Chapter 9

Trajectory Controllability of Hilfer Fractional Systems

Contents

This chapter considers a nonlinear system governed by Hilfer fractional integro-differential equations in a Banach space. Using the concept of operator semigroup and Gronwall's inequality, we have established the trajectory controllability of the integro-differential equation with local and non-local conditions. Finally, we have given an example to illustrate the application of the derived results.

9.1 Introduction

Over the past few decades, the differential equations involving fractional order derivatives have received increasing interest from many researchers due to numerous applications in widespread areas of science and engineering such as in models of epidemiology, medicines, electrical and mechanical engineering, biochemistry, etc. For more applications, one can refer to [14, 64]. It has been verified that the fractional differential equations are more accurate to describe the dynamical behavior of a real-life phenomenon more precisely. Hilfer [64] proposed a new fractional derivative which is a generalization of the Riemann-Liouville and Caputo fractional derivatives. Thereafter many researchers studied the qualitative properties of the solution like existence, uniqueness, and stability of fractional differential equations including Hilfer fractional differential operators. The study of qualitative properties are found in the papers [35, 48, 56, 147, 148, 154] and monographs [64, 78], and reference therein.

Trajectory controllability is finding the control of the system which steers the initial state to the desired final state of the system via prescribed trajectory. There are many physical systems in aerodynamics for which we require trajectory controllability of the system for cost-effectiveness. Therefore, trajectory controllability is stronger than any other controllability. Controllability of linear and non-linear systems in finite and infinite dimensional spaces are found in the articles [16, 17, 32, 25, 26, 66, 94, 95, 96, 104, 128, 127] and reference their in. Singh [139] studied the exact controllability of Hilfer fractional differential systems. Study of trajectory controllability of integer order linear and nonlinear systems are found in article [27, 122] and the same for fractional order finite and infinite dimensional systems are found in [38, 55, 103].

This chapter aims to study the trajectory controllability of the system governed by the Hilfer fractional integro-differential system

$$\mathcal{D}_{0+}^{\lambda,\mu}u(t) + Au(t) = g(t, u(t), \int_0^t a(t, \tau, u(\tau))d\tau) + w(t)$$

over the interval $[0, T]$ with classical condition $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}u(0) = u_0$ and non-local conditions $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[u(0) - h(u)] = u_0$ in the infinite-dimensional Banach.

9.2 Preliminaries

In this section, we will discuss basic definitions and results to derive controllability conditions.

Definition 9.2.1. *[67] For $\lambda > 0$, the fractional integral of order λ of a function $h(t)$ is defined by*

$$\mathcal{I}_{t_0}^{\lambda} h(t) = \frac{1}{\Gamma(\lambda)} \int_{t_0}^{t} (t-\tau)^{\lambda-1} f(\tau) d\tau,$$

provided the integral on the right exists.

Definition 9.2.2. *[67] The Hilfer fractional derivative of order λ, $0 < \lambda < 1$ and type μ, $0 \leq \mu \leq 1$ is defined by*

$$\mathcal{D}_{t_0+}^{\lambda,\mu} h(t) = \mathcal{I}_{t_0+}^{\mu(1-\lambda)} \frac{d}{dt} \mathcal{I}_{t_0+}^{(1-\lambda)(1-\mu)} h(t),$$

provided the right value exists.

Definition 9.2.3. *[67] For all $\theta \in \mathbb{C}$ and $\lambda > 0$, the Wright-type function M_λ is defined as:*

$$M_\lambda(\theta) = \sum_{n\in\mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1-\lambda n)(n-1)!} \tag{9.2.1}$$

provided the sum on the right exists.

The Wright-type function satisfies the following properties:

(1) $M_\lambda(\theta) > 0$ for all $\lambda > 0$.

