Chapter 1

Introduction and Mathematical Preliminaries

Contents

1.1 Introduction & Historical Background

A dynamical problem typically involves studying the behavior and evolution of a system over time. Problems like predicting the orbits of planets, and other celestial bodies based on gravitational interactions, changes in population sizes of species in an ecosystem, predicting economic growth, etc. are called as dynamical problems. Dynamical problems can be studied and analyzed directly through experiments. Experimental studies of dynamical problems can be resource-intensive, requiring significant time, money, and continuous observations. Moreover, problems like carbon dating take a long time and may not be practically amenable to direct experimentation.

Mathematical modeling serves as a crucial approach for studying dynamical problems by transforming them into mathematical expressions which are termed dynamical systems. This process is known as mathematical modeling which involves the identification of relevant variables, establishment of relationships, and formulation of equations that enclose the system's behavior over time. Mathematical models provide a systematic and structured means to analyze dynamical problems, offering insights into the long-term behavior, stability, and other essential characteristics of the system under consideration. One of the important advantages of mathematical modeling is its ability to make predictions and conduct simulations. Researchers can explore diverse scenarios and conditions, enhancing the depth of understanding regarding the system's dynamics. Moreover, mathematical modeling is its cost-effectiveness compared to experimental approaches. Therefore, many researchers are involved in transforming the problem into a mathematical model applying the laws of nature and solving it using various mathematical techniques.

Continuous dynamical problems are expressed using differential equations of the form $\dot{x} = \mathcal{F}(t, x)$ where x represents the state variable of the problem, $\dot{x}$ is the time derivative of x, and $\mathcal{F}$ is a functional relationship between time and state variables. If the state variable lies in finite dimensional space then the system is a finite-dimensional system otherwise is an infinite-dimensional system. If $\mathcal{F}$ in the system is linear then the system is called a linear system otherwise it is called a nonlinear system [6, 15, 111]. Poincare(1899) was the first mathematician to discuss the qualitative theory of dynamical systems. After that many researchers like Hadamard, Birkoff, Andronov, and Pontruagin contributed in developing a qualitative theory of the dynamical systems [65].

Dynamical systems frequently undergo abrupt changes or jumps during specific moments or short intervals. These behaviors can be effectively represented using impulsive differential equations. In recent decades, the theory of impulsive dynamical systems has emerged as a powerful tool for modeling a diverse range of real-world problems in fields such as medicine, biology, engineering, and physics. This modeling approach has proven particularly useful in scenarios where there are sudden alterations in critical parameters or events, such as the removal of biomass and dynamic changes in species populations [39, 84]. The impulsive dynamical system is of the form

$$\begin{aligned} \dot{x}(t) &= \mathcal{F}(t, x(t)), \quad t \in [0, T_0] - \{t_1, t_2, \cdots, t_\rho\} \\ \Delta x(t_k) &= J_k x(t_k), \quad t = t_1, t_2, \cdots, t_\rho \end{aligned} \tag{1.1.1}$$

where J_k are the jumps at the time moments $t = t_k$. Due to the applications of impulsive systems in many real-world phenomena, many researchers are interested in studying qualitative properties like the existence, uniqueness, and stability of impulsive differential equations. A detailed introduction to the impulsive dynamical systems is found in subsequent Chapters of this thesis.

Controllability is one of the fundamental property of control systems appearing in various engineering disciplines. A dynamical system is controllable if we can find a controller, that will steer the system from any initial state to a desired final state in a given finite time interval [125]. The historical development of mathematical control theory spans several decades and it involves contributions from various disciplines such as engineering, mathematics, and physics. The foundations of control theory can be traced back to the early 20th century with the origin of automatic control systems. Engineers like Frederick Taylor and Nicholas Minorsky made significant contributions to ship steering mechanisms. Norbert Wiener in the 1940s laid the theoretical groundwork for control systems. He developed the foundations of cybernetics (a field that studies communication and control in living organisms and machines). Kalman[71] introduced the concept of controllability and observability of finite dimensional linear system in 1960. This concept was extended to semilinear systems and nonlinear systems by Mirza and Womack[101], Balachandran et.al. [8, 9], Joshi and George[70], and Klamka[81]. The classical theory of controllability of finite dimensional space was extended for linear abstract systems defined on infinite dimensional spaces by Triggianni [146]. Further, Quinn and Carmichael [115], Louis and Wexler[93], George[51], Zuazua[33], and many other authors obtained controllability results for nonlinear systems in infinite dimensional spaces. Zuazua discusses various notions of exact controllability of control systems [33]. Results on the approximate controllability of semilinear and nonlinear systems were found in [51, 142, 160]. Partial controllability was discussed by Nandakumaran and George[106, 107].

