Chapter 4

Trajectory Controllability of Impulsive Systems

Contents

This chapter discusses the trajectory controllability of the governed system with classical and non-local initial conditions over the general Banach space. The results of the trajectory controllability for governed systems are obtained through the concept of operator semigroup and Gronwall's inequality. This manuscript is also equipped with examples to illustrate the applications of derived results.

4.1 Introduction

Impulsive differential equations play a vital role in studying the behavior of the phenomenon having abrupt changes in physical problems. If the changes are at a fixed moment then it is called instantaneous impulsive differential equations and if the changes are over small intervals then it is called non-instantaneous impulsive differential equations. There is a wide range of applications of these impulsive evolution equations in all fields of science namely, physical sciences, biological science, and environmental sciences. These applications are found in the monographs [1, 84, 132] and research articles [7, 39, 44, 79, 69] and references there in. Qualitative properties like the existence and uniqueness of solution and continuity of the solutions of instantaneous and noninstantaneous differential or integro-differential or evolution equations are found in research articles with initial conditions found in research articles [5, 24, 28, 29, 30, 46, 63, 91, 92, 100, 123, 124, 134, 135, 141, 144, 149, 152, 154, 158] and references therein.

Nowadays, the concept of controllability plays an important role in the field of applied mathematics. In the notion of controllability, one has to find the control that steers the initial state at the initial time to the desired final state at the final time. The complete controllability of a finite-dimensional linear system using a functional analytic approach was first introduced by Kalmann. Many researchers were involved in developing the different controllability of various linear and nonlinear finite and infinite dimensional impulsive and non-impulsive systems using the functional analytic approach. The notion of controllability are found in the monographs [22, 125, 140] and articles [51, 70, 80, 82, 90] and reference their in. Trajectory controllability is the strongest form of controllability among all other forms of controllability. The study of trajectory controllability of one-dimensional systems was initiated by George [122]. Thereafter Chalishajar, et. al. [27] generalized the concept of trajectory controllability on finite and infinite-dimensional systems.

This chapter established the trajectory controllability of the system:

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t), && t \in [s_k, t_k + 1),\ for\ all\ k = 0, 1, 2 \cdots, p \\ x(t) &= \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), && t \in [t_k, s_k),\ for\ all\ k = 1, 2, \cdots, p, \end{aligned} \tag{4.1.1}$$

with local condition $x(0) = x_0$ and non-local condition $x(0) = x_0 - h(x)$.

4.2 Preliminaries

This section discusses definitions and prepositions to establish trajectory controllability of the system governed by non-instantaneous impulsive evolution equation with classical as well nonlocal conditions.

Definition 4.2.1. *[70] The system* (4.1.1) *is completely controllable on the interval* $\mathcal{J} = [0, T_0]$ *if for any* $x_0, x_1 \in \mathbb{X}$*, if there exist a control* $\mathcal{W}(\cdot)n$ *in* $\mathbb{U}$ *(control space) steers the system from* x_0 *at* $t = 0$ *to* x_1 *at* $t = T_0$*.*

In the definition of complete controllability, there is no information on the path or trajectory on which the given system is to be driven. Sometimes this leads to high cost So to overcome this situation we select the path or trajectory (having minimum cost) under which the control system drives from x_0 to x_1 over the interval $\mathcal{J}$. Searching of controller $\mathcal{W}(\cdot)$ in a way that the system drives from x_0 to x_1 over the interval is called trajectory controllability of the system. Therefore, the trajectory controllability of the system is strongest among all types of controllability.

Definition 4.2.2. *[27] Let,* $\mathcal{C}_\mathcal{T}$ *be the set of all trajectories under which the system* (4.1.1) *drives from* x_0 *to* x_1 *over the interval* $\mathcal{J}$*. The system* (4.1.1) *is trajectory controllable if for any* $z \in \mathcal{C}_\mathcal{T}$*, there is a controller* $\mathcal{W}(\cdot) \in \mathbb{U}$ *such that state of the system* $x(t)$ *drives on prescribed trajectory* $z(t)$*. This means* $x(t) = z(t)$ *a.e. over the interval* $\mathcal{J}$*.*

4.3 T-controllability with local conditions

Consider the system governed by the non-instantaneous impulsive evolution equation

