Chapter 2

Generalized Impulsive Systems

Contents

This chapter derives the existence and uniqueness of a generalized nonlinear impulsive evolution equation. The proposed system is modeled with a nonlinear perturbed force that changes after every impulse. The Banach contraction principle is applied to prove the existence and uniqueness of the mild solution. The existence and uniqueness of classical solution is obtained by fixing the impulse and the conditions in which mild solution becomes classical solution also obtained. Finally, an example is illustrated to the effectiveness of the main results.

2.1 Introduction

From the last few decades, the theory of impulsive differential equations has been extensively used to model different problems such as the removal of insertion of biomass [7], the population of species with abrupt changes of the important biological parameter [44], abrupt harvesting [79], and various real-world problems of medicine, biology, mechanics and control theory [39, 58, 69, 84, 90, 100, 132, 124]. Existence and uniqueness of the solution of the impulsive evolution equation of the form,

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}(t, x(t)), \quad t > 0,\ t \neq t_i \\ x(0) &= x_0, \\ \Delta x(t_i) &= \mathcal{I}_i(x(t_i)), \quad i = 1, 2, 3, \ldots, N. \end{aligned} \tag{2.1.1}$$

is studied by several authors. Rogovchenko [123] discussed existence of solution of (2.1.1) using successive approximations when $\mathcal{A}$ is sectroial operator. Liu [92] used the semigroup property and Banach fixed point theorem to study the existence and uniqueness of classical and mild solutions of evolution equation (2.1.1). Li [89] studies the global solution of the evolution equation (2.1.1) without impulses using the methodology as [92]. Yang [62] replaced a finite number of impulses with an infinite number of impulses and derived the existence of e-positive mild solutions of (2.1.1) with non-compact semigroup in an ordered Banach space. Xiang, Peng, and Wei [152] discussed PWD - α mild solutions of (2.1.1) in which $\mathcal{F}$ contains a integral operator. Anguraj and Arjunan [5] modified $\mathcal{F}$ by introducing two integral operators and obtained similar types of results as Liu [92]. Zhang and Yan [158] studied global mild and positive mild solutions of (2.1.1) without impulses by replacing $\mathcal{F}(t, x(t))$ by $\mathcal{F}(t, x(t), Tx(t), Sx(t))$ where T and S are integral operators. Sattayatham [134] replaced $\mathcal{A}x(t)$ by $\mathcal{A}(t, x(t))$ and discussed the existence and some properties of the solution of (2.1.1) by assuming conditions on resolvent. Liang *at. el.* [91], Fan and Li [46], Wen and Ji [149] and Chen and Yang, and other researchers studied the existence of solution using nonlocal conditions using various approaches [24, 28, 29, 30]. On the other hand Tang and Nieto [144] used variational approach to discussed existence and uniqueness of (2.1.1) with improved impulses $\mathcal{I}_i(t_i, x(t_i))$.

Here a new nonlinear impulsive evolution equation over $[0, T_0]$ is considered with the nonlinear perturbation changes after every impulse in the following form:

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_i(t, x(t), T_i x(t), S_i x(t)),\ t \in [t_{i-1}, t_i)\ t \neq t_i, \\ x(0) &= x_0, \\ \Delta x(t_i) &= \mathcal{I}_i(x(t_i)), \quad i = 1, 2, 3, \ldots, N. \end{aligned} \tag{2.1.2}$$

in a Banach space $\mathbb{X}$, where $\mathcal{A}$ is the infinitesimal generator of C_0 semigroup [110] $\{\mathcal{G}(t)|t \geq 0\}$, $\mathcal{F}_i \in C([0, T_0] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ are operators on Banach space $\mathbb{X}$. The impulses are satisfies $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_i < \cdots < t_N < t_{N+1} = T_0$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-) = \mathcal{I}_i(x(t_i))$, where $\mathcal{I}_i : \mathbb{X} \to \mathbb{X}$ is operator describe the jumps for each $i = 1, 2, \cdots, N$. For $i = 1, 2, \cdots, N+1$ operators $T_i, S_i : X \to X$ defined for all $t \in [t_{i-1}, t_i)$. In case of integro-differential equations one can define $T_i(x(t)) = \int_0^t \phi_i(t, s, x(s))ds$ and $S_i(x(t)) = \int_0^T \psi_i(t, s, x(s))ds$ if $t \in [t_{i-1}, t_i)$. The system (2.1.2) generalizes the system taken by Anguraj and Arjunan [5].

