 - 6(\Omega-1)(\Omega+4)(\Omega+5)r^3 + \Omega(\Omega+4)(3\Omega^2+3\Omega-56)r^2 + (\Omega-1)(\Omega+4)(6\Omega^2+29\Omega+15)r + \Omega(\Omega-1)(\Omega+2)(\Omega+3)]$
	$\{1^2\}$	1	$-\frac{5r(\Omega-1)}{4(\Omega-2)} [10(\Omega+4)r^3 - 10\Omega(\Omega+4)r^2 + \Omega(5\Omega+38)r - 3\Omega(\Omega+4)]$
		2	$-\frac{5r\Omega(\Omega+4)(\Omega-r-1)}{4(\Omega-2)} [3(\Omega-1) - r(\Omega-r-1)(\Omega+3)]$
$\{4^r, 3\}$	$\{2\}$	1	$-\frac{3(r+2)(\Omega+1)}{8(\Omega+2)} [12(r+2)(2r+1)^2 - \Omega^2(12r^2+27r+8) + 4(3r^3-3r^2-19r-4)\Omega]$
		2	$-\frac{3\Omega(r+2)(\Omega+4)(\Omega-r-1)}{8(\Omega+2)} [2r^2(\Omega+5) - 2\Omega r(\Omega+5) - \Omega(\Omega+3)]$
	$\{1^2\}$	1	$-\frac{5r(\Omega-1)}{8(\Omega-2)} [20(\Omega+4)r^3 - 20r^2(\Omega^2+3\Omega-4) + r(-5\Omega^2+12\Omega+20) - 2\Omega(\Omega+4)]$

Table 4.8 – continued

$f_m^{(p)}$	f_2	ν	$\mathcal{Q}^\nu(f_2 : m, f_m^{(p)})$
		2	$-\frac{5r\Omega(\Omega+4)(\Omega-r-1)}{8(\Omega-2)} [2(\Omega+3)r^2 - 2r(\Omega^2 + \Omega - 6) - \Omega(\Omega-3)]$

Table 4.9: $R^{\nu=1}(\{1^2\}\{2\} : m, f_m)$ for $f_m = f_m^{(p)}$. See Eq. (4.5.17) for the definition of $R^{\nu=1}$.

$f_m^{(p)}$	$R^{\nu=1}(\{1^2\}\{2\} : m, f_m^{(p)})$
$\{4^r\}$	$-\frac{15r}{2} \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} (r^2-1)(\Omega-r)(\Omega+4)$
$\{4^r, 1\}$	$\frac{15r}{8} \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} [4r^3(\Omega+4) - 4r^2(\Omega+4)(\Omega-1) - 3r(\Omega+2)^2 + 4\Omega(\Omega+4)]$
$\{4^r, 2\}$	$-\frac{5r}{4} \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} [-6r^3(\Omega+4) - 3\Omega(\Omega+6) + 6r^2(\Omega-2)(\Omega+4) + r(9\Omega^2 + 28\Omega - 8)]$
$\{4^r, 3\}$	$-\frac{15r}{8} (r+2) \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} [-4r^2(\Omega+4) + 4r(\Omega-1)(\Omega+4) + \Omega^2 - 4]$

4.6 Numerical Results for Spectral Variances, Expectation Values of $C_2[SU(4)]$ and Four Periodicity in GS

Employing the analytical formulation described in Secs. 4.4 and 4.5 along with the results in Table 4.6 for $f_m = f_m^{(p)}$ irreps and Table 4.4 for general f_m irreps, numerical calculations are carried out for $\langle H^2 \rangle^{m, f_m}$. In our examples, we have chosen $\Omega = 6$ and $\Omega = 10$ and they correspond to nuclear ($2s1d$) and ($2p1f$) shells, respectively. Results for spectral variances are used to analyze expectation values of $C_2[SU(4)]$ and the four periodicity in the gs energies. Conclusions from these studies are summarized at the end.

