Chapter 7

Fractional Impulsive Cauchy Problem

Contents

In this chapter, we have established conditions for the existence and uniqueness of mild and classical solutions to the fractional order Cauchy problem by including and without including impulses over the completed norm linear space (Banach space). Conditions are established using the concept of generators and the generalized Banach fixed point theorem, which are weaker conditions than the previously derived conditions. We have also established the conditions under which a mild solution to the problem gives rise to a classical solution to the given problem. Finally, illustrations of the existence and uniqueness of the solution are provided to validate our derived results.

7.1 Introduction

The various problems in physics, engineering, and biological sciences that have abrupt changes for a small amount of time are well explained in terms of impulses. Therefore, problems like removal of insertion of biomass, populations of species with abrupt changes, abrupt harvesting, and various problems containing abrupt changes are modeled into impulsive differential equations [7, 39, 44, 69, 79, 84, 90, 100, 124]. Many researchers have studied the qualitative properties like existence, uniqueness, and asymptotic behavior of impulsive differential equations using various techniques. These studies are found in the articles cited [3, 5, 54, 92, 123, 131] and reference therein.

On the other hand, due to the inherited property of the fractional derivative operator [20, 31] many nonlinear complicated problems, such as seepage flow in porous media, anomalous diffusion, wave and transport, and many other problems, are now being remodeled into fractional differential equations [43, 49, 61, 60, 98, 114, 120, 143]. Fractional calculus developed to become one of the most well-liked areas of applied mathematics as a result of the numerous uses of fractional differential equations. This draws a lot of academics interested in differential equations and fractional calculus. Numerous scholars, including [37, 43, 78, 131], have examined the qualitative properties, such as the existence and uniqueness of mild solutions to fractional equations using diverse methodologies. Researchers have looked into the existence and originality of impulsive fractional differential equations. Benchohra and Slimani[18] investigated the presence and distinctiveness of a mild solution to impulsive differential equations in one dimension. To find adequate criteria for the existence and uniqueness of the mild solution, they employed the fixed point theorems of Banach, Schaefer, and Leray-Schauder. With the use of the Banach contraction principle and semigroup theory, Mohphu [102] researched the existence and uniqueness of mild solutions. By assuming the sectorial property of the linear operator A, Ravichandran and Arjunan [119] investigated the existence and uniqueness of the classical and mild solutions of impulsive fractional integro-differential equations on Banach space. Balachandra et al. By omitting the semigroup property from

Mohphu's work, al. [11] examined the existence and uniqueness of mild solutions to impulsive fractional integro-differential equations on a Banach space. The classical solution of a fractional order differential equation of the Caputo type is described by Kataria and Patel [73], who also examine the congruence between the classical and mild solutions of more extended impulsive fractional equations on a Banach space.

Krasnoselskii's fixed point was utilized by Borah and Bora [21] and Kataria et al. [75] to demonstrate the necessary conditions for the existence of mild solutions for the non-local fractional differential equations with non-instantaneous impulses.

In this chapter, we develop the necessary criteria for a mild solution and classical solution of the impulsive fractional evolution problem,

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= \mathcal{I}_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(t_0) &= x_0 \end{aligned} \tag{7.1.1}$$

over the interval $[0, T_0]$ on a Banach space $\mathbb{X}$. Here, ${}^cD^\alpha$ denotes Caputo fractional differential operator of order $0 < \alpha \leq 1$, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is linear operator and $\mathcal{F} : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ is nonlinear function. $\mathcal{I}_k : \mathbb{X} \to \mathbb{X}$ are impulse operator at time $t = t_k$, fro $k = 1, 2, \cdots, p$ and their existence and uniqueness. We also developed conditions under which classical and mild solutions of (7.1.1) coincide.

7.2 Prelimnaries

In this section, we introduce notations, definitions, assumptions, and, preliminary facts used throughout this paper.

Definition 7.2.1. *([78, 112]) The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I^\alpha_{t_0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f(s) ds,$$

provided the integral on the right side exists. Where $\Gamma(\cdot)$ is the gamma function.

Definition 7.2.2. *([78, 112]) The Caputo fractional derivative of order $\alpha > 0$,*

$n-1<\alpha<n$, $n\in\mathbb{N}$, is defined as

$$^{c}D^{\alpha}_{t_0+}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\frac{d^n f(s)}{ds^n}ds,$$

provided the integral on the right exists and $n=[\alpha]+1$.

Definition 7.2.3. *One and two-parameter Mittag-Leffler function is defined as:*

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$$

for all $\alpha,\beta>0$ and $z\in\mathbb{C}$ respectively.