Due to the importance of the impulsive dynamical systems as well as mathematical control theory, this thesis is focused on

- Existence and uniqueness of various impulsive dynamical systems of integer and fractional order considering instantaneous and non-instantaneous impulses.
- Exact controllability of the impulsive systems of integer order by considering instantaneous impulses.
- Trajectory controllability of Impulsive Systems first order and second order systems by considering non-instantaneous impulses.
- Trajectory controllability of fractional order systems.

1.2 Mathematical Preliminaries

In this section, we review some important concepts of functional analysis and the basics of the mathematical control theory of linear and nonlinear systems.

1.2.1 Some Results from Analysis

This subsection deals with some basic results from mathematical analysis and differential equations, including some definitions, lemmas, and theorems that will be of frequent use in the subsequent chapters.

Definition 1.2.1. *[83] (**Normed Spaces**) Let* $(\mathbb{X}, +, \cdot)$ *be the vector space over the field* $\mathbb{F}$*. Then the function* $||\cdot|| : \mathbb{X} \to \mathbb{F}$ *is called a norm on* $\mathbb{X}$ *if*

(i) $||x|| \geq 0, \forall x \in \mathbb{X}$ *and* $||x|| = 0$ *iff* $x = 0$.

(ii) $||cx|| = |c|||x||, \forall x \in \mathbb{X}$ *and* $\forall c \in \mathbb{F}$.

(iii) For all $x, y \in \mathbb{X}$, $||x + y|| \leq ||x|| + ||y||$.

The pair $(\mathbb{X}, ||\cdot||)$ *is called normed space.*

Definition 1.2.2. *[83](**Cauchy Sequence**) The sequence* $\{x_n\}$ *in the normed space is said to be a Cauchy sequence if, for each* $\epsilon > 0$*, there exist* $N_0 \in \mathbb{N}$ *such that* $||x_n - x_m|| < \epsilon$ *for all* $n, m \geq N_0$.

Theorem 1.2.1. *[83](**Banach Contraction Principle**) Let T be a continuous operator on a Banach space $\mathbb{X}$ such that there exists a positive number $n \geq 1$ such that $||T^n x - T^n y|| \leq k||x - y||$ for all $x, y \in \mathbb{X}$ and for some positive number $k < 1$. Then T has a unique fixed point.*
When $n = 1$, the result becomes the Banach contraction principle.

Definition 1.2.3. *[110] A one-parameter family $\{\mathcal{T}(t)\}$ for $t \geq 0$ of bounded linear operator on Banach space $\mathbb{X}$ is a semigroup of bounded linear operator on $\mathbb{X}$ if*

(i) $\mathcal{T}(0)$ is identity operator on $\mathbb{X}$.

(ii) $\mathcal{T}(t + s) = \mathcal{T}(t) \circ \mathcal{T}(s)$, for $t, s \geq 0$.

A semigroup of a bounded linear operator is uniformly continuous if

$$\lim_{t \to 0} ||\mathcal{T}(t) - I|| = 0$$

Definition 1.2.4. *[110] The linear operator A is defined on*

$$D(A) = \left\{ x \in \mathbb{X} : \lim_{t \to 0} \frac{\mathcal{T}(t)x - x}{t} exists\ for\ all\ x \in \mathbb{X} \right\}$$

and defined by

$$Ax = \frac{\mathcal{T}(t)x - x}{t}$$

is the infinitesimal generator of a semigroup $\mathcal{T}(t)$.

Definition 1.2.5. *[110] A semigroup $\{\mathcal{T}(t)\}$ of bounded linear operators on $\mathbb{X}$ is strongly continuous (C_0- semigroup) if*

$$\lim_{t \to 0} \mathcal{T}(t)x = x for\ each\ x \in \mathbb{X}$$

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1.2.2 Controllability of Linear Systems

The concept of controllability is of great importance in mathematical control theory. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to the final state, where the initial state and final state may vary over the entire space [140, 22, 125]. In this section exact controllability of the linear dynamical system has been discussed.