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t), \quad t \in [s_k, t_{k+1}) \\ x(t) &= \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), \quad t \in [t_k, s_k) \\ x(0) &= x_0 \end{aligned} \tag{4.3.1}$$

over the interval $[0, T_0]$. Here, $x(t)$ is the state of the system lies in Banach space $\mathbb{X}$ at any time $t \in [0, T_0]$, $\mathcal{A}$ at any time t is a linear operator on the Banach space $\mathbb{X}$, $\mathcal{F}, \mathcal{G}_k : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions and $\mathcal{W}(t)$ and $\mathcal{W}_k(t)$ are trajectory controller of the system.

To discuss the trajectory controllability of the system governed by non-instantaneous impulsive evolution equation (4.3.1), we have the following theorem:

Theorem 4.3.1. *If,*

(A1) Linear operator $\mathcal{A}$ *in the system* (4.3.1) *infinitesimal generator of* C_0 *semi-group.*

(A2) The non-linear map $\mathcal{F} : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ *is continuous such that there exist a non-decreasing function* $l_F : \mathbb{R}_+ \to \mathbb{R}+$ *and positive real number* r_0 *satisfying*

$$||\mathcal{F}(t, x_1) - \mathcal{F}(t, x_2)|| \leq l_F(r)||x_1 - x_2||$$

, for all $t \in [0, T_0]$, $x_1, x_2 \in B_r(\mathbb{X})$ *and* $r \leq r_0$.

(A3) The non-linear map $\mathcal{G}_k : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ *for all* k *are are continuous such that there exist constants* $0 < l_{gk} < 1$ *satisfying*

$$||\mathcal{G}_k(t, x_1) - \mathcal{G}_k(t, x_2)|| \leq l_{gk}||x_1 - x_2||$$

, for all $t \in [0, T_0]$ *and* $l_g = max\{l_{gk}; \forall k\}$.

Then, the system (4.3.1) *is trajectory controllable over the interval* $[0, T_0]$.

Proof. Let $y(t)$ be any desired state trajectory in $\mathcal{C}_\sqcup$ satisfying $x(t_k^+) = y(t_k^+)$ along which the system (4.3.1) steered from initial state x_0 at $t = 0$ to desired final state x_1 at $t = T_0$.

Over the interval $[0, t_1)$, the system (4.3.1) becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t) \\ x(0) &= x_0 \end{aligned} \tag{4.3.2}$$

Consider

$$\mathcal{W}(t) = y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

over the interval $[0, t_1)$ in and plugging it in the (4.3.2) the system (4.3.2) becomes

$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

with initial condition $x(0) - y(0) = 0$.
Choosing variable $z = x - y$ the equation system reduces to

$$\begin{aligned} z'(t) &= \mathcal{A}(t)z(t) + \mathcal{F}(t, x(t)) - \mathcal{F}(t, y(t)) \\ z(0) &= 0 \end{aligned} \tag{4.3.3}$$

and problem of trajectory controllability of the system (4.3.2) is reduced to the solvability of the system (4.3.3) over the interval $[0, t_1)$. The mild solution of the system (4.3.3) is given by:

$$z(t) = \int_0^t \mathcal{T}(t-\zeta)[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]d\zeta \tag{4.3.4}$$

where, $\mathcal{T}(t)$ is C_0 semigroup generated by linear operator $\mathcal{A}$ satisfying $||\mathcal{T}(t)|| \leq M$ for some positive number M.
Therefore,

$$\begin{aligned} ||z(t)|| &\leq \int_0^t ||\mathcal{T}(t-\zeta)|| \; ||[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]||d\zeta \\ &\leq M \int_0^t l_F(r)||x(\zeta) - y(\zeta)||d\zeta \\ &\leq M \int_0^t l_F(r)||z(\zeta)||d\zeta \end{aligned}$$

and using Gronwall's inequality, we obtain $z(t) = 0$ over the interval $[0, t_1)$. Hence, $x(t) = y(t)$ for all $t \in [0, t_1)$. Therefore, the system is trajectory controllable over the interval $[0, t_1)$.