In this chapter, the author studied the existence and uniqueness of mild and classical solutions using semigroup theory and Banach fixed point theory and proved that if $\mathcal{F}_i$'s are continuously differentiable then mild solution give rise to classical solutions. The equations (2.1.2) are used to model a physical phenomenon having different perturbing force components $\mathcal{F}_i$ after impulses.

2.2 Definitions and Assumptions

Definition 2.2.1. *[135] Let $\mathbb{X}$ be a Banach space. Let $PC([0, T_0], \mathbb{X})$ consist of functions u that are a map from $[0, T_0]$ into $\mathbb{X}$, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$, and the right limit $u(t_i^+)$ exists for $i = 1, 2, 3, \ldots N$. Evidently $PC([0, T_0], \mathbb{X})$ is a Banach space with the norm*

$$\|u\|_{PC} = \sup_{t \in [0, T_0]} \|u(t)\|.$$

2.2.1 Assumptions

Let the impulsive evolution equation (2.1.2) in a Banach space $\mathbb{X}$ where, $\mathcal{F}_i \in C([0, T_0] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$ and ,$T_i,\ S_i : \mathbb{X} \to \mathbb{X}$ for $i = 1, 2, \ldots N$ are operators on

Banach space $\mathbb{X}$. The following are assumptions:

(A1) $\mathcal{F}_i : [0, T_0]\times\mathbb{X}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$, and $\mathcal{I}_i : \mathbb{X} \to \mathbb{X}$, $i = 1, 2, \dots N$. are continuous and there exists constants $L_{i1}, L_{i2}, L_{i3} > 0$, $h_i > 0$, $T_i^*, S_i^* > 0$ for $i = 1, 2, 3, \dots, N$. such that for $t \in [0, T_0]$ and $x, y \in \mathbb{X}$ we have,

$$\|\mathcal{F}_i(t, x_1, x_2, x_3) - \mathcal{F}_i(t, y_1, y_2, y_3)\| \le L_{i1}\|x_1 - y_1\| + L_{i2}\|x_2 - y_2\| + L_{i3}\|x_3 - y_3\|,$$
$$\|\mathcal{I}_i(x) - \mathcal{I}_i(y)\| \le h_i\|x - y\|, \; \|T_i(x) - T_i(y)\| \le T_i^*\|x - y\|,$$
$$\|S_i(x) - S_i(y)\| \le S_i^*\|x - y\|.$$

Let $\mathcal{G}(t)$ be the C_0 semigroup generated by the linear operator $\mathcal{A}$. Let $B(\mathbb{X})$ be the Banach space of all linear and bounded operators on $\mathbb{X}$.
Let

$$M = \max_{t\in[0,T_0]} \|\mathcal{G}(t)\|_{B(X)}, \quad L = \max_{i=1,2,\dot{N}} \{L_{i1}, L_{i2}, L_{i3}\} \quad H = \sum_{i=1}^{N} h_i.$$

(A2) The constants L, H, M, T_i^*, S_i^* satisfy the inequality

$$M\Big[LT_0 \sum_{i=1}^{N}(1 + T_i^* + S_i^*) + H\Big] < 1$$

Definition 2.2.2. *A function $x(t) \in PC([0, T_0], \mathbb{X})$ is a mild solution of equations (2.1.2) if it satisfies*

$$x(t) = \mathcal{G}(t)x_0 + \sum_{t_i<t} \int_{t_i}^{t} \mathcal{G}(t - s)\mathcal{F}_i(s, x(s), T_i x(s), S_i x(s))\mathrm{d}s + \sum_{0<t_i<t} \mathcal{G}(t - t_i)\mathcal{I}_i(x(t_i)) \tag{2.2.1}$$

for all $t \in [0, T_0]$.

Definition 2.2.3. *[5] A classical solution of Equations (2.1.2) is a function $x(t)$ in $PC([0, T_0], \mathbb{X}) \cap C^1((0, T_0)\backslash\{t_1, t_2, \dots, t_N\}, X)$, $x(t) \in D(\mathcal{A})$ (Domain of $\mathcal{A}$) for $t \in (0, T_0)\backslash\{t_1, t_2, \dots, t_N\}$, which satisfies equations (2.2.1) on $[0, T_0]$.*

2.3 Existence and Uniqueness Theorems

Theorem 2.3.1. *Assume that (A1)-(A2) are satisfied. Then for every $x_0 \in X$, for $t \in [0, T_0]$ the equation (2.1.2) has a unique solution.*