Consider the linear dynamical system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) \\ x(0) &= x_0 \end{aligned} \tag{1.2.1}$$

over the interval $[0, T_0],\ 0 < T_0$. Where for each $t \in [0, T_0]$, $x(t) \in \mathbb{X}$ is the state of the system, $u(t)$ is the controller of the system, $\mathcal{A}$ and $\mathcal{B}$ are linear operators respectively.

Definition 1.2.6. *[105](**Exact Controllability**) The system* (1.2.1) *is exactly controllable over the interval* $[0, T_0]$ *if for every* $x_1 \in \mathbb{X}$*, there exists* $u \in L^2([0, T_0], \mathbb{U})$ *such that there exists a differentiable function* $x \in L^2([0, T_0], \mathbb{X})$ *satisfying* (1.2.1) *and the condition* $x(T_0) = x_1$*.*

The mild solution of the system (1.2.1) is given by

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)\mathcal{B}u(s)ds \tag{1.2.2}$$

For each $u \in L^2([0, T_0], \mathbb{U})$ define an operator

$$\mathcal{C}u(t) = \int_0^{T_0} \mathcal{T}(T_0 - s)\mathcal{B}u(s)ds,$$

and its adjoint $\mathcal{C}^* : \mathbb{X} \to L^2([0, T_0], \mathbb{U})$ is

$$\mathcal{C}^* z = \mathcal{B}^* \mathcal{T}^*(T_0 - t)z.$$

The following theorem gives a direct consequence of the exact controllability of the system (1.2.1).

Theorem 1.2.2. *The system* (1.2.1) *is exactly controllable if and only if* $\mathcal{C}$ *is onto.*

Define the operator $\mathcal{W} : L^2([0, T_0], \mathbb{X}) \to L^2([0, T_0], \mathbb{X})$ by

$$\mathcal{W}z = \mathcal{C}\mathcal{C}^* z = \int_0^{T_0} \mathcal{T}(t-s)\mathcal{B}\mathcal{B}^*\mathcal{T}^*(T_0 - s)zds,$$

then foregoing theorem gives a characterization of the exact controllability for the system (1.2.1)

Theorem 1.2.3. *[34] The system* (1.2.1) *is exactly controllable on the subinterval* $[0, T_0]$ *if, any one from below satisfied for some* $\gamma > 0$, *for all* $x \in \mathbb{X}$.

(a) $Range(\mathcal{C}) = \mathbb{X}$,

(b) $||\mathcal{C}^* z||^2_{\mathbb{X}} = \int_0^{T_0} ||(C^* z)(s)||^2_{\mathbb{U}} ds \geq \gamma^2 ||z||^2_{\mathbb{X}}$,

(c) $\langle \mathcal{W} z, z \rangle \geq \gamma^2 ||z||^2_{\mathbb{X}}$,

(d) $\int_0^{T_0} ||\mathcal{B}^* \mathcal{T}^*(T_0 - s) z||^2_{\mathbb{U}} ds \geq \gamma^2 ||z||^2_{\mathbb{X}}$,

(e) $Ker(\mathcal{C}^*) = \{0\}$ *and* $Range(\mathcal{C}^*)$ *is closed.*

Theorem 1.2.4. *[34] The system* (1.2.1) *is exactly controllable on* $[0, T_0]$ *if and only if, the operator* $\mathcal{W}$ *is non-singular. Moreover the control* $u \in L^2(J_0, \mathbb{U})$ *steering an initial state* x_0 *to the final state* x_1 *at time* $t = T_0$ *is given by*

$$u(t) = \mathcal{B}^* \mathcal{T}^*(T_0 - t) \mathcal{W}^{-1} [x_1 - \mathcal{T}(T_0) x_0].$$

1.2.3 Controllability of Semilinear Systems

In this subsection, the Controllability of the semilinear systems

$$\begin{aligned} \dot{x}(t) &= \mathcal{A} x(t) + \mathcal{B} u(t) + \mathcal{F}(t, x(t)) \\ x(0) &= x_0 \end{aligned} \tag{1.2.3}$$

over the interval $[0, T_0]$, $0 < T_0$ is discussed. Where for each $t \in [0, T_0]$, $x(t) \in \mathbb{X}$ is the state of the system, $u(t)$ is the controller of the system, $\mathcal{A}$ and $\mathcal{B}$ are linear operators respectively, and $\mathcal{F}(t, x)$ is the nonlinear term.