4.6.1 Spectral variances

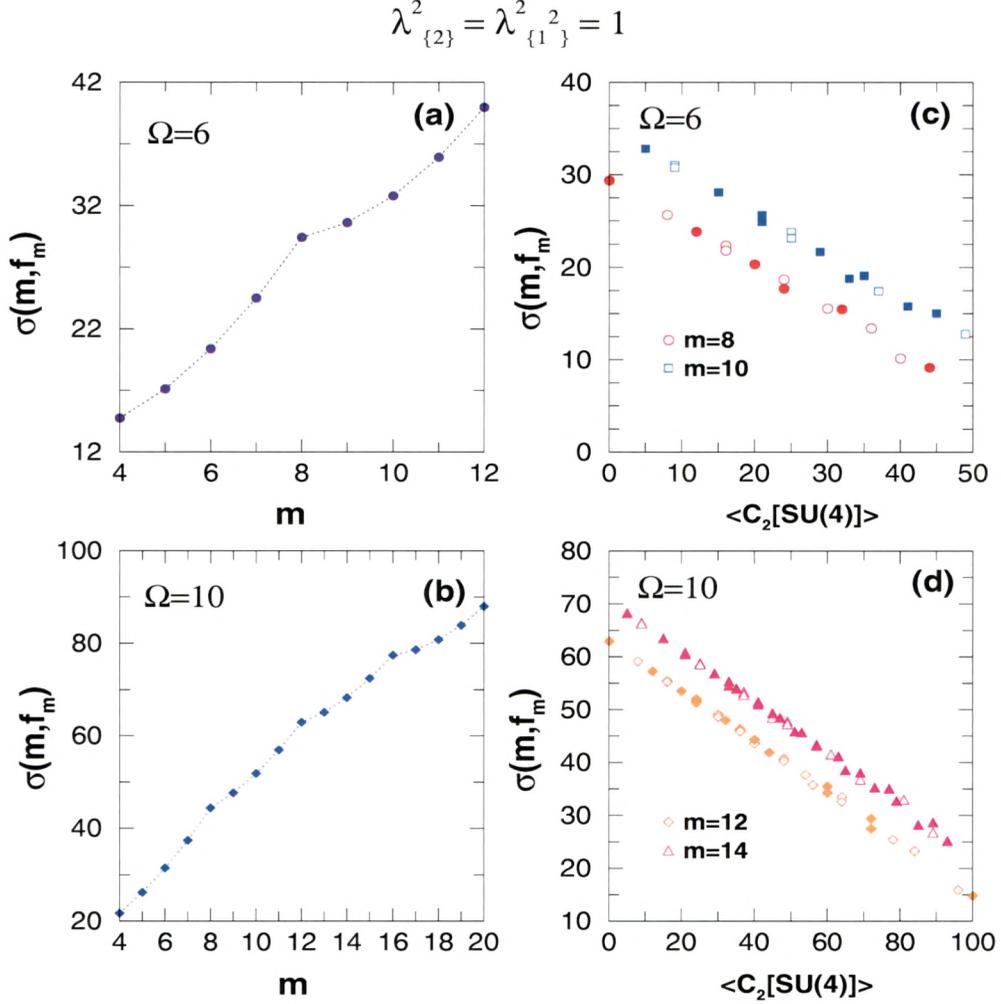


Figure 4.3: Variation of spectral widths $\left[\overline{\langle H^2 \rangle^{m, f_m}} \right]^{1/2}$ as a function of m with fixed f_m and similarly variation as a function of f_m with fixed m . (a) $\Omega = 6$, $f_m = f_m^{(p)}$, (b) $\Omega = 10$, $f_m = f_m^{(p)}$, (c) $\Omega = 6$ and $m = 8$ and 10 , and (d) $\Omega = 10$ and $m = 12$ and 14 . Note that $f_m^{(p)} = \{4^r, p\}$ where $m = 4r + p$. Similarly, instead of showing f_m in (c) and (d) we have used $\langle C_2[SU(4)] \rangle^{f_m}$. We have marked by filled symbols in (c) and (d) the irreps f_m that give $(S, T) = (0, 0)$ for $m = 4r$ systems and $(S, T) = (1, 0) \oplus (0, 1)$ for $m = 4r + 2$ systems. See text for details.

Figures 4.3(a) and (b) show variation in the spectral widths $\sigma(m, f_m) = \left[\overline{\langle H^2 \rangle^{m, f_m}} \right]^{1/2}$ as a function of the particle number m with fixed $f_m = f_m^{(p)}$. Notice the peaks at $m = 4r$; $r = 2, 3, \dots$. Except for this structure, there are no other differences between $\{4^r\}$ and $\{4^r, 2\}$ systems or equivalently between even-even and odd-odd $N=Z$ nuclei. Figures 4.3(c) and (d) show variation in the spectral widths $\sigma(m, f_m)$ as a function of

f_m with fixed m values. Results are shown for $m = 8$ and 10 for $\Omega = 6$ and $m = 12$ and 14 for $\Omega = 10$. In the figures, we have used the physically more appropriate $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ label for the x -axis instead of showing f_m . It is clearly seen that the variation in the spectral widths is almost linear. Considering the eigenvalue density to be Gaussian [extrapolating from the results known for EGUE(2), EGOE(2) and EGOE(2)-s] and neglecting the dimension effects, energy of the lowest state that belong to a given f_m follows from the Jacquod and Stone prescription [Pa-08, Ja-01]. This gives

$$E_{gs}(f_m) - E_c(m, f_m) \propto -\sigma(m, f_m). \quad (4.6.1)$$

This follows from Eq. (4.6.4) given ahead if we restrict it to a given f_m . Combining Eq. (4.6.1) with the results in Figs. 4.3(c) and (d), we can identify the irreps that label the gs generated by EGUE(2)- $SU(4)$. As $\sigma(m, f_m)$ vs $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ curves are linear, clearly EGUE(2)- $SU(4)$ generates gs labeled by the irreps that have lowest $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$. Therefore random interactions, which are $SU(4)$ scalar, carry the properties of $C_2[SU(4)]$, the $SU(4)$ invariant or the Majorana force. In Figs. 4.3(c) and (d), we have marked the irreps that give $(S, T) = (0, 0)$ for $m = 4r$ and $(S, T) = (1, 0) \oplus (0, 1)$ for $m = 4r + 2$ systems. If we restrict to these irreps, the second irrep is forbidden in both cases i.e., there is a gap between the lowest and next allowed irrep. This implies that even with random interactions we obtain gs with $f_m = f_m^{(p)}$. We will further substantiate this result by calculating the expectation values $\langle C_2[SU(4)] \rangle^E$ and also analyzing the four periodicity in E_{gs} .

4.6.2 Expectation values $\langle C_2[SU(4)] \rangle^E$

In order to examine the extent to which random interactions with $SU(4)$ symmetry carry the properties of the Majorana operator, we have studied expectation values (smoothed with respect to E) of the quadratic Casimir invariant of $SU(4)$ using the Hamiltonian H_α ,

$$\{H_\alpha\} = C_2[SU(4)] + \alpha\{H\}. \quad (4.6.2)$$

where $\{H\}$ is defined by Eq. (4.3.1) with $\lambda_{\{2\}}^2 = \lambda_{\{1^2\}}^2 = 1$. In order to study $\langle C_2[SU(4)] \rangle^E$, we decompose it in terms of $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ (see Eq. (3.4.6) and [Pa-78]),

$$\langle C_2[SU(4)] \rangle^E = \sum_{f_m} \frac{I_{\mathcal{G}}^{m,f_m}(E)}{I^m(E)} \langle C_2[SU(4)] \rangle^{\tilde{f}_m}; \quad (4.6.3a)$$

$$I^m(E) = \sum_{f_m} I_{\mathcal{G}}^{m,f_m}(E) = \sum_{f_m} d_{\Omega}(f_m) d_4(\tilde{f}_m) \rho_{\mathcal{G}}^{m,f_m}(E). \quad (4.6.3b)$$

In Eq. (4.6.3a), $I^{m,f_m}(E)$ are partial eigenvalue densities defined over a fixed f_m space, $I^{m,f_m}(E) = \langle \langle \delta(H - E) \rangle \rangle^{m,f_m}$ and $I^m(E)$ is the total eigenvalue density. Equation (4.6.3a) is exact if we remove \mathcal{G} , as f_m (equivalently \tilde{f}_m) label the eigenstates of $C_2[SU(4)]$. For smoothed expectation values, based on the $SU(4)$ partial densities that are studied within the nuclear shell-model (with $\Omega = 6$) [Pa-73, Pa-72], we assume that $\rho^{m,f_m}(E)$ will be close to a Gaussian (\mathcal{G}). Numerical calculations of γ_2 using H matrix construction as discussed in Section 4.3.3 or using the analytical formulation discussed in Appendix F, will verify this assumption. However, at present both these methods are not feasible in practice. For the Hamiltonian in Eq. (4.6.2), the centroid of $\rho_{\mathcal{G}}^{m,f_m}(E)$ is $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ and the variance is $\overline{\langle H^2 \rangle}^{m,f_m}$.

