Appendix J

Bivariate edgeworth expansion

Given the bivariate Gaussian, in terms of the standardized variables $\hat{x}$ and $\hat{y}$,

$$\eta_{\mathscr{G}}(\hat{x},\hat{y}) = \frac{1}{2\pi\sqrt{(1-\zeta^2)}}\, exp\left\{-\frac{\hat{x}^2 - 2\zeta\hat{x}\hat{y} + \hat{y}^2}{2(1-\zeta^2)}\right\}, \tag{J1}$$

the bivariate Edgeworth expansion for any bivariate distribution $\eta(\hat{x},\hat{y})$ follows from,

$$\eta(\hat{x},\hat{y}) = \exp\left\{\sum_{r+s\geq 3}(-1)^{r+s}\frac{k_{rs}}{r!s!}\frac{\partial^r}{\partial\hat{x}^r}\frac{\partial^s}{\partial\hat{y}^s}\right\}\eta_{\mathscr{G}}(\hat{x},\hat{y})\,. \tag{J2}$$

Assuming that the bivariate reduced cumulants k_{r+s} behave as $k_{r_s} \propto \Upsilon^{-(r+s-2)/2}$ where Υ is a system parameter, and collecting in the expansion of Eq. (J2) all the terms that behave as $\Upsilon^{-P/2}$, $P = 1,2,\ldots$, we obtain the bivariate ED expansion to order $1/P$ [Ko-84, St-87],

$$\begin{aligned}
\eta_{biv-ED}(\hat{x},\hat{y}) &= \left\{1 + \left(\frac{k_{30}}{6}He_{30}(\hat{x},\hat{y}) + \frac{k_{21}}{2}He_{21}(\hat{x},\hat{y})\right.\right.\\
&+ \left.\frac{k_{12}}{2}He_{12}(\hat{x},\hat{y}) + \frac{k_{03}}{6}He_{03}(\hat{x},\hat{y})\right)\\
&+ \left(\left\{\frac{k_{40}}{24}He_{40}(\hat{x},\hat{y}) + \frac{k_{31}}{6}He_{31}(\hat{x},\hat{y})\right.\right.\\
&+ \left.\frac{k_{22}}{4}He_{22}(\hat{x},\hat{y}) + \frac{k_{13}}{6}He_{13}(\hat{x},\hat{y}) + \frac{k_{04}}{24}He_{04}(\hat{x},\hat{y})\right\}
\end{aligned} \tag{J3}$$

$$
\begin{aligned}
&+ \left\{\frac{k_{30}^2}{72} He_{60}(\hat{x},\hat{y}) + \frac{k_{30}k_{21}}{12} He_{51}(\hat{x},\hat{y})\right.\\
&+ \left[\frac{k_{21}^2}{8} + \frac{k_{30}k_{12}}{12}\right] He_{42}(\hat{x},\hat{y})\\
&+ \left[\frac{k_{30}k_{03}}{36} + \frac{k_{12}k_{21}}{4}\right] He_{33}(\hat{x},\hat{y})\\
&+ \left[\frac{k_{12}^2}{8} + \frac{k_{21}k_{03}}{12}\right] He_{24}(\hat{x},\hat{y}) + \frac{k_{12}k_{03}}{12} He_{15}(\hat{x},\hat{y})\\
&+ \left.\left.\left.\frac{k_{03}^2}{72} He_{06}(\hat{x},\hat{y})\right\}\right)\right\} \eta_{\mathscr{G}}(\hat{x},\hat{y})\,.
\end{aligned}
$$

The bivariate Hermite polynomials $He_{m_1m_2}(\hat{x},\hat{y})$ in Eq. (J3) satisfy the recursion relation,

$$
\begin{aligned}
(1-\zeta^2)He_{m_1+1,m_2}(\hat{x},\hat{y}) &= (\hat{x}-\zeta\hat{y})He_{m_1,m_2}(\hat{x},\hat{y})\\
&- m_1 He_{m_1-1,m_2}(\hat{x},\hat{y}) + m_2\zeta He_{m_1,m_2-1}(\hat{x},\hat{y})\,.
\end{aligned}
\tag{J4}
$$

The polynomials $He_{m_1m_2}$ with $m_1+m_2 \le 2$ are

$$
\begin{aligned}
He_{00}(\hat{x},\hat{y}) &= 1\,,\\
He_{10}(\hat{x},\hat{y}) &= \left(\hat{x}-\zeta\hat{y}\right)/\left(1-\zeta^2\right),\\
He_{20}(\hat{x},\hat{y}) &= \frac{(\hat{x}-\zeta\hat{y})^2}{(1-\zeta^2)^2} - \frac{1}{(1-\zeta^2)}\,,\\
He_{11}(\hat{x},\hat{y}) &= \frac{(\hat{x}-\zeta\hat{y})(\hat{y}-\zeta\hat{x})}{(1-\zeta^2)^2} + \frac{\zeta}{1-\zeta^2}\,.
\end{aligned}
\tag{J5}
$$

Note that $He_{m_1m_2}(\hat{x},\hat{y}) = He_{m_2m_1}(\hat{y},\hat{x})$.