Definition 7.2.4. *[135] Let X be Banach space. Then the set*
$PC([t_0,T],X)=\Big\{u:[t_0,T]\to X; u\text{ is continuous at } t\neq t_k,\ left\ continuous\ at\ t=t_k,\ right\ limit\ at\ t=t_k\ exist\ for\ all\ k=1,2,\cdots,p\Big\}.$
This set $PC([t_0,T],X)$ is Banach space under the norm defined by $||u||_{PC}=sup\{||x(t)||;\ t\in[t_0,T]\}$.

7.3 Motivation

This section is devoted to the motivation behind studying the existence and uniqueness of solutions for the Caputo Cauchy problem. Consider the non-homogeneous diffusion equation without impulses

$$\begin{aligned} ^{c}D^{\alpha}z(t,x)&=z_{xx}(t,x)+F(t,x),\\ z(t,0)&=z(t,\pi)=0,\\ z(0,x)&=z_0(x)\end{aligned}\tag{7.3.1}$$

over the rectangle $[0,T_0]\times[0,\pi]$. The solution of this equation (7.3.1) using the Laplace transform and Fourier series at any time $t\in[0,T_0]$ is given by

$$z(t,x)=T_\alpha(t)z_0(x)+\int_0^t (t-s)^{\alpha-1}T_{\alpha,\alpha}(t-s)F(s,x)ds\tag{7.3.2}$$

where, the families of operators $T_\alpha(t), T_{\alpha,\beta}(t) : \mathbb{X} \to \mathbb{X}$ for all $t \in [0, T_0]$ are defined as

$$T_\alpha(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2 t^\alpha) < z, \phi_n > \phi_n$$

and

$$T_{\alpha,\beta}(t)z = \sum_{n=1}^{\infty} E_{\alpha,\beta}(-n^2 t^\alpha) < z, \phi_n > \phi_n$$

in the space

$$\mathbb{X} = \big\{ z : [0, \pi] \to \mathbb{R} : z'' \textit{ exists and } z(0) = z(\pi) = 0 \big\}$$

the functions $E_\alpha(\cdot)$ and $E_{\alpha,\beta}(\cdot)$ are Mittag-Leffler functions of one and two-parameter family respectively and $\phi_n(x)$ are orthonormal Fourier basis corresponding to eigenvalues.

In view of the equation (7.3.2) we can define a mild solution of a semilinear diffusion equation

$$\begin{aligned} {}^cD^\alpha z(t, x) &= z_{xx}(t, x) + F(t, z), \\ z(t, 0) &= z(t, \pi) = 0, \\ z(0, x) &= z_0(x) \end{aligned} \tag{7.3.3}$$

as a function u satisfy the equation

$$z(t, x) = T_\alpha(t) z_0(x) + \int_0^t (t - s)^{\alpha - 1} T_{\alpha,\alpha}(t - s) F(s, z) ds \tag{7.3.4}$$

where, the families of operators $T_\alpha(t)$, $T_{\alpha,\beta}(t)$ are defined above.

Observe that the operator $\mathcal{A} = \partial_{xx}$ in equation (7.3.3) is neither bounded nor semigroup property but solutions of equation (7.3.3) exist under certain conditions (derived in the Section -7.4). From this, we can say that the function u is the mild solution of diffusion equation (7.3.3) if u satisfies the integral equation (7.3.4). Using this concept we can easily study the various qualitative properties like existence and uniqueness of solution, various types of stability and controllability of the Caputo fractional evolution system (7.1.1) with and without impulses. This motivates us to study the existence and uniqueness of solutions of Caputo fractional evolution equation (7.1.1).

7.4 Mild and Classical solutions without Impulses

In this section, we will discuss the existence and uniqueness of classical and mild solutions of the fractional order evolution equation (7.1.1) without impulses by using the concept of generators, motivated by the previous section.
Consider the fractional order evolution equation without impulses over the interval $[0, T_0]$ of the form:

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(0) &= x_0 \end{aligned} \tag{7.4.1}$$

in the general Banach space $\mathbb{X}$, where $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is linear operator, ${}^cD^\alpha$ is fractional differential operator of Caputo type for $0 < \alpha \leq 1$ and $\mathcal{F} : [0, T_0] \times \mathbb{X} \to \mathbb{X}$ is nonlinear function.
We define the operators generated by the linear operator $\mathcal{A}$.