As an example, for $\Omega = 6$ and $m = 8$ and 10 , the expectation values are calculated as a function of energy for various values of α in Eq. (4.6.2) and the results are shown in Figs. 4.4(a) and (b). It is seen that with the increase in the strength α , fluctuations decrease and the staircase form for $\alpha \sim 0$ turns into a smooth curve for $\alpha \gtrsim \alpha_c = 0.3$. This conclusion remains same even when we consider $U(\Omega)$ irreps with $(S, T) = (0, 0)$ for m even and $(S, T) = (1, 0) \oplus (0, 1)$ for m odd. Then the normalization for $I_{\mathcal{G}}^{m,f_m}(E)$ is $d_{\Omega}(f_m) \times d_{gs}$. Note that the degeneracy $d_{gs} = 1, 6$, and 4 , respectively for $m = 4r$ (even-even nuclei), $m = 4r + 2$ (odd-odd nuclei) and $m = 4r + 1$ or $4r + 3$ (odd-A nuclei). Just as for EGOE(1+2) and EGOE(1+2)-s, it is expected that the transition point $\alpha_c \propto \Omega/K(m, f_m)$ and the variance propagator $K(m, f_m)$, as mentioned in Section 4.3.2, follows from the formulas in Table 4.6 for $f_m^{(p)}$ irreps and for general irreps from Eq. (4.5.5) and Table 4.4 with $\lambda_{\{1^2\}}^2 = \lambda_{\{2\}}^2 = 1$. From the results in Table 4.6, for the $f_m^{(p)}$ irreps, it follows that in the dilute limit, $K(m, f_m) \rightarrow m^2 \Omega^2$. Thus, $\alpha_c \propto 1/m^2 \Omega$ and this result is same as those derived before for EGOE(1+2) and EGOE(1+2)-s; see Chapter 2 for details. Therefore, with fixed m , $\alpha_c = 0.3$ for $\Omega = 6$ corresponds to $\alpha_c \sim 0.2$ for

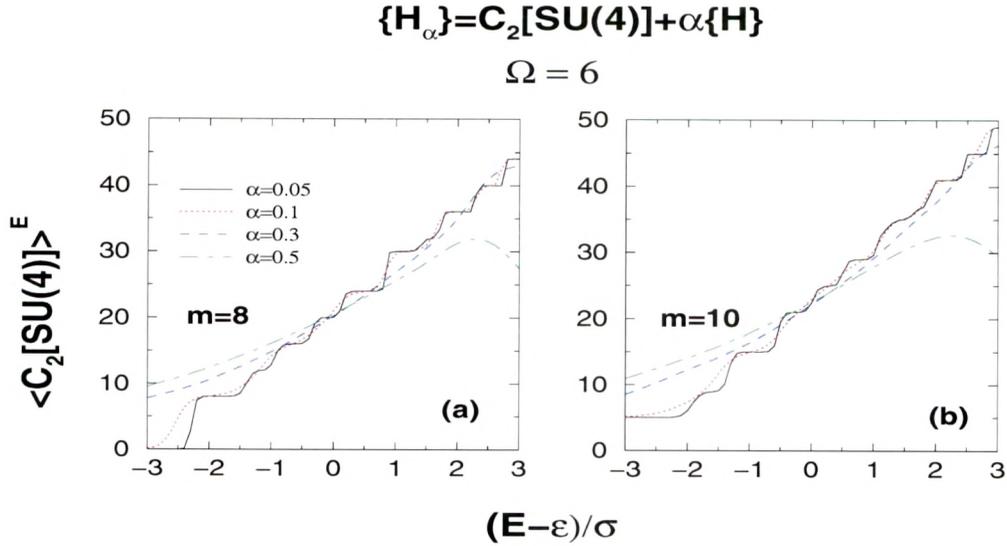


Figure 4.4: Expectation values of the quadratic Casimir invariant of $SU(4)$ as a function of excitation energy for the H_α Hamiltonian ensemble defined in Eq. (4.6.2). Results are shown for four values of interaction strength α : (a) for $m = 8$ and (b) for $m = 10$. Note that the energies are zero centered with respect to the centroid ϵ and scaled with the width σ defined by first and second moments of the total density of states. All the results are for $\Omega = 6$. Similar results are obtained even when we consider, in Eq. (4.6.3b), the irreps f_m that give $(S, T) = (0, 0)$ for $m = 8$ and $(S, T) = (1, 0) \oplus (0, 1)$ for $m = 10$.

$\Omega = 10$. We have verified this by comparing the numerical results for $\Omega = 6$ and 10.

Results in Figs. 4.4(a) and (b) confirm that even with random interactions that are $SU(4)$ scalar, ground states have lowest value for $\langle C_2[SU(4)] \rangle^E$ and therefore they carry the property of the Majorana force. Also beyond a critical strength (α_c) of the random part in Eq. (4.6.2), expectation values will be smooth with respect to energy.

4.6.3 Four-periodicity in the ground state energies

An evidence for effective space symmetry for nuclear ground states is derived from the four periodicity in the gs energies E_{gs} per particle [Pa-78]. An important question is: will this feature survive even in the presence of random interactions. To test this, as a model, we consider the Hamiltonian H_α in Eq. (4.6.2) where α is the strength of the random interaction with $SU(4)$ symmetry. For the strength $\alpha = 0$, H reduces to the quadratic Casimir invariant of the $SU(4)$ group and this, as it is well-known, produces oscillations in $E_{gs}(m)/m$ with minima at $m = 4r$ (this is called four periodicity) as seen clearly from Fig. 4.5. When the strength α is non-zero, for given number of particles m , all the irreps f_m , with $(S, T) = (0, 0)$ for $m = 4r$, $(S, T) = (1, 0) \oplus (0, 1)$ for