Proof. Let $x_0 \in \mathbb{X}$ be fixed. Define an operator $\mathcal{F}$ on $PC([0, T_0], \mathbb{X})$ by

$$(\mathcal{F}u)(t) = \mathcal{G}(t)x_0 + \sum_{t_i<t} \int_{t_i}^{t} \mathcal{G}(t-s)\mathcal{F}_i(s, u(s), T_i u(s), S_i u(s))\mathrm{d}s + \sum_{0<t_i<t} \mathcal{G}(t-t_i)\mathcal{I}_i(u(t_i))$$

for $t \in [0, T_0]$. Then $\mathcal{F} : PC([0, T_0], \mathbb{X}) \to PC([0, T_0], \mathbb{X})$. To prove (2.2.1) has unique solution, $\mathcal{F}$ must be a contraction. For any $x, y \in PC([0, T_0], \mathbb{X})$,

$$\begin{aligned} \|(\mathcal{F}x)(t) &- (\mathcal{F}y)(t)\| \\ &\leq \sum_{t_i<t} \int_{t_i}^{t} \|\mathcal{G}(t-s)\| \|\mathcal{F}_i(s, x(s), T_i x(s), S_i u(s) - \mathcal{F}_i(s, y(s), T_i y(s), S_i y(s))\|\mathrm{d}s \\ &+ \sum_{0<t_i<t} \|\mathcal{G}(t-t_i)\| \|\mathcal{I}_i(u(t_i)) - \mathcal{I}_i(v(t_i))\| \end{aligned}$$

By using assumption (A1), we have

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \leq M[T_0 L \sum_{i=1}(1 + T_i^* + S_i^*) + H]\|x - y\|$$

Taking $\alpha = M[T_0 L \sum_{i=1}(1+T_i^*+S_i^*)+H]$ and assuming (A2), $\|(\mathcal{F}x)(t)-(\mathcal{F}t)(t)\| \leq \alpha\|x - y\|$, for $x, y \in PC([0, T_0], \mathbb{X})$ with $\alpha < 1$.
Therefore, $\mathbb{F}$ is a contraction operator on $PC([0, T_0], \mathbb{X})$ and applying Banach fixed point theorem [83], one get unique solution of (2.2.1). □

Remark 2.3.1. *Conditions (A1) and (A2) guarantee the existence of the mild solution of (2.2.1) but if conditions are not satisfied then we may get a unique solution. Let us consider the impulsive evolution equation over* $[0, 2.5]$

$$x'(t) = \frac{x}{t}, \quad x(0) = 0, \quad t \in [0, 2.5], \quad t \neq 1, 2, \quad \Delta x(1) = 2x(1), \quad \Delta x(2) = 3x(2)$$

has unique solution by $x(t) = 0$ but $\mathcal{F} = \frac{x}{t}$ is not continuous at $t = 0$. Therefore conditions (A1) and (A2) are sufficient but not necessary.

Remark 2.3.2. *If $\mathcal{I}_i$'s are constants then equation (2.2.1) has unique solution if we* *reduce inequality in (A2) by* $M[T_0L\sum_{i=1}(1+T_i^*+S_i^*)]<1.$

The following lemmas are necessary to prove a unique classical solution of equation (2.1.2).

Lemma 2.3.1. *Consider the evolution equation,*

$$\begin{aligned} x'(t) &= \mathcal{A}x(t)+\mathcal{F}(t,x(t),Tx(t),Sx(t)), \quad 0<t<T_0 \\ x(0) &= x_0, \end{aligned} \tag{2.3.1}$$

If $x_0\in D(\mathcal{A})$*, and* $\mathcal{F}\in C^1((0,T_0)\times\mathbb{X}\times\mathbb{X}\times\mathbb{X},\mathbb{X})$ *and* $T,\ S$ *are operator on* $\mathbb{X}$*, and there are positive constants* L_1,L_2,L_3,P,Q,M *satisfies*

(B1) For all $t\in[0,T_0]\quad x,y\in\mathbb{X}$*.*

$$\begin{aligned} \|\mathcal{F}(t,x_1,x_2,x_3)-\mathcal{F}(t,y_1,y_2,y_3)\| &\le L_1\|x_1-y_1\|+L_2\|x_2-y_2\|+L_3\|x_3-y_3\| \\ \|Tx-Ty\|\le P\|x-y\|,\ \|Sx-Sy\| &\le Q\|x-y\|,\ M=\max_{t\in[0,T_0]}\|\mathcal{G}(t)\|. \end{aligned}$$