(2) For $-1 < \iota < \infty$ the integral, $\int_0^\infty \theta^\eta M_\lambda(\theta) d\theta = \frac{\Gamma(1+\eta)}{\Gamma(1+\lambda\eta)}$

(3) For $r > 0$ the integral, $\int_0^\infty \frac{\lambda}{\theta^{\lambda+1}} e^{-r\theta} M_\lambda(\theta^{-\lambda}) d\theta = e^{-r^\lambda}$

Let, $\mathcal{C}_\mathcal{T} = C([0,T], X)$, the set of all continuous functions from $[0,T]$ into Banach space X under the norm given by $||\Psi||_T = \sup_{0\leq t\leq T} ||\Psi(t)||$.

Let, $\mathcal{T}(t)$ be the family of semi-group generated by the linear operator $-A$. We define two two linear operators $\mathcal{S}_\lambda$ and $\mathcal{Q}_\lambda$ as:

$$\mathcal{S}_\lambda(t) = \int_0^\infty M_\lambda(\theta)\mathcal{T}(t^\lambda\theta)d\theta \tag{9.2.2}$$

$$\mathcal{Q}_\lambda(t) = \int_0^\infty \lambda\theta M_\lambda(\theta)\mathcal{T}(t^\lambda\theta)d\theta \tag{9.2.3}$$

We have following result for the operators $\mathcal{S}_\lambda$ and $\mathcal{Q}_\lambda$.

Lemma 9.2.1. *If $\mathcal{T}(t)$ be the family of C_0-semigroup generated by the linear operator $-A$ for all $t \in [0,T]$ then the families of operators $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ defined by* (9.2.2) *and* (9.2.3) *are:*
(1) continuous and bounded for all $t \in [0,T]$.
(2) strongly continuous over the interval $t \in (0,T]$

Proof. Since, the family $\mathcal{T}t$ is C_0-semigroup generated by the linear operator $-A$ therefore there exist $M \geq 0$ such that $||\mathcal{T}(t)|| \leq M$.

For any $u \in X$

$$||\mathcal{S}_\lambda(t)u|| \leq \int_0^\infty M_\lambda(\theta)||\mathcal{T}(t^\lambda\theta)||||u||d\theta \leq M\int_0^\infty M_\lambda(\theta)d\theta||u|| \leq M||u||$$

and

$$||\mathcal{Q}_\lambda(t)u|| \leq \int_0^\infty \lambda\theta M_\lambda(\theta)||\mathcal{T}(t^\lambda\theta)||||u||d\theta \leq M\int_0^\infty \lambda\theta M_\lambda(\theta)||u||d\theta \leq \frac{M}{\Gamma(\lambda)}||u||$$

Therefore the operators $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ are bounded with bound M and $M/\Gamma(\lambda)$ respectively.

Let $\{u_n\}$ be any sequence in a Banach space X converges to $u \in X$ and consider,

$$||\mathcal{S}_\lambda(t)u_n - \mathcal{S}_\lambda(t)u|| = ||\mathcal{S}_\lambda(t)[u_n - u]|| \leq M||u_n - u||$$

Therefore, $\mathcal{S}_\lambda(t)$ is continuous. Similarly, we can prove that $\mathcal{Q}_\lambda(t)$ is continuous for all $t \in [0,T]$. Clearly, families $\mathcal{S}_\lambda(t)$ and $\mathcal{Q}_\lambda(t)$ are strongly continuous as

$$||\mathcal{S}_\lambda(t_2)u - \mathcal{S}_\lambda(t_1)u|| \leq \int_0^\infty M_\lambda(\theta)||\mathcal{T}(t_2^\lambda\theta) - \mathcal{T}(t_1^\lambda\theta)||||u||d\theta \leq ||\mathcal{T}(t_2^\lambda\theta) - \mathcal{T}(t_1^\lambda\theta)||||u||$$

which tends to zero as $t_2 \to t_1$ for all $0 < t_1 < t_2 \leq T$ and $u \in X$. Similarly, family

$\mathcal{Q}_\lambda(t)$ is also strongly continuous. This completes the proof of the lemma. □