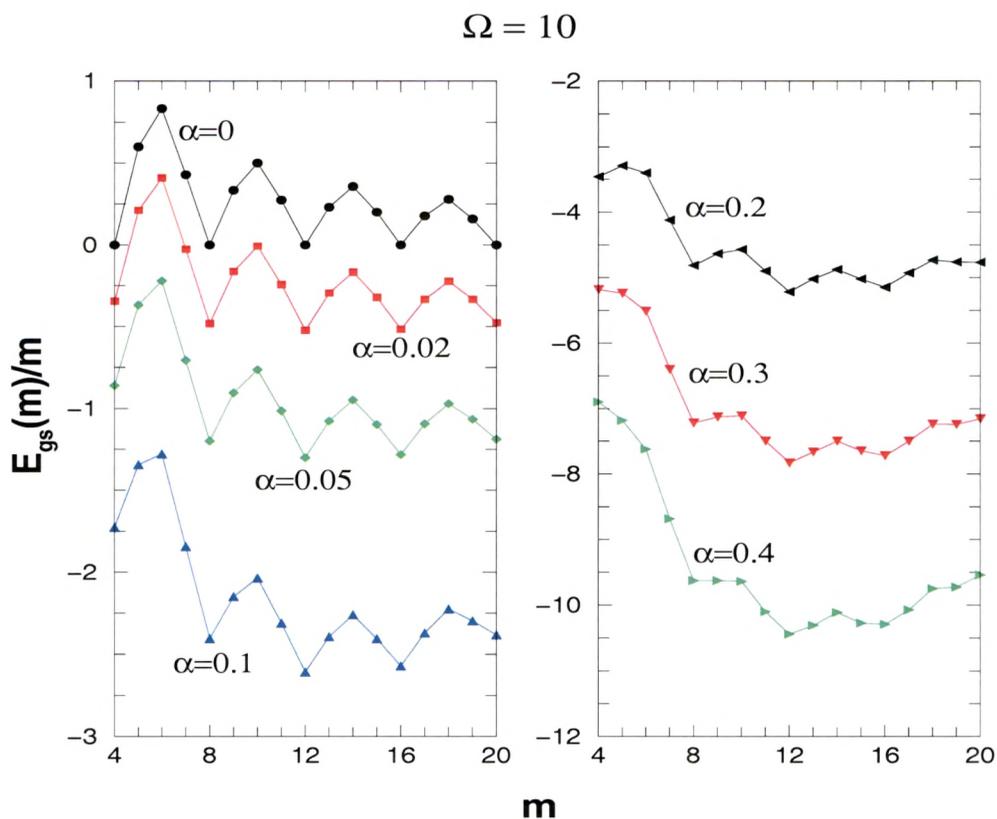


Figure 4.5: Ground state energy $E_{gs}(m)$ per particle m as a function of m for different values of the interaction strength α in Eq. (4.6.2). Results are shown for $\alpha \leq 0.4$. The variation of $E_{gs}(m)/m$ shown in the figure brings out the four periodicity effect in the gs energies. See text for details.

$m = 4r + 2$ and $(S, T) = (\frac{1}{2}, \frac{1}{2})$ for $m = 4r + 1$ and $4r + 3$, contribute to the sum in Eq. (4.6.3b) in generating the total density of states. Using Eq. (4.6.3b), $E_{gs}(m)$ for a fixed m is determined numerically by inverting the integral,

$$\frac{1}{2} = \int_{-\infty}^{E_{gs}(m)} \sum_{f_m} d_{\Omega}(f_m) \rho_{cg}^{m, f_m}(E) dE. \quad (4.6.4)$$

This is known as ‘‘Ratcliff procedure’’ in nuclear physics literature [Ra-71, Wo-86]. We show in Fig. 4.5, the variation of $E_{gs}(m)/m$ vs m for different interaction strengths α . In the calculations, $\Omega = 10$ and $m = 4 - 20$. It is clearly seen that the four periodicity produced by $C_2[SU(4)]$ is preserved by random Hamiltonian H_{α} for $\alpha \leq 0.2$. The kinks in the spectral widths at $m = 4r$ as a function of m as seen from Fig. 4.3(a) and (b) and similarly, their monotonic decrease with $\langle C_2[SU(4)] \rangle^{\bar{f}_m}$ as seen from Fig. 4.3(c) and (d), together explain the four periodicity in the gs energies.

Beyond $\alpha = 0.2$, this structure starts disappearing as the difference Δ between the centroids, produced by $C_2[SU(4)]$ for the lowest two irreps, becomes comparable to the width of the gs irrep $\{4^r\}$; $m = 4r$. Therefore, with a regular part that is close to $C_2[SU(4)]$, random interactions that are not too strong [$\alpha \lesssim \alpha'_c = 0.2$ in Eq. (4.6.2)] generate, in the $\Omega = 10$ example, ground states that are spatially symmetric. Thus, $\alpha_c \sim \alpha'_c$ (see Section 4.6.2 for α_c) and therefore, the region of onset of smooth behavior for $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ also marks the onset of diminishing four periodicity effect in the gs energies. As $\alpha_c \propto 1/m^2\Omega$, the four periodicity effect should diminish faster for large m and this is clearly seen from Fig. 4.5.

4.6.4 Conclusions

Thus, ensemble averaged spectral variances $\overline{\langle H^2 \rangle^{m, f_m}}$, expectation values $\langle C_2[SU(4)] \rangle^E$ and the four periodicity in $E_{gs}(m)/m$ discussed in Secs. 4.6.1-4.6.3 establish that random interactions with $SU(4)$ symmetry keep intact all the essential features of the Majorana force (see Section 4.8 for further discussion on the importance of this result). Therefore the EGUE(2)- $SU(4)$ and the corresponding EGOE(2)- $SU(4)$ ensemble should be useful in nuclear structure.

4.7 Numerical Results for Correlations in Energy Centroids and Spectral Variances

Using the results in Tables 4.3, 4.6, 4.8 and 4.9 for $f_m = f_m^{(p)}$ irreps and Tables 4.4 and 4.7 for general f_m irreps, the self and cross-correlations in energy centroids and spectral variances [i.e., Σ_{11} and Σ_{22} in Eq. (4.4.8)] are calculated. See [Br-81, Fl-00, Pa-00] for a detailed discussion on the significance of self-correlations (they affect level motion in the ensemble) and [Pa-07, Ko-07, Ko-06] on the significance of the cross-correlations (they will vanish for GE's) generated by embedded ensembles. Results for Σ_{11} and Σ_{22} are discussed in Secs. 4.7.1-4.7.3 and a summary is given at the end.

4.7.1 Self-correlations

Results for self-correlations ($m = m'$, $f_m = f_{m'}$) are shown in Table 4.10 for $f_m = f_m^{(p)}$ and $\Omega = 6$ and 10. For $\Omega = 6$ we have, $[\Sigma_{11}]^{1/2} \sim 12 - 28\%$ and $[\Sigma_{22}]^{1/2} \sim 7 - 15\%$ as

m changes from 6 to 12. Similarly, for $\Omega = 10$ and m ranging from 12 to 20, they decrease to 10 – 22% for $[\Sigma_{11}]^{1/2}$ and 4 – 9% for $[\Sigma_{22}]^{1/2}$. We can also infer from Table 4.10 that as m increases, the self-correlations also increase. Therefore, fluctuations in the level motion in the ensemble increase with m and as a result the ensemble averages deviate from spectral averages with increasing m . This feature has been studied before for EGOE(2) and EGOE(2)-s [Br-81, Fl-00, Le-08].