The mild solution of the system (1.2.3) over the interval $[0, T_0]$ is given by

$$x(t) = \mathcal{T}(t) x_0 + \int_0^t \mathcal{T}(t - s)[\mathcal{B} u(s) + \mathcal{F}(s, x(s))] ds \tag{1.2.4}$$

where, $\mathcal{T}(\cdot)$ is the operator semigroup generated by the linear part $\mathcal{A}$.

Definition 1.2.7. *[86] The system* (1.2.3) *is said to be exactly controllable over the interval* $[0, T_0]$ *if for every* $x_1 \in \mathbb{X}$*, there exists* $u \in L^2([0, T_0], \mathbb{U})$ *such that there exists a differentiable function* $x \in L^2([0, T_0], \mathbb{X})$ *satisfying* (1.2.4) *satisfies* $x(T_0) = x_1$*.*

The mild solution of the equation (1.2.3) at $t = T_0$ becomes

$$x_1 = x(T_0) = \mathcal{T}(T_0)x_0 + \int_0^{T_0} \mathcal{T}(T_0 - s)[\mathcal{B}u(s) + \mathcal{F}(s, x(s))]ds. \tag{1.2.5}$$

Define the operator $\mathcal{G} : L^2([0, \tau], \mathbb{U}) \to \mathbb{X}$ by

$$\mathcal{G}u = \int_0^{T_0} \mathcal{T}(T_0 - s)[\mathcal{B}u(s) + \mathcal{F}(s, x(s))]ds$$

then the system (1.2.3) exactly controllable over the interval $[0, T_0]$ if and only if $\mathcal{G}$ is onto. If the corresponding linear system (1.2.1) is controllable then there exists a steering operator $\mathcal{S} = \mathcal{B}^*\mathcal{T}^*(T_0 - t)\mathcal{W}^{-1}$ such that $\mathcal{C} \circ \mathcal{S} = I$(identity operator).

Define the operator $\bar{\mathcal{G}} : \mathbb{X} \to \mathbb{X}$ by

$$\bar{\mathcal{G}}\zeta = (\mathcal{G} \circ \mathcal{S})\zeta = \zeta + \int_0^{T_0} \mathcal{T}(T_0 - s)\mathcal{F}(s, \zeta(s))ds = (I + \mathcal{K})\zeta$$

then system (1.2.3) is controllable if and only if $\bar{\mathcal{G}}$ is non-singular where, $\mathcal{K}\zeta = \int_0^{\tau} \mathcal{T}(\tau - s)\mathcal{F}(s, \zeta(s))ds$.

The following theorems give sufficient conditions for the exact controllability of the system (1.2.3).

Theorem 1.2.5. *[157] If the corresponding linear system* (1.2.1) *is controllable and* $\mathcal{K}$ *is globally Lipschitz continuous with Lipschitz constant* $0 < k < 1$ *then system* (1.2.3) *is exactly controllable over the interval* $[0, T_0]$ *and state* x *of the systems* (1.2.3) *steers form initial state* x_0 *to final state* x_1 *at* $t = T_0$*.*

Theorem 1.2.6. *[157] If* $\mathcal{F}(t, x)$ *is Lipchitz continuous with respect to the second argument then the system* (1.2.3) *is exactly controllable over the interval* $[0, T_0]$ *and controller* $u(t)$ *which steers initial state* x_0 *to final state at* $t = \tau$ *is given by* $u(t) = \mathcal{B}^*\mathcal{T}^*(T_0 - t)\mathcal{W}^{-1}(I + \mathcal{K})\left[x_1 - \mathcal{T}(T_0 - t)x_0 - \int_0^{T_0} \mathcal{T}(T_0 - s)\mathcal{F}(s, x(s))ds\right]$*.*

1.3 Layout of the thesis

The thesis contains nine chapters focusing on the existence of solutions, exact controllability, and trajectory controllability of the different types of systems. The detailed layout of the thesis is as follows:

Chapter 1 deals with the introduction and historical background as well as mathematical preliminaries.