With the help of families of the operators we define two linear operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ as:

$$\mathcal{S}_{\lambda,\mu}(t) = \mathcal{I}_0^{\mu(1-\lambda)}\mathcal{K}_\lambda(t) \tag{9.2.4}$$

$$\mathcal{K}_\lambda(t) = t^{\lambda-1}\mathcal{Q}_\lambda \tag{9.2.5}$$

The operators satisfy following properties:

Lemma 9.2.2. *If $\mathcal{T}(t)$ be the family of C_0-semigroup generated by the linear operator $-A$ for all $t \in [0,T]$ then the families of operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ defined by* (9.2.4) *and* (9.2.5) *are:*
(1) continuous and bounded for all $t \in [0,T]$.
(2) strongly continuous over the interval $t \in (0,T]$

Proof. Clearly,

$$||\mathcal{K}_\lambda(t)u|| \leq t^{\lambda-1}||\mathcal{Q}_\lambda(t)u|| \leq \frac{t^{\lambda-1}M}{\Gamma(\lambda)}||u||$$

and

$$||\mathcal{S}_{\lambda,\mu}(t)u|| \leq \frac{1}{\Gamma(\mu(1-\lambda))}\int_0^t (t-\tau)^{\mu(1-\lambda)-1}\tau^{\lambda-1}\frac{M}{\Gamma(\lambda)}||u||d\tau \leq \frac{M(\mu(1-\lambda))t^{\lambda+\mu-\lambda\mu-1}}{\Gamma(\lambda+\mu-\lambda\mu-1)}||u||$$

for all $t \in [0,T]$ and $u \in X$. Therefore the operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ are bounded for $t \in [0,T]$.

The continuity and strong continuity of the operators $\mathcal{S}_{\lambda,\mu}(t)$ and $\mathcal{K}_\lambda(t)$ are achieved using Lemma-9.2.1. This completes the proof of the Lemma. □

Consider the control system governed by Hilfer fractional integro-differential equation with the classical condition in a Banach space X over the interval $[0,T]$

$$\begin{aligned} &\mathcal{D}_{0+}^{\lambda,\mu}u(t) + Au(t) = g(t,u(t),\int_0^t a(t,\tau,u(\tau))d\tau) + w(t) \\ &\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}u(0) = u_0, \end{aligned} \tag{9.2.6}$$

where, $\mathcal{D}_{0+}^{\lambda,\mu}$ is Hilfer fractional derivative operator. A is closed linear operator which is infinitesimal generator of C_0 semigroup and $u_0 \in X$. $w \in U$, a Hilbert space.

Definition 9.2.4. *A function $u \in \mathcal{C}_{\mathcal{T}}$ is called mild solution of integro-differential equation* (9.2.6) *if u is solution of the integral equation*

$$u(t) = \mathcal{S}_{\lambda,\mu} u_0 + \int_0^t \mathcal{K}_\lambda(t-\tau)[g(\tau, u(\tau), Su(\tau)) + w(\tau)]d\tau \qquad (9.2.7)$$

where, $Su(t) = \int_0^t a(t, \tau, u(\tau))d\tau$, operators $\mathcal{S}_{\lambda,\mu}$ and $\mathcal{K}_\lambda(t)$ are defined as (9.2.4) *and* (9.2.5) *respectively.*

Definition 9.2.5. *The system* (9.2.6) *is said to be completely controllable on the interval $[0, T]$ if for any $u_0, u_1 \in X$ there exit a control $w(t) \in U$ such that the solution* (9.2.7) *of* (9.2.6) *satisfies $u(T) = u_1$.*

Definition 9.2.6. *The system* (9.2.6) *is totally controllable on the interval $[0, T]$ if it is completely controllable over all its sub intervals $[\tau_k, \tau_{k+1}]$.*

Let $\mathcal{T}$ be the set of all functions $y(\cdot)$ defined over the interval $[0, T]$ satisfying $y(0) = u_0$ and $y(T) = u_1$ and Hilfer fractional derivative exist everywhere. The set $\mathcal{T}$ is called the set of all feasible trajectories.