Further significance of the magnitude of the self-correlations follows by comparing the results with the corresponding ones for EGUE(2) and EGUE(2)-s for fixed number of sp states (N). Using the analytical formulas given in [Ko-05] for EGUE(2), $[\Sigma_{11}(m, m)]^{1/2}$ and $[\Sigma_{22}(m, m)]^{1/2}$ are calculated for various values of m with $N = 24$ and 40 and the results are shown in third and sixth columns of Table 4.10. Similarly, using the formulas in [Ko-07] for EGUE(2)-s, $[\Sigma_{11}(m, S; m, S)]^{1/2}$ and $[\Sigma_{22}(m, S; m, S)]^{1/2}$ with $S = 0$ for even m and $S = 1/2$ for odd m are calculated for various values of m with $N = 24$ ($\Omega = 12$) and 40 ($\Omega = 20$) and the results are shown in fourth and seventh columns of Table 4.10. It is seen from Table 4.10 that the magnitude of the covariances in energy centroids and spectral variances increases by a factor 3 when we go from EGUE(2) \rightarrow EGUE(2)-s \rightarrow EGUE(2)- $SU(4)$.

As discussed in Section 4.3, the fraction of independent matrix elements \mathcal{I} increases with symmetry and also the sparsity (\mathbf{S}) decreases and therefore the EGUE(2)- $SU(4)$ matrices will be dense leading to a more complete mixing of the basis states compared to EGUE(2) and EGUE(2)-s. Therefore there is a correlation between (i) increase in fluctuations defined by Σ_{11} and Σ_{22} and (ii) the matrices $H_{f_m}(m)$ becoming more dense as we go from EGUE(2) \rightarrow EGUE(2)-s \rightarrow EGUE(2)- $SU(4)$. See Section 4.8 for further discussion.

4.7.2 Cross-correlations

Results for cross-correlations in energy centroids $\Sigma_{11}(m, f_m; m', f_{m'})$ and spectral variances $\Sigma_{22}(m, f_m; m', f_{m'})$ with $f_m = f_m^{(p)}$ as a function of m and m' are shown in Fig. 4.6 for both $\Omega = 6$ and 10. It is seen that $[\Sigma_{11}]^{1/2}$ and $[\Sigma_{22}]^{1/2}$ increase almost linearly with m . At $m = 4r$, $r = 2, 3, \dots$ there is a slight dip in $[\Sigma_{11}]^{1/2}$ as well as in $[\Sigma_{22}]^{1/2}$. For $\Omega = 6$ we have, $[\Sigma_{11}]^{1/2} \sim 10 - 24\%$ and $[\Sigma_{22}]^{1/2} \sim 6 - 12\%$. Similarly, for $\Omega = 10$ these decrease to 5 – 16% for $[\Sigma_{11}]^{1/2}$ and 2 – 6% for $[\Sigma_{22}]^{1/2}$. The decrease in Σ 's

Table 4.10: Variation in the self-correlations in energy centroids (Σ_{11}) and spectral variances (Σ_{22}) with symmetry. See text for details.

N	m	$[\Sigma_{11}]^{1/2}$				$[\Sigma_{22}]^{1/2}$			
		EGUE(2)	EGUE(2)-s	EGUE(2)-SU(4)	EGUE(2)	EGUE(2)-s	EGUE(2)-SU(4)	EGUE(2)-SU(4)	
24	6	0.017	0.043	0.125	0.0056	0.017	0.069	0.069	
	7	0.021	0.055	0.144	0.0059	0.019	0.076	0.076	
	8	0.026	0.066	0.160	0.0064	0.021	0.083	0.083	
	9	0.031	0.081	0.196	0.0069	0.025	0.099	0.099	
	10	0.037	0.094	0.229	0.0077	0.028	0.117	0.117	
	11	0.044	0.112	0.256	0.0087	0.034	0.134	0.134	
	12	0.051	0.128	0.276	0.0099	0.039	0.148	0.148	
40	12	0.0139	0.038	0.105	0.00222	0.0079	0.035	0.035	
	13	0.0157	0.044	0.120	0.00234	0.0086	0.039	0.039	
	14	0.0176	0.048	0.134	0.00247	0.0093	0.044	0.044	
	15	0.0196	0.054	0.146	0.00262	0.0103	0.049	0.049	
	16	0.0218	0.06	0.156	0.0028	0.0112	0.053	0.053	
	17	0.0241	0.067	0.174	0.003	0.0125	0.061	0.061	
	18	0.0267	0.073	0.192	0.00324	0.0138	0.069	0.069	
	19	0.0294	0.081	0.206	0.00352	0.0156	0.078	0.078	
	20	0.0325	0.088	0.218	0.00385	0.0174	0.085	0.085	