Definition 7.4.1. *The families of operators $T_\alpha(t), T_{\alpha,\beta}(t) : \mathbb{X} \to \mathbb{X},\ t \geq 0$ are generated by a linear operator $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ satisfies the following properties:*

(1) $T_\alpha(0) = I$ *and* $T_{\alpha,\beta}(0) = I$ *where,* I *is identity operator*

(2) $T(t)$ *satisfies the linear fractional equation* ${}^cD^\alpha x(t) = \mathcal{A}(t)x(t)$ *in Banach space* $\mathbb{X}$

(3) $\lim_{\beta \to 1} T_{\alpha,\beta}(t) = T_\beta(t)$

Example 7.4.1. *The operators $T_\alpha(t), T_{\alpha,\beta}(t) : \mathbb{X} \to \mathbb{X}$ for all $t \in [0, T_0]$ are defined as*

$$T_\alpha(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2 t^\alpha) < z, \phi_n > \phi_n$$

and

$$T_{\alpha,\beta}(t)z = \sum_{n=1}^{\infty} E_{\alpha,\beta}(-n^2 t^\alpha) < z, \phi_n > \phi_n$$

defined on the space

$$\mathbb{X} = \left\{ z : [0, \pi] \to \mathbb{R} : z'' \text{ exists and } z(0) = z(\pi) = 0 \right\}$$

are generated by the linear operator $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ satisfies the above properties.

With the operators $T_\alpha(\cdot)$ and $T_{\alpha,\beta}$,the mild and classical solutions of Caputo fractional evolution equation (7.4.1) is defined as follows

Definition 7.4.2. *The function $x \in \mathbb{X}$ is called mild solution of Caputo fractional order $(0 < \alpha \leq 1)$ evolution equations* (7.4.1) *over the interval $[0, T_0]$ if u satisfies the equation of the form:*

$$x(t) = T_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_{\alpha,\alpha}(t-s)\mathcal{F}(s,x)ds \tag{7.4.2}$$

where, $T(t)$ and $T_\alpha(t)$ are generated by the linear operator $\mathcal{A}$.

Definition 7.4.3. *The solution $x \in \mathbb{X}$ is a classical solution of semi-linear fractional order evolution equation* (7.4.1) *of α order Caputo fractional derivative with respect to t exists and continuous.*

Theorem 7.4.1. *The fractional order Caputo fractional evolution equation* (7.4.1) *has a unique mild solution over the interval $[0, T_0]$ if the following properties are satisfied.*

(1) The families of operators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ generated by the operator $\mathcal{A}$ are continuous and bounded over $[0, T_0]$. That is, there exist positive constants M and M_α such that $||T_\alpha(t)|| \leq M$ and $||T_{\alpha,\beta}(t)|| \leq M_\alpha$ for all $t \in [0, T_0]$.

(2) The nonlinear function $\mathcal{F}$ is continuous with respect to t and there exist r_0 such that $\mathcal{F}$ Lipchitz continuous with respect to x in $B_{r_0} = \{x \in \mathbb{X}; ||x|| \leq r_0\}$. That is, there exist positive constant L such that $||\mathcal{F}(t,x) - \mathcal{F}(t,y)|| \leq L||x-y||$ for all $t \in [0, T_0]$ and $x, y \in B_{r_0}$.

Proof. Define the operator $\mathcal{P} : \mathbb{X} \to \mathbb{X}$ as:

$$\mathcal{P}x(t) = T_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} T_{\alpha,\alpha}(t-s)\mathcal{F}(s,x)ds.$$

To show (7.4.1) has a unique mild solution it is sufficient to show $\mathcal{P}^{(m)}$ is a contraction for some $m \in \mathbb{N}$.

For any $u, v \in B_{r_0}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&||\mathcal{P}^{(n)}u(t) - \mathcal{P}^{(n)}v(t)|| \\
&\quad \leq M_\alpha L \int_0^t (t-s)^{\alpha-1} ||\mathcal{P}^{(n-1)}x(s) - \mathcal{P}^{(n-1)}v(s)|| ds \\
&\quad \leq M_\alpha^2 L^2 \int_0^t \int_0^{s_1} (t-s_1)^{\alpha-1}(s_1-s)^{\alpha-1} ||\mathcal{P}^{(n-2)}x(s) - \mathcal{P}^{(n-2)}v(s)|| ds ds_1
\end{aligned}$$

