# Appendix G

# Further extensions of BEGOE(1+2)

For completeness, we briefly outline here extension of BEGOE(1+2) to BEGOE(1+2)-$M_S$ and BEGOE(1+2)-$p$; here $p$ corresponds to spin $\mathbf{s} = 1$ bosons and $M_S$ is the $S_z$ quantum number for spin $\mathbf{s} = \frac{1}{2}$ bosons. We restrict our discussion to the definition and construction of these ensembles using the results for spinless BEGOE(1+2) discussed in Chapter 1.

**BEGOE(1+2)-$M_S$**

Consider a system of $m$ bosons occupying $\Omega$ number of sp orbitals each with spin $\mathbf{s} = \frac{1}{2}$ so that the number of sp states $N = 2\Omega$. The sp states are denoted by $|\,v_i, m_{\mathbf{s}}\rangle$, $i = 1, 2, \ldots, \Omega$ and $m_{\mathbf{s}} = \pm\frac{1}{2}$. The average spacing between the $v_i$ states is assumed to be $\Delta$ and between two $m_{\mathbf{s}}$ states for a given $v_i$ to be $\Delta_{m_{\mathbf{s}}}$. For constructing the $H$ matrix in good $M_S$ representation, we arrange the sp states $\left|i, m_{\mathbf{s}} = \pm\frac{1}{2}\right\rangle$ in such a way that the first $\Omega$ states have $m_{\mathbf{s}} = \frac{1}{2}$ and the remaining $\Omega$ states have $m_{\mathbf{s}} = -\frac{1}{2}$. Many-particle states for $m$ bosons in the $2\Omega$ sp states, arranged as explained above, can be obtained by distributing $m_1$ bosons in the $m_{\mathbf{s}} = \frac{1}{2}$ sp states ($\Omega$ in number) and similarly, $m_2$ fermions in the $m_{\mathbf{s}} = -\frac{1}{2}$ sp states ($\Omega$ in number) with $m = m_1 + m_2$. Thus, $M_S = (m_1 - m_2)/2$. Let us denote each distribution of $m_1$ fermions in $m_{\mathbf{s}} = \frac{1}{2}$ sp states by $\mathbf{m_1}$ and similarly, $\mathbf{m_2}$ for $m_2$ fermions in $m_{\mathbf{s}} = -\frac{1}{2}$ sp states. Many-particle basis defined by $(\mathbf{m_1}, \mathbf{m_2})$ with $m_1 - m_2 = 2M_S$ will form the basis for BEGOE(1+2)-$M_S$. As the two-particle $m_s$ can take values $\pm 1$ and 0, the two-body part of the Hamiltonian preserving $M_S$ will be $\widehat{V}(2) = \lambda_0 \widehat{V}^{m_s=0}(2) + \lambda_1 \widehat{V}^{m_s=1}(2) + \lambda_{-1} \widehat{V}^{m_s=-1}(2)$ with the corresponding two-particle matrix being a direct sum matrix generated by $\widehat{V}^{m_s}(2)$. Therefore, the

Hamiltonian is

$$\widehat{H} = \widehat{h}(1) + \lambda_0 \left\{ \widehat{V}^{m_s=0}(2) \right\} + \lambda_1 \left\{ \widehat{V}^{m_s=1}(2) \right\} + \lambda_{-1} \left\{ \widehat{V}^{m_s=-1}(2) \right\} . \tag{G1}$$

In Eq. (G1), the $\{\widehat{V}^{m_s}(2)\}$ ensembles in two-particle spaces are represented by independent GOE(1)'s [see Eq. (1.2.4)] and $\lambda_{m_s}$'s are their corresponding strengths. The action of the Hamiltonian operator defined by Eq. (G1) on the $(\mathbf{m_1},\mathbf{m_2})$ basis states with a given $M_S$ generates the BEGOE(1+2)-$M_S$ ensemble in $m$-particle spaces. Therefore, BEGOE(1+2)-$M_S$ is defined by six parameters $(\Omega, m, \Delta_{m_s}, \lambda_0, \lambda_1, \lambda_{-1})$ [we put $\Delta = 1$ so that $\Delta_{m_s}$ and $\lambda_{m_s}$'s are in the units of $\Delta$]. In the $(\mathbf{m_1},\mathbf{m_2})$ basis with a given $M_S$, the $H$ matrix construction reduces to the matrix construction for spinless boson systems; see Chapter 1. The $H$ matrix dimension for a given $M_S$ is $\sum_{S \geq M_S} d_b(\Omega, m, S)$. Finally, pairing can also be introduced in this ensemble using the algebra $U(2\Omega) \supset SO(2\Omega) \supset SO(\Omega) \otimes SO(2)$ with $SO(2)$ generating $M_S$; see [Ko-06c].