$$\lambda_{\{2\}}^2 = \lambda_{\{1\}^2} = 1$$

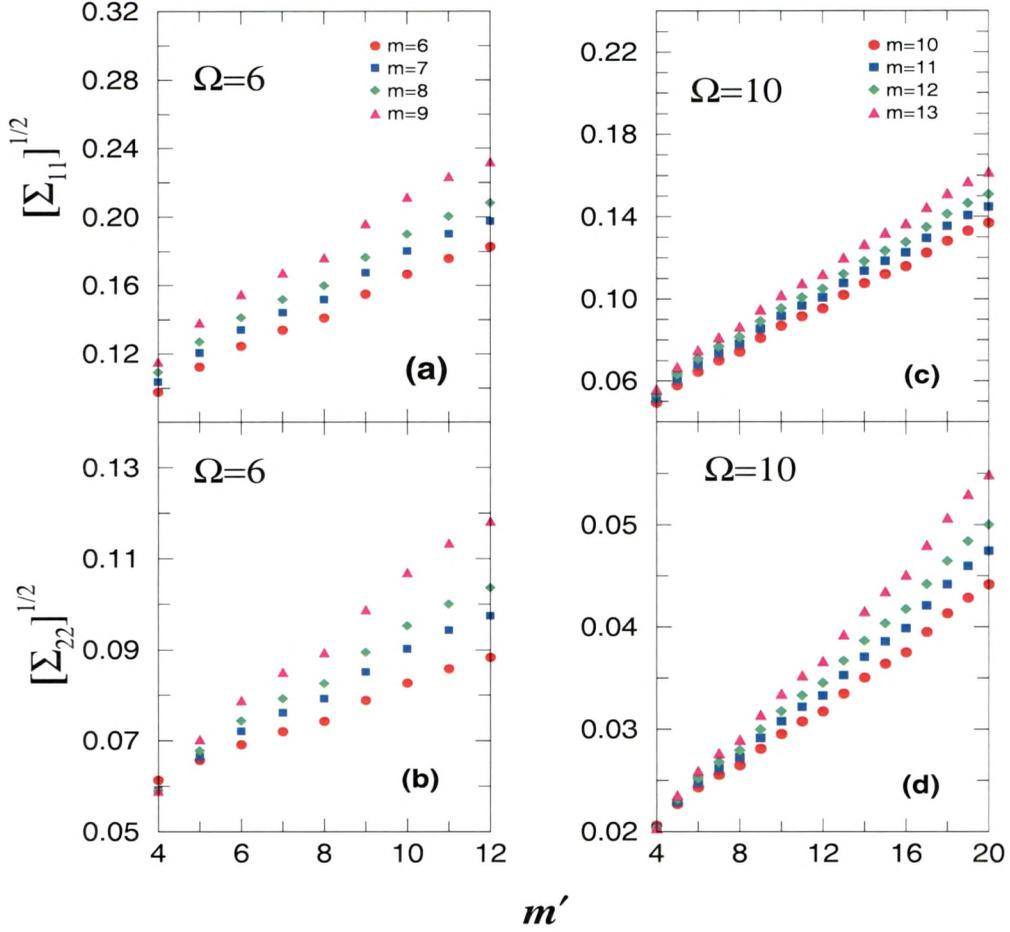


Figure 4.6: Self and cross-correlations in energy centroids and spectral variances as a function of m and m' (with fixed f_m and $f_{m'}$) for $\Omega = 6$ and $\Omega = 10$ examples: (a) $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ for $\Omega = 6$; (b) $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ for $\Omega = 6$; (c) $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ for $\Omega = 10$; (d) $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ for $\Omega = 10$. Results in the figure are for $f_m = f_m^{(p)}$ and $f_{m'} = f_{m'}^{(p)}$. See text for details.

with increasing Ω is in agreement with the results obtained for EGOE(2) for spinless fermions and EGOE(2)-s. Similarly, the covariances in spectral variances are always smaller compared to those for energy centroids.

Figures 4.7(a) and (b) show cross-correlations in energy centroids Σ_{11} and spectral variances Σ_{22} as a function of f_m and $f_{m'}$ with fixed $m = m'$. Results are shown for the first, second and fourth lowest $U(\Omega)$ irreps, ordered according to $\langle C_2[SU(4)] \rangle_{\bar{f}_m}$, with all other f_m 's for $m = 8$ and 10 with $\Omega = 6$. The correlations grow with increase in $\langle C_2[SU(4)] \rangle_{\bar{f}_m}$. It is important to note that there is no correlation between variation

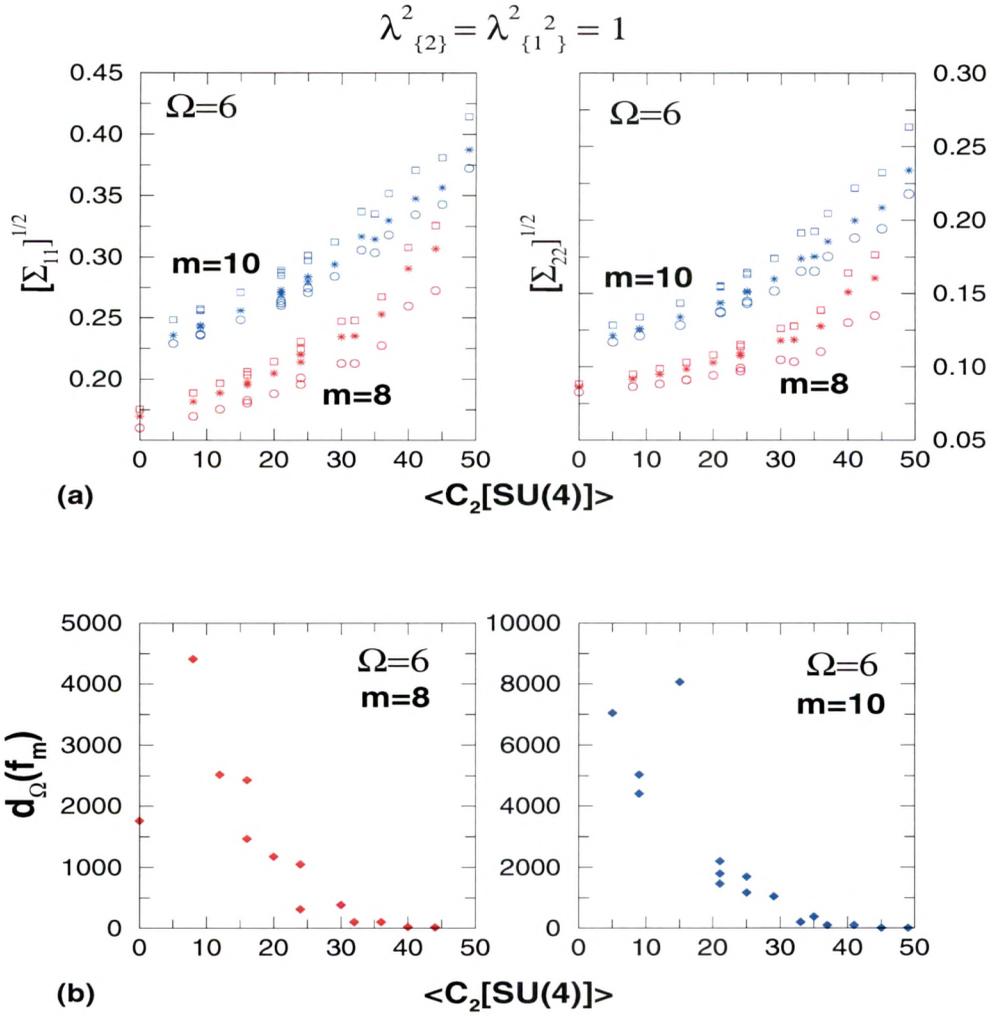


Figure 4.7: (a) Self and cross-correlations in energy centroids $[\Sigma_{11}(m, f_m; m', f_{m'})]^{1/2}$ and spectral variances $[\Sigma_{22}(m, f_m; m', f_{m'})]^{1/2}$ as a function of f_m and $f_{m'}$ (with fixed $m = m'$). Results are shown for the first (circle), second (star) and fourth (square) lowest $U(\Omega)$ irreps (ordered according to $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$) with all other irreps for $m = m' = 8$ (red) and 10 (blue) as a function of $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$. (b) Dimension $d_{\Omega}(f_m)$ for $m = 8, 10$ vs the eigenvalue of $C_2[SU(4)]$ in the corresponding $SU(4)$ irrep. Note that for a given value of the eigenvalue of $C_2[SU(4)]$ in some cases there are more than one f_m with the same eigenvalue. All results are for $\Omega = 6$.

in covariances with the variation in the f_m dimensions; see Figs. 4.7(a) and (b).