(B2) The constants $L,P,Q,$ *and* M *satisfies* $\alpha=MLT_0(1+P+Q)<1$ *where* $L=\max\{L_1,L_2,L_3\}$*.*

then it has a unique classical solution, which satisfies

$$x(t)=\mathcal{G}(t)x_0+\int_0^t\mathcal{G}(t-s)\mathcal{F}(s,x(s),Tx(s),Sx(s))\mathrm{d}s \tag{2.3.2}$$

for all $t\in[0,T_0)$*.*

Proof. Let, x_0 be fixed and define an operator $\mathcal{F}$ on $C^1[0,T_0]$ by

$$(\mathcal{F}x)(t)=\mathcal{G}(t)x_0+\int_0^t\mathcal{G}(t-s)\mathcal{F}(s,x(s),Tx(s),Sx(s))\mathrm{d}s$$

Then to show (2.3.1) has a unique classical solution (2.3.2), it is sufficient to show that $\mathcal{F}$ is contraction.

$$\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\|\le\int_0^t\|\mathcal{G}(t-s)\|\|\mathcal{F}(s,x(s),Tx(s),Sx(s))-\mathcal{F}(s,y(s),Ty(s),Sy(s))\|\mathrm{d}s.$$

Applying conditions in of lemma to get,$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \leq \alpha\|x - y\|$ and α is less than 1.
So, $\mathcal{F}$ is a contraction operator. Therefore $x(t)$ is unique solution of (2.3.2).
Let, $y(t)$ be the classical solution of the evolution equation (2.3.1) and $\mathcal{F} \in C^1((0, T_0)\times \mathbb{X} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$. Therefore,

$$y(t) = \mathcal{G}(t)x_0 + \int_0^t \mathcal{G}(t-s)\mathcal{F}(s, y(s), Ty(s), Sy(s))\mathrm{d}s. \tag{2.3.3}$$

Thus, $u(t)$ also satisfies mild solution of (2.3.2). But the mild solution is unique therefore $u(t) = x(t)$. Which completes the proof of the lemma. □

Lemma 2.3.2. *For the unique classical solution $x(t)$ on $[t_{i-1}, t_i)$ of equation (2.1.2) without impulsive conditions, one can define $x(t_i)$ in a such way that $x(t)$ is left continuous at t_i and $x(t_i) \in D(\mathcal{A})$ for $i = 1, 2, \cdots, N+1$.*

Proof. On the interval $[t_{i-1}, t_i)$, equation (2.1.2) becomes

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_i(t, x(t), T_i x(t), S_i x(t)) \\ x(t_{i-1}) &= x_{t_{i-1}} \end{aligned} \tag{2.3.4}$$

without impulsive condition. Then by lemma 2.3.1 equation (2.3.1) has unique classical solution

$$x(t) = \mathcal{G}(t)x_{t_{i-1}} + \int_{t_{i-1}}^t \mathcal{G}(t-s)f_{i-1}(s, x(s), T_{i-1}x(s), S_{i-1}x(s))\mathrm{d}s \tag{2.3.5}$$

on the interval $[t_{i-1}, t_i)$ and $x(t) \in D(\mathcal{A})$ for all $t \in (t_{i-1}, t_i)$. Therefore we can define

$$x(t_i) = \mathcal{G}(t_i - t_{i-1})x_{i-1} + \int_{t_{i-1}}^{t_i} \mathcal{G}(t_i - s)f_i(s, x(s), T_i x(s), S_i x(s))\mathrm{d}s \tag{2.3.6}$$

One can complete the proof by showing $x(t)$ is left continuous att_i.
Let s_k be increasing sequence of real numbers such that s_k converges to t_i, then

$$\begin{aligned} \|x(t_i) - x(s_k)\| \leq{}& \|\mathcal{G}(t_i - t_{i-1}) - \mathcal{G}(s_k - t_{i-1})\|\|x_{i-1}\| \\ &+ \int_{s_k}^{t_i} \|\mathcal{G}(t_i - s) - \mathcal{G}(s_k - s)\|\|\mathcal{F}_i(s, x(s), T_i x(s), S_i x(s))\|\mathrm{d}s \end{aligned}$$

Since, $\mathcal{G}(t)$, and $\mathcal{F}_i, T_i, S_i$'s are continuous for $i = 1, 2, \cdots, N+1$. Therefore, $\|x(t_i) - x(s_k)\| \to 0$ as $k \to \infty$. Therefore $x(t)$ is left continuous at t_i and $x(t_i) \in D(\mathcal{A})$. Which completes the proof of the lemma. □

Theorem 2.3.2. *Assume that $x_0 \in D(\mathcal{A})$, $q_i \in D(\mathcal{A})$, $i = 1, 2, \cdots, N$. and that $\mathcal{F}_i \in C^1((0, T_0) \times \mathbb{X} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$. Then the impulsive equation*