**BEGOE(1+2)-*p***

Let us begin with a system of $m$ bosons distributed say in $\Omega$ number of sp orbitals each with spin $\mathbf{s} = 1$ so that the number of sp states $N = 3\Omega$. The sp states are denoted by $|i, m_\mathbf{s}\rangle$ with $m_\mathbf{s} = 0, \pm 1$ and $i = 1, 2, \ldots, \Omega$. For a one plus two-body Hamiltonians preserving $m$-particle spin $S$, the one-body Hamiltonian $\widehat{h}(1)$ is defined by the sp energies $\epsilon_i$; $i = 1, 2, \ldots, \Omega$, with average spacing $\Delta$. Similarly the two-body Hamiltonian $\widehat{V}(2)$ is defined by the two-body matrix elements $\lambda_s V^s_{ijkl} = \left\langle (kl)s, m_s \mid \widehat{V}(2) \mid (ij)s, m_s \right\rangle$ with the two-particle spins $s = 0, 1$ and 2. These matrix elements are independent of the $m_s$ quantum number. Note that the $\lambda_s$ are parameters. For generating the many-particle states, firstly, the sp states are arranged such that the first $\Omega$ number of sp states have $m_\mathbf{s} = 1$, next $\Omega$ number of sp states have $m_\mathbf{s} = 0$ and the remaining $\Omega$ sp states have $m_\mathbf{s} = -1$. Now, the many-particle states for $m$ bosons can be obtained by distributing $m_1$ bosons in the $m_\mathbf{s} = 1$ sp states, $m_2$ bosons in the $m_\mathbf{s} = 0$ sp states and similarly, $m_3$ bosons in the $m_\mathbf{s} = -1$ sp states with $m = m_1 + m_2 + m_3$. Thus, $M_S = (m_1 - m_3)$. Let us denote each distribution of $m_1$ bosons in $m_\mathbf{s} = 1$ sp states by $\mathbf{m_1}$, $m_2$ bosons in $m_\mathbf{s} = 0$ sp states by $\mathbf{m_2}$ and similarly, $\mathbf{m_3}$ for $m_3$ bosons in $m_\mathbf{s} = -1$ sp states. Many-particle basis defined by $(\mathbf{m_1},\mathbf{m_2},\mathbf{m_3})$

will form a basis for BEGOE(1+2)-$p$. The $V$ matrix in two-particle spaces will be a direct sum matrix and the $V(2)$ operator is $\widehat{V}(2) = \lambda_0 \widehat{V}^{s=0}(2) + \lambda_1 \widehat{V}^{s=1}(2) + \lambda_2 \widehat{V}^{s=2}(2)$ with three parameters $(\lambda_0, \lambda_1, \lambda_2)$. Now, BEGOE(1+2)-$p$ for a given $(m, S)$ system is generated by defining the three parts of the two-body Hamiltonian to be independent GOE(1)'s in two-particle spaces and then propagating the $V(2)$ ensemble $\{\widehat{V}(2)\} = \lambda_0\{\widehat{V}^{s=0}(2)\} + \lambda_1\{\widehat{V}^{s=1}(2)\} + \lambda_2\{\widehat{V}^{s=2}(2)\}$ to the $m$-particle spaces with a given spin $S$ by using the geometry (direct product structure) of the $m$-particle spaces. The embedding algebra is $U(3\Omega) \supset G \supset G1 \otimes SO(3)$ with SO(3) generating spin S. Thus BEGOE(1+2)-$p$ is defined by the operator

$$\widehat{H} = \widehat{h}(1) + \lambda_0 \{\widehat{V}^{s=0}(2)\} + \lambda_1 \{\widehat{V}^{s=1}(2)\} + \lambda_2 \{\widehat{V}^{s=2}(2)\} . \tag{G2}$$

The sp levels defined by $\widehat{h}(1)$ will be triply degenerate. The action of the Hamiltonian operator defined by Eq. (G2) on $(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3})$ basis states with fixed-$(m, M_S = M_S^{min})$ generates the ensemble in $(m, M_S)$ spaces. It is important to note that the construction of the $m$-particle $H$ matrix in fixed-$(m, M_S = M_S^{min})$ spaces reduces to the problem of BEGOE(1+2) for spinless boson systems and hence Eqs. (1.3.1)- (1.3.3) of Chapter 1 will apply. Then the $\hat{S}^2$ operator is used for projecting states with good $S$. Therefore, BEGOE(1+2)-$p$ ensemble is defined by five parameters $(\Omega, m, \lambda_0, \lambda_1, \lambda_2)$ with $\lambda_s$ in units of $\Delta$. Finally, it is important to mention that it is also possible to study the pairing symmetry in the space defined by BEGOE(1+2)-$p$ ensemble. For this, there are two possible algebras (each defining a particular type of pairing), $U(3\Omega) \supset [U(\Omega) \supset SO(\Omega)] \otimes [U(3) \supset SO(3)]$ and $U(3\Omega) \supset SO(3\Omega) \supset SO(\Omega) \otimes SO(3)$ and they can be studied in detail by extending the results for IBM-3 model in nuclear structure where $\Omega = 6$ [Ga-99, Ko-96]. Exploiting the group chain $U(3\Omega) \supset U(\Omega) \otimes [U(3) \supset SO(3)]$, it is possible to write the dimension formulas for the $H$ matrices for a given $(m, S)$ as it was done in Sec. 4.2.3 for $SU(4) - ST$ reductions.