The increase in the cross-correlations with m' for fixed f_m and similar increase with $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ with fixed m , seen from Figs. 4.6 and 4.7, could possibly be exploited in deriving experimental signatures for cross-correlations. Note that the cross-correlations will be zero if we replace EGUE by GUE for $H_{f_m}(m)$ matrix.

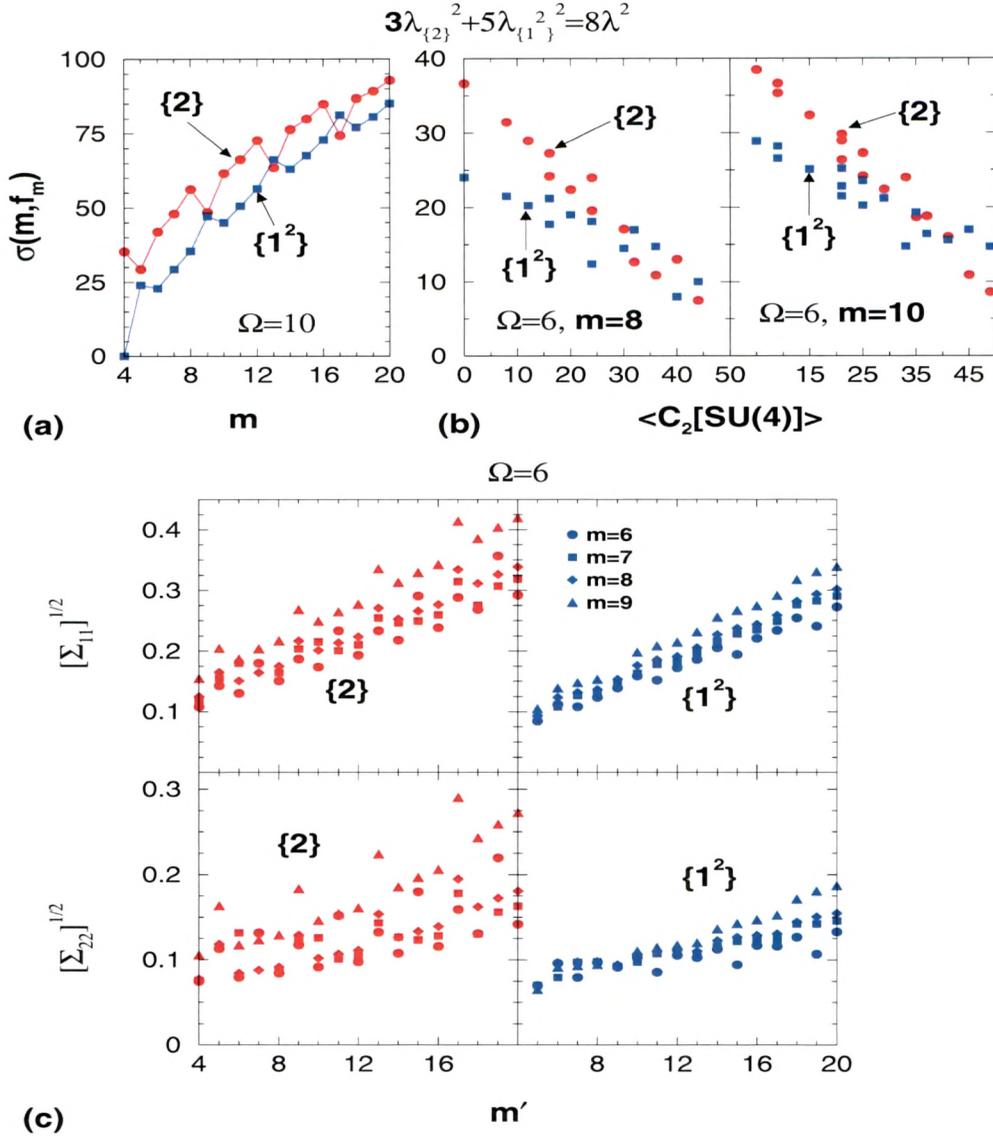


Figure 4.8: (a) Variation of spectral widths $\sigma(m, f_m)$ as a function of m with $f_m = f_m^{(p)}$. (b) Variation of spectral widths as a function of $\langle C_2[SU(4)] \rangle^{f_m}$ for $m = 8$ and 10 . In (c), results are shown for the covariances in energy centroids $[\Sigma_{11}]^{1/2}$ and spectral variances $[\Sigma_{22}]^{1/2}$ for some values of m and m' with $f_m = f_m^{(p)}$ and $f_{m'} = f_{m'}^{(p)}$. For the calculations in (a), $\Omega = 10$ and for (b) and (c), $\Omega = 6$. Note that in the figures $\lambda_{\{2\}}^2 = 8/3$, $\lambda_{\{1^2\}}^2 = 0$ is denoted as ' $\{2\}$ ' and similarly $\lambda_{\{2\}}^2 = 0$, $\lambda_{\{1^2\}}^2 = 8/5$ is denoted as ' $\{1^2\}$ '. See text for details.

4.7.3 Results for $\lambda_{\{2\}}^2 \neq \lambda_{\{1^2\}}^2$

All the discussion in the Secs. 4.6, 4.7.1, and 4.7.2 is restricted to $\lambda_{\{1^2\}}^2 = \lambda_{\{2\}}^2$, i.e., for equal strengths for the symmetric and anti-symmetric parts of the interaction. For completeness, we have studied the variation of widths and covariances when $\lambda_{\{1^2\}}^2 \neq \lambda_{\{2\}}^2$ by fixing the value for the ensemble averaged two-particle spectral vari-