Over the interval $[t_k, s_k)$, the system becomes

$$x(t) = \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), \tag{4.3.5}$$

and at $t = t_k^+$ value of the state x is given by $x(t_k^+) = \mathcal{G}_k(t_k^+, x(t_k^+)) + \mathcal{W}_k(t_k^+)$.
Plugging the trajectory controller $\mathcal{W}_k(t) = y(t) - \mathcal{G}_k(t, y(t))$ over the interval $[t_k, s_k)$ in the system (4.3.5) the system becomes:

$$x(t) - y(t) = \mathcal{G}_k(t, y(t)) - \mathcal{G}_k(t, y(t))$$

Choosing $z(t) = x(t) - y(t)$ we obtain

$$z(t) = \mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, y(t))$$

and the value of the z at $t = t_k^+$ is zero. Therefore, we have

$$||z(t)|| \leq ||\mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, y(t))|| \leq l_g||z(t)||,$$

using (A3) $l_g < 1$ we obtain $z(t) = 0$ for all $t \in [t_k, s_k)$ Therefore, system (4.3.1) is T-Controllable over the interval $[t_k, s_k)$. Moreover, $z(s_k) = 0$ as $\mathcal{G}_k's$ are continuous.

Over the interval $[s_k, t_{k+1})$ the system (4.3.1) becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t) \\ x(s_k) &= y(s_k) \end{aligned} \tag{4.3.6}$$

Choose the controller over the interval $[s_k, t_{k+1}$ as:

$$\mathcal{W}(t) = y'(t) - \mathcal{A}(t)y(r) - \mathcal{F}(t, y(t))$$

and plugging it in the equation (4.3.6) we get,

$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

considering $z(t) = x(t) - y(t)$, above expression becomes:

$$\begin{aligned} z'(t) &= \mathcal{A}(t)z(t) + \mathcal{F}(t, x(t)) - \mathcal{F}(t, y(t)) \\ z(s_k) &= 0, \end{aligned} \tag{4.3.7}$$

Therefore

$$\begin{aligned} ||z(t)|| &\leq \int_{s_k}^{t} ||\mathcal{T}(t-\zeta)|| \; ||[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]||d\zeta \\ &\leq M \int_0^t l_F(r)||x(\zeta) - y(\zeta)||d\zeta \\ &\leq M \int_0^t l_F(r)||z(\zeta)||d\zeta \end{aligned}$$

and using Gronwall's inequality, we obtain $z(t) = 0$ over the interval $[s_k, t_{k+1})$. Hence, $x(t) = y(t)$ for all $t \in [s_k, t_{k+1})$. Therefore, the system is T- controllable over the interval $[s_k, t_{k+1})$.

Since, the system is T-controllable over the intervals $[0, t_1), [s_k, t_{k+1})$ and $[t_k, s_k)$ for all k. Hence, the system is controllable over the entire interval $[0, T_0]$. This completes the proof of the theorem. □

Example 4.3.1. *Let, $\mathbb{X} = L^2([0,\pi], \mathbb{R})$ and consider the system governed by a non-instantaneous impulsive evolution equation:*

$$\begin{aligned}
\frac{\partial H(t,\psi)}{\partial t} &= \partial_\psi^2 H(t,\psi) + F(t, H(t,\psi)) + w(t,\psi) && t \in [0,1/3) \cup [2/3,1],\\
H(t,\psi) &= G_1(t, H(t,\psi)) + w_1(t,\psi) && t \in [1/3,2/3),\\
H(t,0) &= 0 \quad H(t,\pi) = 0 && t > 0,\\
H(0,\psi) &= H_0(\psi) && 0 < \psi < \pi,
\end{aligned} \tag{4.3.8}$$

over the interval $[0,1]$.
Defining the operator on the space $\mathbb{X}$ as $\mathcal{A}(t) = \partial_\psi^2$, $\mathcal{A}(t)$ is the infinitesimal generator of the C_0 semigroup $\mathcal{T}(t)$. The representation of $\mathcal{T}(t)$ is

$$\mathcal{T}(t)z = \sum_{m=0}^{\infty} \exp(\mu_m t) < z, \phi_m > \phi_m$$

where, $\phi_m = \sqrt{2} sin(n\psi)$ for all $m = 1, 2, \cdots$ is the orthonormal basis corresponding to eigenvalue $\mu_m = -m^2$ of the operator $\mathcal{A}$.