Continuing this process to get

$$\begin{aligned}
&||\mathcal{F}^{(P)}u(t) - \mathcal{P}^{(n)}v(t)|| \\
&\quad \leq M_\alpha^n L^\alpha \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (t-s_1)^{\alpha-1}(s_1-s)^{\alpha-1} \cdots (s_{n-1}-s)^{\alpha-1} ||u(s) - v(s)|| \\
&\qquad\qquad ds ds_{n-1} \cdots ds_1 \\
&\quad \leq M_\alpha^n L^n \int_0^{T_0} \int_0^{T_0} \cdots \int_0^{T_0} (T_0-s_1)^{\alpha-1}(T_0-s_2)^{\alpha-1} \cdots (T_0-s)^{alpha-1} ||u-v|| \\
&\qquad\qquad ds ds_{n-1} \cdots ds_1 \\
&\quad \leq M_\alpha^n L^n \int_0^{T_0} (T_0-s)^{n(\alpha-1)} \frac{(T_0-s)^n}{(n-1)!} ||u-v|| ds \\
&\quad \leq \frac{(M_\alpha L)^n T_0^{n\alpha}}{n!\alpha} ||u-v||
\end{aligned}$$

Therefore, for any fixed T_0 and sufficiently large integer n say m the operator $\mathcal{P}^{(m)}$ is contraction therefore, by generalized Banach fixed point theorem $\mathcal{P}$ has a unique fixed point. Hence, (7.4.1) has a unique mild solution given by (7.4.2). □

Example 7.4.2. *The operators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ generated for the equation* (7.3.4) *are continuous and bounded. Hence, there exist positive constants M and M_α such that $||T_\alpha(t)|| \leq M$ and $||T_{\alpha,\beta}(t)|| \leq M_\alpha$. Therefore, the equation* (7.3.3) *has a unique mild solution given by* (7.3.4) *since $\mathcal{F}$ is continuous with respect to t and Lipchitz continuous with respect to u in a given Banach space over the interval* $[0, T_0]$.

Remark 7.4.1. *We have the following observations from the theorem-7.4.1.*

(1) Conditions derived in the Theorem-7.4.1 are more liberal than previously derived conditions by the author for a similar system.

(2) The conditions obtained in Theorem-7.4.1 are sufficient but not necessary.

Now we consider a system in which the initial time is taken $t = t_0$ instead of $t = 0$. Thus the theorem-7.4.1 can be extended as follows:

Corollary 7.4.1. *The fractional evolution equation*

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(t_0) &= x_0 \end{aligned} \tag{7.4.3}$$

has unique mild solution over interval $[t_0, T_0]$ *given by*

$$x(t) = T_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x)ds \tag{7.4.4}$$

if the following conditions are satisfied:

(1) *The families of operators* $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *generated by the operator* A *are continuous and bounded over* $[t_0, T_0]$*. That is, there exist positive constants* M *and* M_α *such that* $||T_\alpha(t)|| \leq M$ *and* $||T_{\alpha,\beta}(t)|| \leq M_\alpha$ *for all* $t \in [t_0, T_0]$

(2) *The nonlinear function* $\mathcal{F}$ *is continuous with respect to* t *and Lipchitz continuous with respect to* x*. That is, there exist positive constant* L *such that* $||\mathcal{F}(t, x) - \mathcal{F}(t, y)|| \leq L||x - y||$ *for all* $t \in [t_0, T_0]$ *for* $x, y \in B_{r_0}$*.*

Condition for the classical solution of the system (7.4.1) is given by the following theorem:

Theorem 7.4.2. *The mild solution of* (7.4.1) *is the classical solution if*

(1) $x_0 \in \mathcal{D}(\mathcal{A})$ *(Domain of A)*

(2) *The generators* $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *are continuously differentiable for all* $t > 0$*.*

(3) *The function* $\mathcal{F}$ *is differentiable with respect to* t *and continuous with respect to* x*.*

Proof. Let $x(t)$ be the mild solution of (7.4.1). Therefore $x(t)$ satisfies the corresponding integral equation (7.4.2). Assuming conditions (1),(2), and (3) of the hypothesis, the fractional Caputo derivative of $x(t)$ in equation (7.4.2) exists and is continuous. Moreover for all $t \in [0, T_0]$ the function $x(t) \in \mathcal{D}(\mathcal{A})$. Hence the mild solution $x(t)$ defined by (7.4.2) is a classical solution of the equation (7.4.1). This completes the proof of the theorem. □

Similarly one has the classical solution for the system (7.4.4).

Corollary 7.4.2. *The mild solution given by* (7.4.4) *of* (7.4.3) *is the classical solution if*

(1) $x_0 \in \mathcal{D}(\mathcal{A})$ *(Domain of* $\mathcal{A}$*)*

(2) The generators $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *are continuously differentiable for all* $t \in [t_0, T_0]$

(3) The function $\mathcal{F}$ *is differentiable with respect to* t *and continuous with respect to* x.

The following theorems give the uniqueness of the classical solution of both systems.