ance $\sigma_{H(2)}^2(2)$ to a constant and then varying $\lambda_{\{2\}}$ (or equivalently $\lambda_{\{1^2\}}$). The two-particle spectral variance for $\Omega \gg 1$ is $\sigma_{H(2)}^2(2) = \Omega^2[3\lambda_{\{2\}}^2 + 5\lambda_{\{1^2\}}^2]/16$. Therefore calculations are carried out with the constraint $[3\lambda_{\{2\}}^2 + 5\lambda_{\{1^2\}}^2] = 8\lambda^2$. All our previous results correspond to $\lambda_{\{2\}}^2 = \lambda_{\{1^2\}}^2 = \lambda^2 = 1$. Now we will discuss some results for the extreme cases: (i) $\lambda_{\{1^2\}}^2 = 0$, $\lambda_{\{2\}}^2 = 8/3$ (denoted by $\{2\}$ in Fig. 4.8 and this corresponds to $H = H_{\{2\}}$) and (ii) $\lambda_{\{1^2\}}^2 = 8/5$, $\lambda_{\{2\}}^2 = 0$ (denoted by $\{1^2\}$ in Fig. 4.8 and this corresponds to $H = H_{\{1^2\}}$). Figure 4.8(a) shows that the spectral widths have peaks at $m = 4r$ and $m = 4r + 1$ for $H_{\{2\}}$ and $H_{\{1^2\}}$, respectively. The peak for $H_{\{2\}}$ is much larger and for $H_{\{1^2\}}$ it appears at a wrong place when compared to the results shown in Fig. 4.3 for $H = H_{\{2\}} \oplus H_{\{1^2\}}$. Similarly, it is seen from Fig. 4.8(b) that the variation in the spectral widths $\sigma(m, f_m) = [\langle H^2 \rangle^{m, f_m}]^{1/2}$ as a function of f_m show more fluctuations as compared to a good linear behavior for $\lambda_{\{1^2\}}^2 = \lambda_{\{2\}}^2$. Figure 4.8(c) shows self and cross-correlations $\Sigma_{11}(m, f_m; m', f_{m'})$ and $\Sigma_{22}(m, f_m; m', f_{m'})$ with $f_m = f_m^{(p)}$ as a function of m and m' . Results for $H_{\{2\}}$ and $H_{\{1^2\}}$ show more fluctuations and more importantly, the magnitude of correlations for $H_{\{2\}}$ is much larger and for $H_{\{1^2\}}$ somewhat smaller compared to the results for $H = H_{\{2\}} \oplus H_{\{1^2\}}$. From this exercise, we can conclude that the results for spectral widths and lower order correlations will deviate strongly from those reported in Secs. 4.6 and 4.7 ($\lambda_{\{1^2\}}^2 = \lambda_{\{2\}}^2 = \lambda^2$) when $\lambda_{\{1^2\}}^2$ differs significantly from $\lambda_{\{2\}}^2$.

4.7.4 Conclusions

Increase in the magnitude of self-correlations in energy centroids and spectral variances, defined by Σ_{11} and Σ_{22} and the matrices $H_{f_m}(m)$ becoming more dense as we go from EGUE(2) \rightarrow EGUE(2)-s \rightarrow EGUE(2)-SU(4) is an important result that deserves more investigation. The cross-correlations increase with m' for fixed f_m and also with $\langle C_2[SU(4)] \rangle^{\tilde{f}_m}$ with fixed m . For $\lambda_{\{2\}}^2 \neq \lambda_{\{1^2\}}^2$, results for spectral widths and lower order correlations will deviate strongly from those with $\lambda_{\{2\}}^2 = \lambda_{\{1^2\}}^2$ only when $\lambda_{\{1^2\}}^2$ differs significantly from $\lambda_{\{2\}}^2$.

4.8 Summary

We have introduced in this chapter a new embedded ensemble, EGUE(2)-SU(4), and it is defined for two-body Hamiltonians preserving SU(4) symmetry for a system of

m fermions in Ω number of levels each four-fold degenerate. We have developed, for this ensemble, an analytical formulation based on the Wigner-Racah algebra of the embedding $U(\Omega) \otimes SU(4)$ algebra. Explicit formulas are derived for spectral variances and covariances in energy centroids and spectral variances for $U(\Omega)$ irreps of the type $f_m^{(p)} = \{4^r, p\}$, $p = 0, 1, 2$ and 3 . Results in Tables 4.3, 4.6, 4.8 and 4.9 allow one to calculate these for any m and Ω . For general $U(\Omega)$ irreps f_m , the analytical formulation in Secs. 4.3-4.5 and the formulas in the Tables 4.4 and 4.7 (obtained by simplifying the tabulations due to Hecht [He-74a]), allows one to carry out numerical calculations and codes for the same are developed. The analytical formulas in the Tables led to simple expressions for the covariances in energy centroids and spectral variances in the dilute limit for the irreps $f_m^{(p)}$. Using the formulation in Secs. 4.3-4.5 and the results in Tables 4.3-4.9, several numerical calculations are carried out and the results are presented in Secs. 4.6 and 4.7 and in Figs. 4.3-4.8. Main conclusions from these are as follows:

- (i) Expectation values $\langle C_2[SU(4)] \rangle^E$ studied in Section 4.6.2 by constructing Gaussian partial densities with centroids given by $\langle C_2[SU(4)] \rangle^{\bar{f}_m}$ and variances given by $\langle H^2 \rangle^{m, f_m}$ and similarly, the four periodicity in the gs energies studied in Section 4.6.3, establish that random interactions with $SU(4)$ symmetry keep intact the essential features of the Majorana force. This conclusion is quite similar to the result derived for EGOE(2)- J (also called TBRE some times), the embedded ensemble with angular-momentum J symmetry. This ensemble is generated by (see also Sec. 7.4) random interactions that are J scalar [$SO(3)$ scalar] and it is found that, for systems with even number of fermions, there is $J^\pi = 0^+$ preponderance in the ground states. This feature has been investigated in many different ways [Zh-04, Zh-04a, Ze-04, Pa-04]. It should be noted that the $SO(3)$ invariant operator is J^2 and it gives (with $H = J^2$) $J = 0$ as gs, a property generated also by random interactions.
- (ii) As shown in Section 4.7.1, there is increase in the magnitude of self-correlations in energy centroids and spectral variances, defined by Σ_{11} and Σ_{22} in direct correlation with the $H_{f_m}(m)$ matrices becoming more dense (implying stronger mixing) as we go from EGUE(2) \rightarrow EGUE(2)- \mathbf{s} \rightarrow EGUE(2)- $SU(4)$. Further inves-

tigation of this feature may provide additional justification for the recent claim by Papenbrock and Weidenmüller [Pa-05] that symmetries are responsible for chaos in nuclear shell-model spaces.

- (iii) As shown in Section 4.7.2, there is a significant increase in cross-correlations with particle number m for a fixed $U(\Omega)$ irrep f_m and similarly with $\langle C_2[SU(4)] \rangle_{\bar{f}_m}$ for fixed m . This could be used as a signature for experimental detection of cross-correlations generated by EGUE(2)- $SU(4)$.

Finally, we conclude that the results presented in the present chapter represent a first detailed analytical study of an embedded ensemble with a non-trivial symmetry that is relevant in nuclear structure.