With this concept the equation (4.3.8) can be rewritten as an abstract equation on the space $\mathbb{X}$ as

$$\begin{aligned}
x'(t) &= \mathcal{A}(t) + \mathcal{F}(t,x) + \mathcal{W}(t) && t \in [0,1/3) \cup [2/3,1],\\
x(t) &= \mathcal{G}_1(t,x) + \mathcal{W}_1 && t \in [1/3,2/3),\\
x(0) &= x_0,
\end{aligned} \tag{4.3.9}$$

where, $x(t) = H(t,\cdot), \mathcal{W}(t) = w(t,\cdot)$ and $W_1(t) = w_1(t,\psi)$. The system (4.3.9) is trajectory controllable over the interval $[0,1]$ if the functions $\mathcal{F}$ and $\mathcal{G}_1$ satisfy the hypotheses of the theorem.

4.4 T-controllability with non-local conditions

Consider the system governed by the non-instantaneous impulsive evolution equation

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t), \quad t \in [s_k, t_{k+1}) \\ x(t) &= \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), \quad t \in [t_k, s_k) \\ x(0) &= h(x) \end{aligned} \tag{4.4.1}$$

over the interval $[0, T_0]$. Here, $x(t)$ is the state of the system lies in Banach space $\mathbb{X}$ at any time $t \in [0, T_0]$, $\mathcal{A}(t)$ at any time t is a linear operator on the Banach space $\mathbb{X}$, $\mathcal{F}, \mathcal{G}_k : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions, $\mathcal{W}_k(t)$ are trajectory controller of the system and $h : \mathbb{X} \to \mathbb{X}$ is the operator representing the non-local conditions. The mild solution of the equation (4.4.1) is given by

$$x(t) = \begin{cases} \mathcal{T}(t)h(x) + \displaystyle\int_0^t \mathcal{T}(t-\zeta)\Big[\mathcal{F}(\zeta, x(\zeta)) + \mathcal{W}(\zeta)\Big]d\zeta, & t \in [0, t_1) \\ \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), & t \in [t_k, s_k) \\ \mathcal{T}(t)\mathcal{G}_k(s_k, x(s_k)) + \displaystyle\int_{s_k}^t \mathcal{T}(t-\zeta)\Big[\mathcal{F}(\zeta, x(\zeta)) + \mathcal{W}(\zeta)\Big]d\zeta, & t \in [s_k, t_{k+1}), \end{cases} \tag{4.4.2}$$

where, $\mathcal{T}(t)$ is semigroup generated by the linear operator $\mathcal{A}(t)$.

The following theorem discusses the trajectory controllability of the system governed by the equation (4.4.1).

Theorem 4.4.1. *If,*

(A1) Linear operator $\mathcal{A}$ in the system (4.3.1) *infinitesimal generator of C_0 semi-group.*

(A2) The non-linear map $F : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ is continuous such that there exist a non-decreasing function $l_F : \mathbb{R}_+ \to \mathbb{R}+$ and positive real number r_0 satisfying

$$||\mathcal{F}(t, x_1) - \mathcal{F}(t, x_2)|| \leq l_F(r)||x_1 - x_2||$$

, for all $t \in [0, T_0]$, $x_1, x_2 \in B_r(\mathbb{X})$ and $r \leq r_0$.

(A3) The non-linear map $\mathcal{G}_k : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ for all k are are continuous such that

there exist constants $0 < l_{gk} < 1$ *satisfying*

$$||\mathcal{G}_k(t, x_1) - \mathcal{G}_k(t, x_2)|| \leq l_{gk}||x_1 - x_2||,$$

for all $t \in [0, T_0]$ *and* $l_g = max\{l_{gk}; \forall k\}$.

(A4) The function $h : \mathbb{X} \to \mathbb{X}$ *is Lipchitz continuous with Lipchitz constant* $0 \leq l_h \leq 1$.

Then, the system (4.4.1) *is trajectory controllable over the interval* $[0, T_0]$.

Proof. Let $y(t)$ be any desired state trajectory in $\mathcal{C}_{\sqcup}$ satisfying $x(t_k^+) = y(t_k^+)$ along which the system (4.4.1) steered from initial state $x(0) = h(x)$ at $t = 0$ to desired final state x_1 at $t = T_0$.