Definition 9.2.7. *The system* (9.2.6) *is said to be trajectory controllable (T- Controllable) any $y \in \mathcal{T}$, there exist L^2 control function $w \in U$ such that the solution $u(t)$* (9.2.7) *satisfy $u(t) = y(t)$ almost everywhere.*

The trajectory controllability of the system is the strongest in comparison with other controllability.

9.3 Trajectory Controllability with Classical Condition:

In this section, we are going to discuss the trajectory controllability of the system governed by Hilfer fractional integro-differential equation (9.2.6). To discuss it we make following conditions.

(A1) Linear operator $-A$ is the infinitesimal generator of C_0-semigroup.

(A2) The nonlinear function $g : [0, T] \times X \times X \to X$ satisfies

$$||g(t, u_1, v_1) - g(\tau, u_2, v_2)|| \leq L_1(r)||u_1 - v_1|| + L_2(r)||u_2 - v_2||$$

for all $t, \tau \in [0, T]$, $u_1, u_2, v_1, v_2 \in B_r(X)$. The functions $L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing and $B_r(X)$ is a closed ball in the Banach space X of radius r.

(A3) The nonlinear operator $S : X \to X$ satisfies

$$||Su(t) - Sv(\tau)|| \leq L_3(r)||u - v||$$

for all $t, \tau \in [0, T]$, $u, v \in B_r(X)$ and the function $L_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing function.

to write

Theorem 9.3.1. *If conditions* (A1) *to* (A3) *satisfies then, the system* (9.2.6) *is trajectory controllable over the interval* $[0, T]$.

Proof. Let, $y(t)$ be any trajectory in $\mathcal{T}$ then we define feedback control as:

$$w(t) = \mathcal{D}_{0+}^{\lambda,\mu} y(t) + Ay(t) - g(t, y(t), Sy(t)))d\tau). \qquad (9.3.1)$$

Putting the value of feedback control $w(t)$ from equation (9.3.1) in equation (9.2.7) and simplifying we get,

$$\mathcal{D}_{0+}^{\lambda,\mu}[ut - y(t)] = -A[u(t) - y(t)] + g(t, u(t), Su(t)) - g(t, y(t), Sy(t)) \qquad (9.3.2)$$

Choosing $z(t) = u(t) - y(t)$, equation 9.3.2 becomes:

$$\begin{aligned} \mathcal{D}_{0+}^{\lambda,\mu} z(t) &= -Az(t) + [g(t, u(t), Su(t)) - g(t, y(t), Sy(t))] \\ \mathcal{I}_{0+}^{(1-\lambda)(1-\mu)} z(0) &= 0. \end{aligned} \qquad (9.3.3)$$

The mild solution of the equation (9.3.3) is given by

$$z(t) = \int_0^t \mathcal{K}_\lambda(t - \tau)[g(\tau, u(\tau), Su(\tau)) - g(\tau, y(\tau), Sy(\tau))]d\tau \qquad (9.3.4)$$

Therefore,

$$\begin{aligned}||z(t)|| &\leq \int_0^t ||\mathcal{K}_\lambda(t-\tau)||||g(\tau,u(\tau),Su(\tau))-g(\tau,y(\tau),Sy(\tau))||d\tau \\ &\leq \int_0^t \frac{\tau^{\lambda-1}M}{\Gamma(\lambda)}\big[L_1(r)||u(\tau)-y(\tau)||+L_2(r)L_3(r))||u(\tau)-y(\tau)||\big]d\tau \\ &\leq L\int_0^t \tau^{\lambda-1}||z(\tau)||d\tau,\end{aligned}$$

where, $L = \frac{M(L_1(r)+L_2(r)L_3(r))}{(\Gamma(\lambda))}$ and using the Gronwal's inequality we get $z(t) = 0$ almost everywhere. Therefore, $u(t) = y(t)$ almost everywhere. Hence system equation (9.2.6) is Trajectory controllable over the interval $[0,T]$. □

Example 9.3.1. *Let $X = L^2([0,1],\mathbb{R})$ and consider the partial differential equation*

$$\mathcal{D}_t^{\lambda,\mu}u(t,x) = u_{xx}(t,x) + 2u(t,x)u_x(t,x) + \int_0^t e^{-u(s,x)}ds + w(t) \tag{9.3.5}$$

with initial condition $\mathcal{I}^{(1-\lambda)(1-\mu)}u(0,x) = u_0$ and boundary conditions $u(t,0) = u(t,1) = 0$.