Theorem 7.4.3. *Equation* (7.4.1) *has unique classical solution over the interval* $[0, T_0]$ *if*

(1) $x_0 \in \mathcal{D}(\mathcal{A})$ *(Domain of* A*).*

(2) The generators $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *of the linear operator* $\mathcal{A}$ *are continuously differentiable and bounded over the interval* $[0, T_0]$.

(3) The function $\mathcal{F}$ *is differentiable with respect to* t *and Lipchitz continuous with respect to* x *in* B_{r_0}.

Proof. Using condition (2) the generators are continuously differentiable and bounded over $[0, T_0]$ so, they are continuous and bounded over $[0T_0]$. This means there exist positive constants M and M_α such that $||T_\alpha(t)|| \leq M$ and $||T_{\alpha,\beta}(t)|| \leq M_\alpha$ and condition (3) the function $\mathcal{F}$ is continuous with respect t and Lipchitz continuous with respect to x and applying theorem-7.4.1 the equation (7.4.1) and has unique mild solution given by (7.4.2). Assuming (1), (2) and (3) this mild solution becomes a classical solution of the equation (7.4.1). Since the mild solution is unique, the classical solution is also unique. □

Corollary 7.4.3. *Equation* (7.4.3) *has unique classical solution over the interval* $[t_0, T_0]$ *if*

(1) $x_0 \in \mathcal{D}(\mathcal{A})$ *(Domain of* $\mathcal{A}$*).*

(2) The generators $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *of the linear operator* $\mathcal{A}$ *are continuously differentiable and bounded over the interval* $[t_0, T_0]$*.*

(3) The function $\mathcal{F}$ *is differentiable with respect to* t *and Lipchitz continuous with respect to* x *in* B_{r_0}*.*

Example 7.4.3. *Consider the fractional order equation*

$$^cD^\alpha w(t,x) + w\frac{\partial w}{\partial x}(t,x) + \frac{\partial^2 w}{\partial x^2}(t,x) = \mathcal{F}(t, w(t,x)) \tag{7.4.5}$$

on the domain $[0, T_0]$ *boundary conditions*

$$w(t,0) = w(t,2\pi) = 0 \tag{7.4.6}$$

with initial condition $w(0,x) = w_0$*. The domain of the operator* $\mathcal{A}w = -\frac{\partial^2 w}{\partial x^2}$ *is* $\mathcal{D}(\mathcal{A}) = \{z \in L^2[0,2\pi] : z''$ *continuous and satisfies boundary conditions* $\}$*. Then the mild solution in the interval* $[0, T_0]$ *of the equation* (7.4.5) *with conditions* (7.4.6) *is given by*

$$w(t,x) = T_\alpha(t)w_0 + \int_0^t (t-s)^{\alpha-1}T_{\alpha,\alpha}(t-s)\{\frac{1}{2}\frac{\partial w^2}{\partial x} + f(s,w)\}ds \tag{7.4.7}$$

where,

$$T_\alpha(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2t^\alpha) < z, \phi_n > \phi_n$$

and

$$T_{\alpha,\beta}(t)z = \sum_{n=1}^{\infty} E_{\alpha,\beta}(-n^2t^\alpha) < z, \phi_n > \phi_n$$

are the generators of the linear operator A*.* $\phi_n(x)$ *are orthogonal Fourier basis functions in* $L^2[0,2\pi]$*.*
We have the following observations:

(1) The generators $T_\alpha(t)$ *and* $T_{\alpha,\beta}(t)$ *are defined in equation* (7.4.7) *are continuously differentiable with respect to* t*. Therefore there exists positive constants*

M and M_α such that $||T_\alpha(t)|| \leq M$ and $||T_{\alpha,\beta}(t)|| \leq M_\alpha$ respectively.

(2) The first non linear term in (7.4.5) $\frac{1}{2}\frac{\partial w^2}{\partial x}$ is composition of two continuous operators $Pw = \frac{1}{2}\frac{\partial w}{\partial x}$ and $Qw = w^2$ which are continuous with respect to t and Lipchitz continuous with respect to w in finite closed ball B_{r_0} as the operator P is linear and the partial derivative of Q with respect to w exist for every w. Moreover, P and Q are differentiable with respect to arguments t and w.

Therefore equation (7.4.5) has a unique mild solution given by (7.4.7) if the second term $\mathcal{F}(t, w)$ is continuous with respect to t and Lipchitz continuous with respect to w The mild solution (7.4.7) is unique classical solution of (7.4.5) if $\mathcal{F}(t, w)$ is differentiable and $w_0 \in \mathcal{D}(\mathcal{A})$.

7.5 Mild and Classical solutions with Impulses

In this section, we are going to derive a set of sufficient conditions for the existence and uniqueness of classical as well mild solution of impulsive fractional evolution equation (7.1.1). We are also deriving the conditions in which the classical and mild solutions coincide.