Chapter 2, contains the existence and uniqueness of the classical and mild solutions of the generalized impulsive evolution equation

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_i(t, x(t), T_i x(t), S_i x(t)), \quad t \in [t_{i-1}, t_i), \quad t \neq t_i \\ x(0) &= x_0 \\ \Delta x(t_i) &= \mathcal{I}_i x(t_i), t = t_i, \quad for \;\; i = 1, 2, 3, \ldots, N. \end{aligned} \tag{1.3.1}$$

over the finite interval $[0, T_0]$ in the Banach space $\mathbb{X}$. Here, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is the linear part of the evolution equation, for all i, $\mathcal{F}_i : [0, T_0] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions operated over the interval $[t_{i-1}, t_i)$, operators $T_i, S_i : \mathbb{X} \times \mathbb{X}$ are operators operated over the interval $[t_{i-1}, t_i)$, and $\mathcal{I}_i : \mathbb{X} \to \mathbb{X}$ are the jumps at the time moments $t = t_i$.

Chapter 3 contains the Exact controllability of the system

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_k(t, x(t), u(t)) + \mathcal{B}_k u(t) \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \cdots, \rho \\ x(0) &= x_0 \\ \Delta x(t_k) &= \mathcal{M}_k x(t_k) + \mathcal{N}_k u(t_k), \quad t = t_k, \quad k = 1, 2, \cdots, \rho \end{aligned} \tag{1.3.2}$$

over the interval $[0, T_0]$ using the concept of operator semigroup, linear and nonlinear functional analysis. In equation (1.3.2) the state $x(t)$ in the Hilbert space $\mathbb{X}$ for all $t \in J_0 = [0, T_0]$, $\mathcal{A}$, and $\mathcal{M}_k$ are linear operators on $\mathbb{X}$, $u \in L^2([0, T_0], \mathbb{U})$ $\mathbb{B}_k, \mathbb{N}_k$: $\mathbb{X} \times \mathbb{U}$ are bounded linear functions between Hilbert spaces $\mathbb{X}$ and $\mathbb{U}$, and $\mathcal{F}_k$: $[0, T_0] \times \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ are nonlinear functions.

Chapter 4 discussed the trajectory controllability of first-order non-instantaneous impulsive systems

$$\begin{aligned} x'(t) &= \mathcal{A}(t)x(t) + \mathcal{F}(t, x(t)) + \mathcal{W}(t), \quad t \in [s_k, t_k + 1),\ for\ all\ k = 0, 1, 2 \cdots, p \\ x(t) &= \mathcal{G}_k(t, x(t)) + \mathcal{W}_k(t), \quad t \in [t_k, s_k),\ for\ all\ k = 1, 2, \cdots, p, \end{aligned} \tag{1.3.3}$$

with local condition $x(0) = x_0$ and non-local condition $x(0) = x_0 - h(x)$ over the interval $[0, T]$ in the Banach space $\mathbb{X}$ where, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is linear operator, $\mathcal{F}$, and $\mathcal{G}_k$ are nonlinear functions on $[0, T] \times \mathbb{X}$, and $\mathcal{W}, \mathcal{W}_k$ is trajectory controller.

Chapter 5 discussed the trajectory controllability of second-order systems

$$\begin{cases} x''(t) = \mathcal{A}x + \mathcal{F}(t, x(t), x'(t)) + \mathcal{W}(t), \\ x(0) = x_{10}, \quad x'(0) = x_{20} \end{cases} \tag{1.3.4}$$

by considering non-instantaneous impulses into account over the finite time interval $\Omega = [0, T_0]$. Here, at each time t, the state lies in $\mathbb{X}$, $\mathcal{A}$ is the linear on $\mathbb{X}$, $\mathcal{F} : \Omega \times \mathbb{X}^2 \to \mathbb{X}$ is a non-linear function, and $\mathcal{W}(t)$ is the trajectory controller of the system.