Define an operator A as $Au = u''$ over the domain $D(A) = H^2(0,1)\cap H^1(0,1)$ then the operator A is generates infinitesimal generator of $C-0$ strongly continuous semigroup $\mathcal{T}(t)$ given by

$$\mathcal{T}(t) = \sum_{n=1}^{\infty} exp(-n^2\pi^2 t) < u,\phi_n > \phi_n$$

where, ϕ_n are orthonormal Fourier basis for X.

The equation (9.3.5) can be rewritten as the abstract equation in $X = L^([0,1],\mathbb{R})$ as:

$$\begin{aligned}\mathcal{D}_t^{\lambda,\mu}u(t) &= Au(t) + g(t,u(t),Su(t)) + w(t) \\ \mathcal{I}^{(1-\lambda)(1-\mu)}u(0) &= u_0\end{aligned} \tag{9.3.6}$$

Clearly, $g(t,u(t),Su(t)) = 2uu_x + \int_0^t e^-u(\tau)d\tau$ is smooth function therefore for there exist $L_1, L2$ and L_3 such that $||g(t,u(t),Su(t)) - g(t,v(t),Sv(t))||[\leq L_1(r) + L_2(r)L_3(r)]||u-v||$ for all $u,v\in B(X)$. Hence, by Theorem-9.3.1, the system (9.3.5) is trajectory controllable over the interval $[0,1]$.

9.4 Trajectory Controllability with Nonlocal Condition

In this section, we are going to discuss the trajectory controllability of the system governed by the Hilfer fractional integro-differential equation in the Banach space X.

$$\begin{aligned} \mathcal{D}_{0+}^{\lambda,\mu}u(t) + Au(t) &= g(t, u(t), \int_0^t a(t,\tau,u(\tau))d\tau) + w(t) \\ \mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[u(0) - h(u)] &= u_0, \end{aligned} \tag{9.4.1}$$

where, $\mathcal{D}_{0+}^{\lambda,\mu}$ is Hilfer fractional derivative operator. A is a closed linear operator which is an infinitesimal generator of C_0-semigroup and $u_0 \in X$. $w \in U$, a Hilbert space.

Definition 9.4.1. *A function $u \in \mathcal{C}_\mathcal{T}$ is called mild solution of integro-differential equation* (9.4.1) *if u is solution of the integral equation*

$$u(t) = \mathcal{S}_{\lambda,\mu}\big[u_0 + \mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}h(u)\big] + \int_0^t \mathcal{K}_\lambda(t-\tau)[g(\tau,u(\tau),Su(\tau)) + w(\tau)]d\tau \tag{9.4.2}$$

where, $Su(t) = \int_0^t a(t,\tau,u(\tau))d\tau$, operators $\mathcal{S}_{\lambda,\mu}$ and $\mathcal{K}_\lambda(t)$ are defined as (9.2.4) *and* (9.2.5) *respectively.*

To discuss the trajectory controllability of the system (9.4.1) we required the following condition on h.

(A4) The function h satisfies

$$||h(u) - h(v)|| \leq L_h||u - v||$$

for all $u, v \in C([0,T], X)$.

Theorem 9.4.1. *If conditions* (A1) *to* (A4) *satisfies then, the system* (9.4.1) *is trajectory controllable over the interval* $[0,T]$ *provided* $L^* \neq 1$.