Definition 7.5.1. ***Classical Solution****[73]*
A solution $x(t)$ is a classical solution of the equation (7.1.1) for $0 < \alpha < 1$ if $x(t) \in PC([0, T_0], \mathbb{X}) \cap C^\alpha(J', \mathbb{X})$ where, $J' = [0, T_0] - \{t_1, t_2, \cdots, t_p\}$ and $C^\alpha(J', \mathbb{X}) = \big\{u : J' \to \mathbb{X} :^c D^\alpha x(t)$ exist and continuous at each $t \in J'\big\}$, $x(t) \in \mathcal{D}(\mathcal{A})$ (Domain of $\mathcal{A}$) for $t \in J'$ and satisfies (7.1.1) on $[0, T_0]$.

Definition 7.5.2. ***Mild Solution***
A function $x(t) \in PC([0, T_0], \mathbb{X})$ is a mild solution of the equation (7.1.1) if it satisfies

$$x(t) = \begin{cases} T_\alpha(t-t_i)\big(\prod_{k=i}^{1} T_\alpha(t_k - t_{k-1})\big)x_0 + T_\alpha(t-t_i)\sum_{j=1}^{i}\big(\prod_{k=j}^{2} T_\alpha(t_k - t_{k-1}\big) \\ \int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}T_{\alpha,\alpha}(t_j - s)\mathcal{F}(s, x(s))ds + \int_{t_i}^{t}(t-s)^{\alpha-1}T_{\alpha,\alpha}(t-s)F(s, x(s))ds \\ +T_\alpha(t-t_i)\sum_{j=1}^{i}\big(\prod_{k=i}^{3} T_\alpha(t_k - t_{k-1})\big)\mathcal{I}_k x(t_k) \end{cases} \tag{7.5.1}$$

for each $t \in [t_i, t_{i+1})$.

Here, the families of operators $T(t)$ and $T_\alpha(t)$ are generated by the linear operator A.

Theorem 7.5.1. *The fractional order semi-linear impulsive evolution equation* (7.1.1) *has a unique mild solution over the interval* $[0, T_0]$ *if the following properties are satisfied.*

(1) The families of operators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ generated by the operator $\mathcal{A}$ are continuous and bounded over $[0, T_0]$. That is there exist positive constants M and M_α such that $||T_\alpha(t)|| \leq M$ and $||T_{\alpha,\beta}(t)|| \leq M_\alpha$ for all $t \in [0, T_0]$.

(2) The nonlinear function $\mathcal{F}$ is continuous with respect to t and Lipchitz continuous with respect to u in B_{r_0}. That is there exist positive constant L such that $||\mathcal{F}(t, u) - \mathcal{F}(t, v)|| \leq L||u - v||$ for all $t \in [0, T_0]$ and $u, v \in B_{r_0}$.

(3) Impulses $\mathcal{I}_k$ at $t = t_k$ for $k = 1, 2, \cdots, k$ are continuous and bounded.

Proof. Over the interval $[0, t_1]$ the equation (7.1.1) becomes,

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(t_0) &= x_0 \end{aligned} \tag{7.5.2}$$

Assuming conditions (1) and (2) of the hypotheses and using theorem-7.4.1 the equation (7.5.2) has a unique mild solution over the interval $[0, t_1)$ given by

$$x(t) = T_\alpha(t - t_0)x_0 + \int_{t_0}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds. \tag{7.5.3}$$

At $t = t_1$ the mild solution $x(t_1^-)$ becomes:

$$x(t_1^-) = T_\alpha(t_1 - t_0)x_0 + \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} T_{\alpha,\alpha}(t_1 - s)\mathcal{F}(s, x(s))ds.$$

Over the interval $[t_1, t_2)$ the equation (7.1.1) becomes:

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(t_1^+) &= x_1 = x(t_1^-) + \mathcal{I}_1 u(t_1) \end{aligned} \tag{7.5.4}$$

Here, the impulse operator $\mathcal{I}_1$ is continuous and bounded. Assuming conditions (1) and (2) and applying corollary 7.4.1, the equation (7.5.4) has a unique mild solution over the interval $[t_1, t_2)$ given by

$$x(t) = T_\alpha(t - t_1)u_1 + \int_{t_1}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds. \tag{7.5.5}$$

Continuing in this way the equation (7.1.1) over the interval $[t_i, t_{i+1})$ becomes

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(t_i^+) &= x_i = x(t_i^-) + \mathcal{I}_i x t_i). \end{aligned} \tag{7.5.6}$$