Chapter 6 derive a set of sufficient conditions for the existence of a mild solution for the generalized fractional order impulsive system

$$\begin{aligned} {}^cD^\lambda x(t) &= \mathcal{A}x(x) + \mathcal{F}_k\left(t, x(t), \int_0^t a_k(t, \tau, x(\tau))d\tau\right),\ t \in [s_{k-1}, t_k),\ k = 1, 2, \cdots, p \\ x(t) &= \mathcal{G}_k(t, x(t)),\ t \in [t_k, s_k) \end{aligned}$$

with local condition $x(0) = x_0$ and non-local condition $x(0) = x_0 + h(x)$ over the interval $[0, T]$ in a Banach space $\mathbb{X}$. Here $A : \mathbb{X} \to \mathbb{X}$ is linear operator, $P_k x = \int_0^t a_k(t, \tau, x(t))d\tau$ are nonlinear Volterra integral operator on $\mathbb{X}$, $\mathcal{F}_k : [0, T] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are nonlinear functions applied in the intervals $[s_{k-1}, t_k)$ and $\mathcal{G}_k : [0, T] \times \mathbb{X}$ are set of nonlinear functions applied in the interval $[t_k, s_k)$ for all $k = 1, 2, \cdots, p$.

Chapter 7 developed the necessary criteria for a mild solution and classical solution of the impulsive fractional evolution problem,

$$\begin{aligned} {}^cD^\alpha x(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)) \quad t \neq t_k,\ k = 1, 2, \cdots, p \\ \Delta x(t_k) &= \mathcal{I}_k(x(t_k)), \quad t = t_k,\ k = 1, 2, \cdots, p \\ x(t_0) &= u_0 \end{aligned} \tag{1.3.5}$$

over the interval $[0, T_0]$ on a Banach space $\mathbb{X}$. Here, ${}^cD^\alpha$ denotes Caputo fractional differential operator of order $0 < \alpha \leq 1$, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is linear operator and $\mathcal{F} : [0, T_0] \times \mathbb{X} \to$ is nonlinear function. $I_k : \mathbb{X} \to \backsim$ are impulse operator at time $t = t_k$, fro $k = 1, 2, \cdots, p$ and their existence and uniqueness. We also developed conditions under which classical and mild solutions of (1.3.5) coincide.

Chepter 8 considered non-instantaneous impulsive integro-differential fractional order ($0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$) evolution system of Hilfer type

$$\begin{aligned} D_{0+}^{\lambda,\mu} x(\zeta) &= -\mathcal{A}x(t) \\ &\quad + \mathcal{F}\left(t, u(t), \int_0^t a(t, \tau, x(\tau)) d\tau\right),\ t \in \left[\cup [s_i, t_{i+1})\right] \cup [s_p, T_0] \\ u(t) &= \mathcal{G}_k(t, x(t)),\ t \in [t_1, s_1) \cup [t_2, s_2) \cup \cdots \cup [t_p, s_p), \end{aligned}$$

and discussed the existence of solutions with local condition $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)} x(0) = x_0$ and non-local $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[x(0) - h(x)] = x_0$ initial conditions over the finite interval $[0, T_0]$ in a Banach space $\mathbb{X}$. $D^{\lambda,\mu}$ differential operators of Hilfer type, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is a linear part of the integrodifferential evolution equation, $Kx = \int_0^t a(t, \tau, x(\tau)) d\tau$ is nonlinear Volterra integral operator on $\mathbb{X}$, $\mathcal{F} : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is nonlinear function and $\mathcal{G}_k : [0, T_0] \times \mathbb{X}$ are set of non-linear functions applied in the interval $[t_k, s_k)$ for all $i = 1, 2, \cdots, p$.

Chapter 9 discussed the Trajectory controllability of infinite dimensional Hilfer fractional control systems

$$\mathcal{D}_{0+}^{\lambda,\mu} x(t) + Ax(t) = \mathcal{F}(t, x(t), \int_0^t a(t, \tau, x(\tau)) d\tau) + \mathcal{W}(t)$$

over the interval $[0, T_0]$ with classical condition $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)} x(0) = x_0$ and non-local conditions $\mathcal{I}_{0+}^{(1-\lambda)(1-\mu)}[x(0) - h(x)] = x_0$ in the Banach space $\mathbb{X}$ Where, $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ is a linear part of the integrodifferential evolution equation, $Kx = \int_0^t a(t, \tau, x(\tau)) d\tau$ is nonlinear Volterra integral operator on $\mathbb{X}$, $\mathcal{F} : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is nonlinear function.