Over the interval $[0, t_1)$, the system (4.4.1) becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t) \\ x(0) &= h(x) \end{aligned} \tag{4.4.3}$$

Consider trajectory controller

$$\mathcal{W}(t) = y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

over the interval $[0, t_1)$ and plugging it in the system (4.4.3), the system becomes

$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

with initial condition $x(0) - y(0) = h(x) - h(y)$.
Choosing variable $z = x - y$ the equation system reduces to

$$\begin{aligned} z'(t) &= \mathcal{A}(t)z(t) + \mathcal{F}(t, x(t)) - F(t, y(t)) \\ z(0) &= h(x) - h(y) \end{aligned} \tag{4.4.4}$$

and problem of trajectory controllability of the system (4.4.3) is reduced to the solvability of the system (4.4.4) over the interval $[0, t_1)$. The mild solution of the system (4.4.4) is given by:

$$z(t) = \mathcal{T}(t)[h(x) - h(y)] + \int_0^t \mathcal{T}(t-\zeta)[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]d\zeta \tag{4.4.5}$$

where, $\mathcal{T}(t)$ is C_0 semigroup generated by linear operator $\mathcal{A}$ satisfying $||\mathcal{T}(t)|| \leq M$ for some positive number M.
Therefore,

$$\begin{aligned} ||z(t)|| &\leq ||\mathcal{T}(t)||||h(x) - h(y)|| + \int_0^t ||\mathcal{T}(t-\zeta)||\ ||[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]||d\zeta \\ &\leq Ml_h||x(t) - u(t)|| + M\int_0^t l_F(r)||x(\zeta) - y(\zeta)||d\zeta \\ &\leq Ml_h||z(t)|| + M\int_0^t l_F(r)||z(\zeta)||d\zeta \end{aligned}$$

This implies

$$||z(t)|| \leq \frac{Ml_F(r)}{1 - Ml_h}\int_0^t ||z(\zeta)||d\zeta$$

Using Gronwall's inequality, we obtain $z(t) = 0$ over the interval $[0, t_1)$. Hence, $x(t) = y(t)$ for all $t \in [0, t_1)$. Therefore, the system (4.4.1) is trajectory controllable over the interval $[0, t_1)$.

Over the interval $[t_k, s_k)$, the system becomes

$$x(t) = \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), \tag{4.4.6}$$

and at $t = t_k^+$ value of the state x is given by $x(t_k^+) = \mathcal{G}_k(t_k^+, x(t_k^+)) + \mathcal{W}_k(t_k^+)$.
Plugging the trajectory control $\mathcal{W}_k(t) = y(t) - \mathcal{G}_k(t, y(t))$ over the interval $[t_k, s_k)$ in the system (4.4.6) the system becomes:

$$x(t) - y(t) = \mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, y(t))$$

Choosing $z(t) = x(t) - y(t)$ we obtain

$$z(t) = \mathcal{G}_k(t, x(t)) - \mathcal{G}_k(ty(t))$$

and the value of the z at $t = t_k^+$ is zero. Therefore, we have

$$||z(t)|| \leq ||\mathcal{G}_k(t, x(t)) - \mathcal{G}_k(t, y(t))|| \leq l_g||z(t)||,$$

using (A3) $l_g < 1$ we obtain $z(t) = 0$ for all $t \in [t_k, s_k)$ and using continuity of $\mathcal{G}_k$ leads to $z(s_k) = 0$. Therefore, system (4.4.6) is T-Controllable over the interval $[t_k, s_k)$.
Over the interval $[s_k, t_{k+1})$ the system (4.3.1) becomes:

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t) \\ x(s_k) &= y(s_k) \end{aligned} \tag{4.4.7}$$

Choose the control over the interval $[s_k, t_{k+1})$ as:

$$\mathcal{W}(t) = y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

and plugging it in the equation (4.4.7) we get,

$$x'(t) = \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + y'(t) - \mathcal{A}(t)y(t) - \mathcal{F}(t, y(t))$$