Assuming condition (1) and (2) of the hypotheses and applying corollary-7.4.1 the equation (7.5.6) has unique mild solution over the interval $[t_i, t_{i+1})$ given by

$$x(t) = T_\alpha(t - t_i)u_i + \int_{t_i}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds. \tag{7.5.7}$$

Finally over the interval $[t_p, T_0]$ the equation (7.1.1) becomes:

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \\ x(t_p^+) &= x_1 = x(t_p^-) + \mathcal{I}_p x(t_p). \end{aligned} \tag{7.5.8}$$

Assuming condition (1) and (2) of the hypotheses and applying corollary-7.4.1 the equation (7.5.8) has unique mild solution over the interval $[t_p, T_0]$ given by

$$x(t) = T_\alpha(t - t_p)u_p + \int_{t_p}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds. \tag{7.5.9}$$

Therefore for any $t \in [t_i, t_{i+1})$ for $i = 1, 2, \cdots, p$ the equation (7.1.1) has unique mild solution given by

$$\begin{aligned} x(t) &= T_\alpha(t - t_i)x_i + \int_{t_i}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds \\ &= T_\alpha(t - t_i)[x(t_i^-) + \mathcal{I}_i x(t_i)] + \int_{t_i}^{t} (t - s)^{\alpha-1} T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds \end{aligned}$$

Substituting the values of x_k's for $k = 1, 2, \cdots, i$ we obtained,

$$x(t) = T_\alpha(t - t_i)\big(\prod_{k=i}^{1} T_\alpha(t_k - t_{k-1})\big)x_0 + T_\alpha(t - t_i)\sum_{j=1}^{i}\big(\prod_{k=j}^{2} T_\alpha(t_k - t_{k-1}\big)$$
$$\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}T_{\alpha,\alpha}(t_j - s)\mathcal{F}(s, x(s))ds$$
$$+ \int_{t_i}^{t}(t - s)^{\alpha-1}T_{\alpha,\alpha}(t - s)\mathcal{F}(s, x(s))ds + T_\alpha(t - t_i)\sum_{j=1}^{i}\big(\prod_{k=i}^{3} T_\alpha(t_k - t_{k-1})\big)\mathcal{I}_k x(t_k).$$

We complete the proof by showing $x(t) \in PC([0, T_0], \mathbb{X})$ for all $t \in [0, T_0]$.
If $t \in [0, T_0]$ for all $j = 1, 2, \cdots, p$ then $t \in [t_i, t_{i+1})$ for atleast one i. Assuming conditions (1), (2) and (3) we get the continuity of u at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit exists at $t = t_i$. Therefore $x(t) \in PC([0, T_0], \mathbb{X})$. Hence, equation (7.1.1) has unique mild solution in $PC([0, T_0], \mathbb{X})$. □

Theorem 7.5.2. *The mild solution* (7.5.1) *of* (7.1.1) *is the classical solution if*
(1) The generators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ are continuously differentiable for all $t > 0$.
(2) The function $\mathcal{F}$ is diffrentiable with respect to t and continuous with respect to x.
(3) Impulses $\mathcal{I}_k$ at $t = t_k$ are for $k = 1, 2, \cdots, k$ differentiable and bounded.
(4) x_0 and $\mathcal{I}_k x(t_k)$ are in $\mathcal{D}(\mathcal{A})$ (Domain of $\mathcal{A}$).

Proof. Over the interval $[0, t_1)$ the equation (7.1.1) becomes (7.5.2) which is evolution equation without impulses. Applying theorem-7.4.2 the mild solution (7.5.3) becomes a classical solution of (7.1.1) over the interval $[0, t_1)$ by assuming the conditions (1), (2) and (4).
In the interval $[t_1, t_2)$ the equation (7.1.1) becomes (7.5.3) and $\mathcal{I}_1$ is differentiable and bounded with $\mathcal{I}_1 x(t_1) \in \mathcal{D}(\mathcal{A})$ therefore, $x_1 \in \mathcal{D}(\mathcal{A})$. Again assuming the conditions (1), (2), and (4) and using corollary- 7.4.2 the mild solution (7.5.5) becomes a classical solution of (7.1.1) over the interval $[t_1, t_2)$.
Continuing in same manner the mild solution (7.5.7) of equation (7.1.1) over the interval $[t_i, t_{i+1})$ becomes classical solution of (7.1.1).
Finally, the mild solution (7.5.9) of the equation (7.1.1) becomes classical solution of equation (7.1.1) over the interval $[t_p, T_0]$ proceeding in similar manner.
Hence the mild solution (7.5.1) of equation (7.1.1) becomes classical solution of (7.1.1) over the whole interval $[0, T_0]$. This completes the proof. □

Now we discuss the uniqueness of a classical solution of impulsive evolution equation (7.1.1).