Proof. Let, $x(t)$ any trajectory in $\mathcal{T}$ then define

$$w(t) = \mathcal{D}_{0+}^{\lambda,\mu}x(t) + Ax(t) - g(t, x(t), Sx(t)))d\tau). \tag{9.4.3}$$

Plugging the value of $w(t)$ in equation (9.4.1) from equation (9.4.3) and simplifying we get,

$$\mathcal{D}_{0+}^{\lambda,\mu}[u(t)-x(t)] = -A[u(t)-x(t)] + g(t,u(t),Su(t)) - g(t,x(t),Sx(t)) \tag{9.4.4}$$

Choosing $z(t) = u(t) - y(t)$, equation 9.4.4 becomes:

$$\begin{aligned} \mathcal{D}_{0+}^{\lambda,\mu} z(t) &= -Az(t) + [g(t,u(t),Su(t)) - g(t,x(t),Sx(t))] \\ \mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}&[u(0) - x(0) - h(u) + h(x)] = 0 \end{aligned} \tag{9.4.5}$$

The mild solution of the equation (9.4.5) is given by

$$z(t) = \mathcal{S}_{\lambda,\mu} I^{(1-\lambda)(1-\mu)}[h(u)-h(x)] + \int_0^t \mathcal{K}_\lambda(t-\tau)[g(\tau,u(\tau),Su(\tau)) - g(\tau,x(\tau),Sx(\tau))]d\tau \tag{9.4.6}$$

Therefore,

$$\begin{aligned} ||z(t)|| &\leq ||\mathcal{S}_{\lambda,\mu} I^{(1-\lambda)(1-\mu)}[h(u)-h(x)]|| \\ &\quad + \int_0^t ||\mathcal{K}_\lambda(t-\tau)|| ||g(\tau,u(\tau),Su(\tau)) - g(\tau,x(\tau),Sx(\tau))|| d\tau \\ &\leq L^*||u(t)-x(t)|| \\ &\quad + \int_0^t \frac{\tau^{\lambda-1}M}{\Gamma(\lambda)} \big[L_1(r)||u(\tau)-y(\tau)|| + L_2(r)L_3(r))||u(\tau)-y(\tau)||\big] d\tau \\ &\leq L^*||z(t)|| + L\int_0^t \tau^{\lambda-1}||z(\tau)||d\tau, \\ &\leq \frac{L}{1-L^*}\int_0^t \tau^{\lambda-1}||z(\tau)||d\tau \end{aligned}$$

where, $L = \frac{M(L_1(r)+L_2(r)L_3(r))}{(\Gamma(\lambda))}$ and $L^* = \frac{ML_hT}{\Gamma(\lambda)\Gamma(\mu(1-\lambda))\Gamma((1-\lambda)(1-\mu)+1)}$ and using Gronwall's inequality get $z(t) = 0$ almost everywhere. Hence, system (9.4.1) is trajectory controllable over $[0,T]$. □

Example 9.4.1. *Let $X = L^2([0,1],\mathbb{R})$ and consider the partial differential equation*

$$\mathcal{D}_t^{\lambda,\mu} u(t,x) = u_{xx}(t,x) + 2u(t,x)u_x(t,x) + \int_0^t e^{-u(s,x)}ds + w(t) \tag{9.4.7}$$

with initial condition $\mathcal{I}^{(1-\lambda)(1-\mu)}[u(0,x)+h(u]) = u_0$, $h(u) = \sum_{i=1}^{2}\frac{1}{3^i}u(1/i,x)$ and boundary conditions $u(t,0) = u(t,1) = 0$. The equation (9.4.7) is converted into *abstract equation*

$$\begin{aligned} \mathcal{D}_t^{\lambda,\mu}u(t) &= Au(t) + g(t, u(t), Su(t)) + w(t) \\ \mathcal{I}^{(1-\lambda)(1-\mu)}[u(0) - h(u)] &= u_0 \end{aligned} \tag{9.4.8}$$

Since h is Lipchitz continuous with Lipschitz constant L_h, the system (9.4.7) is trajectory controllable over the interval $[0, 1]$.

9.5 Conclusion

In this chapter, we have discussed sufficient conditions for the trajectory controllability of a system governed by integro-differential systems with local and nonlocal conditions on general Banach space. We have also added illustrations to validate the derived results.