, considering $z(t) = x(t) - y(t)$, above expression becomes:

$$\begin{aligned} z'(t) &= \mathcal{A}(t)z(t) + \mathcal{F}(t, x(t)) - \mathcal{F}(t, y(t)) \\ z(s_k) &= 0, \end{aligned} \tag{4.4.8}$$

Therefore,

$$\begin{aligned} ||z(t)|| &\leq \int_{s_k}^{t} ||\mathcal{T}(t-\zeta)|| \; ||[\mathcal{F}(\zeta, x(\zeta)) - \mathcal{F}(\zeta, y(\zeta))]|| d\zeta \\ &\leq M \int_0^t l_F(r)||x(\zeta) - y(\zeta)||d\zeta \\ &\leq M \int_0^t l_F(r)||z(\zeta)||d\zeta \end{aligned}$$

and using Gronwall's inequality, we obtain $z(t) = 0$ over the interval $[s_k, t_{k+1})$. Therefore, $x(t) = y(t)$ for all $t \in [s_k, t_{k+1})$. Thus, the system is trajectory controllable over the interval $[s_k, t_{k+1})$.

Since, the system is T-controllable over the intervals $[0, t_1), [s_k, t_{k+1})$ and $[t_k, s_k)$ for all k. Hence, the system is controllable over the entire interval $[0, T_0]$. This completes the proof of the theorem. □

Example 4.4.1. *Let,* $\mathbb{X} = L^2([0, \pi], \mathbb{R})$ *and consider the system governed by a non-*

instantaneous impulsive evolution equation:

$$\begin{aligned} \frac{\partial H(t,\psi)}{\partial t} &= \partial^2_\psi H(t,\psi) + F(t, H(t,\psi)) + w(t,\psi) && t \in [0,1/3) \cup [2/3,1], \\ H(t,\psi) &= G_1(t, H(t,\psi)) + w_1(t,\psi) && t \in [1/3, 2/3), \\ H(t,0) &= 0 \quad H(t,\pi) = 0 && t > 0, \\ H(0,\psi) &= H(t,\psi) && 0 < \psi < \pi, \end{aligned} \tag{4.4.9}$$

over the interval $[0,1]$. *Here,* $H(t,\psi)$ *is nonlocal opertor defined by* $\sum_{i=1}^{n} \alpha_i H(t_i,\psi)$.

Defining the operator on the space $\mathbb{X}$ *as* $\mathcal{A}(t) = \partial^2_\psi$, $\mathcal{A}(t)$ *is the infinitesimal generator of the* C_0 *semigroup* $\mathcal{T}(t)$. *The representation of* $\mathcal{T}(t)$ *is*

$$\mathcal{T}(t)z = \sum_{m=0}^{\infty} \exp(\mu_m t) < z, \phi_m > \phi_m$$

where, $\phi_m = \sqrt{2} sin(n\psi)$ *for all* $m = 1, 2, \cdots$ *is the orthonormal basis corresponding to eigenvalue* $\mu_m = -m^2$ *of the operator* $\mathcal{A}$.

With this concept the equation (4.4.9) *can be rewritten as an abstract equation on the space* $\mathbb{X}$ *as*

$$\begin{aligned} x'(t) &= \mathcal{A}(t) + \mathcal{F}(t,x) + \mathcal{W}(t) && t \in [0,1/3) \cup [2/3,1], \\ x(t) &= \mathcal{G}_1(t,x) + \mathcal{W}_1(t) && t \in [1/3, 2/3), \\ x(0) &= h(x), \end{aligned} \tag{4.4.10}$$

where, $x(t) = H(t,\cdot), W(t) = w(t,\cdot), W_1(t) = w_1(t,\psi)$ *and* $h(x) = \sum_{i=1}^{n} \alpha_i x(t_i)$. *The system* (4.4.10) *is trajectory controllable over the interval* $[0,1]$ *if the functions* $F, G_1,$ *and* h *satisfy the hypotheses of the theorem.*

4.5 Conclusion

In this chapter, we have discussed the trajectory controllability of the system governed by a non-instantaneous impulsive evolution equation with classical and nonlocal conditions on the infinite-dimensional Banach space. Results for the trajectory controllability were obtained through the concept of nonlinear functional analysis, Lipschitz conditions and Gronwall's inequality. Illustrations were discussed as an application of the derived results.