Theorem 7.5.3. *Classical solution of* (7.1.1) *is unique if*

(1) The generators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ are continuously differentiable for all $t > 0$.

(2) The function $\mathcal{F}$ is differentiable with respect to t and Lipschitz continuous with respect to u on B_{r_0}.

(3) Impulses $\mathcal{I}_k$ at $t = t_k$ are for $k = 1, 2, \cdots, k$ differentiable and bounded.

(4) x_0 and $\mathcal{I}_k x(t_k)$ are in $\mathcal{D}(A)$ (Domain of $\mathcal{A}$).

Proof. Under the assumption (1), (2), (3), and (4) the mild solution (7.5.1) of equation (7.1.1) becomes a classical solution. Lipchitz continuity of $\mathcal{F}$ with respect to x leads to the uniqueness of mild solution. Since a mild solution of (7.1.1) is unique therefore a classical solution of (7.1.1) is unique. □

Example 7.5.1. *Consider the semi-linear fractional order impulsive heat equation*

$$\begin{aligned}
{}^cD_t^\alpha w(t,x) &= \frac{\partial^2 w(t,x)}{\partial x^2} + w\frac{\partial u}{\partial x}(t,x), \quad t \neq t_1, t_2 \cdots, t_p \\
w(t,0) &= w(t,\pi) = 0 \\
w(0,t) &= w_0 = x(\pi - x) \\
\Delta w(t_k) &= \mathcal{I}_k(t_k) = a_k w(t_k^-), \quad t = t_k, \quad (a_k\text{'s are constants}) \quad k = 1, 2, \cdots, p
\end{aligned} \tag{7.5.10}$$

over the interval $[0, T_0]$. Here t_k's are time points where impulses are applied.
We have the following observations:

(1) The operator $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ over the domain $\mathcal{D}(\mathcal{A}) = \big\{z : [0,\pi] \to \mathbb{R} : z''$ exists and $z(0) = z(\pi) = 0\big\}$ generates the continuously differentiable and bounded families of operators $T_\alpha(t)$ and $T_{\alpha,\beta}(t)$ defined by

$$T(t)z = \sum_{n=1}^{\infty} E_\alpha(-n^2 t^\alpha) < z, \phi_n > \phi_n$$

and

$$T_{\alpha,\beta}(t)z = \sum_{n=1}^{\infty} E_{\alpha,\beta}(-n^2t^\alpha) < z, \phi_n > \phi_n$$

respectively.

(2) The nonlinear function $\mathcal{F}(t,u) = u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}u^2$ *is differentiable with respect to* t *and Lipchitz continuous with respect to* u *on* B_{r_0}.

(3) Impulses $\mathcal{I}_k u(t_k) = a_k u(t_k^-)$ *are differentiable such that* $\mathcal{I}_k x(t_k) \in \mathcal{D}(\mathcal{A})$.

(4) $u_0 \in \mathcal{D}(\mathcal{A})$.

Therefore by Theorem 7.5.1, 7.5.2 and 7.5.3 the equation (7.5.10) *has unique mild solution given by*

$$\begin{aligned} w(t) &= T_\alpha(t-t_i)\Big(\prod_{k=i}^{1} T_\alpha(t_k - t_{k-1})\Big)u_0 \\ &+ T_\alpha(t-t_i)\sum_{j=1}^{i}\Big(\prod_{k=j}^{2} T_\alpha(t_k - t_{k-1}\Big)\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}T_{\alpha,\alpha}(t_j - s)\frac{\partial}{\partial x}w^2 ds \\ &+ \int_{t_i}^{t}(t-s)^{\alpha-1}T_{\alpha,\alpha}(t-s)\frac{\partial}{\partial x}w^2 ds + T_\alpha(t-t_i)\sum_{j=1}^{i}\Big(\prod_{k=i}^{3} T_\alpha(t_k - t_{k-1})\Big)a_k w(t_k) \end{aligned} \tag{7.5.11}$$

for all $t \in [0, T_0]$. *Moreover, this mild solution* (7.5.11) *becomes a classical solution of* (7.5.10). *Since the mild solution is unique therefore the classical solution is unique.*

7.6 Conclusion

The fractional semi-linear evolution equation over general Banach space without and with impulses has a set of mild and classical solutions, which are deduced in this article. We developed the novel notion of generators and derived the adequate requirements—which are more lax criteria and apply to a broader class of fractional evolution equations using the generalized Banach fixed point theorem.