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_i(t, x(t), T_i x(t), S_i x(t)), \\ x(0) &= x_0, \\ \Delta x(t_i) &= q_i, \quad i = 1, 2, 3, \ldots, N. \end{aligned} \tag{2.3.7}$$

has a unique classical solution $x(t)$ which satisfies,

$$x(t) = \mathcal{G}(t)x_0 + \sum_{t_i < t} \int_{t_i}^{t} \mathcal{G}(t-s)\mathcal{F}_i(s, x(s), T_i x(s), S_i x(s))\mathrm{d}s + \sum_{0 < t_i < t} \mathcal{G}(t - t_i) q_i. \tag{2.3.8}$$

for $t \in [0, T_0)$.

Proof. Consider the interval $I_1 = [t_0, t_1)$ then by Lemma 2.3.2 for equation

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_1(t, x(t), T_1 x(t), S_1 x(t)), \\ x(0) &= x_0 \end{aligned}$$

has a unique classical solution $x_1(t)$ which satiesfies

$$x_1(t) = \mathcal{G}(t)x(0) + \int_0^t \mathcal{G}(t-s)\mathcal{F}_1(s, x(s), T_1 x(s), S_1 x(s))\mathrm{d}s, \quad t \in [0, t_1)$$

and $x_1(t) \in D(\mathcal{A})$ for all $t \in [0, t_1)$. We define $x_1(t_1)$ as

$$x_1(t_1) = \mathcal{G}(t_1)x(0) + \int_0^{t_1} \mathcal{G}(t_1 - s)\mathcal{F}_1(s, x(s), T_1 x(s), S_1 x(s))\mathrm{d}s$$

such that $x_1(t)$ is left continuous at $t = t_1$ and $x_1(t_1) \in D(\mathcal{A})$.
On the interval $[t_1, t_2)$ consider the equation

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_2(t, x(t), T_2 x(t), S_2 x(t)), \\ x(t_1) &= x_1(t_1) + q_1. \end{aligned}$$

Then $x(t_1) \in D(\mathcal{A})$ and applying lemma (2.3.2) above equation has unique classical solution $x_2(t)$ satisfying

$$x_2(t) = \mathcal{G}(t - t_1)x(t_1) + \int_{t_1}^{t} \mathcal{G}(t - s)\mathcal{F}_2(s, x(s), T_2x(s), S_2x(s))\mathrm{d}s, \quad t \in [t_1, t_2)$$

and we define $x_2(t)$ at $t = t_2$ as

$$x_2(t_2) = \mathcal{G}(t_2 - t_1)x(t_1) + \int_{t_1}^{t_2} G(t_2 - s)\mathcal{F}_2(s, x(s), T_2x(s), S_2x(s))\mathrm{d}s$$

such that $x_2(t)$ is left continuous at t_2 and $x_2(t_2) \in D(\mathcal{A})$.
Continuing this process on each interval $[t_{k-1}, t_k)$ for $k = 1, 2, \ldots, N$, get classical solution $x_k(t)$ which satisfies the mild solution,

$$x_k(t) = \mathcal{G}(t_k - t_{k-1})x(t_{k-1}) + \int_{t_{k-1}}^{t} \mathcal{G}(t - s)\mathcal{F}_{k-1}(s, x(s), T_{k-1}x(s), S_{k-1}x(s))\mathrm{d}s$$

such that $x_k(t)$ is left continuous at $t = t_k$ and $x_k(t_k) \in D(\mathcal{A})$.
On interval $[t_N, T_0)$ the system become,

$$\begin{aligned} x'(t) &= \mathcal{A}x(t) + \mathcal{F}_{N+1}(t, x(t), T_{N+1}x(t), S_{N+1}x(t)), \\ x(t_N) &= x_N(t_N) + q_N \end{aligned}$$

Again applying Lemma 2.3.2 above system has solution $x_{N+1}(t)$ which satisfies equation on interval $[t_N, T_0]$

$$x_{N+1}(t) = \mathcal{G}(t - t_N)x(t_N) + \int_{t_N}^{t} \mathcal{G}(t - s)\mathcal{F}_{N+1}(s, x(s), T_{N+1}x(s), S_{N+1}x(s))\mathrm{d}s$$

such that $x_{N+1}(t)$ is continuous at T_0 and $x_{N+1}(T_0) \in D(\mathcal{A})$.
Define

$$x(t) = \begin{cases} x_1(t), & 0 \le t < t_1, \\ x_k(t), & t_k - 1 \le t < t_k, k = 2, 3, \ldots, N. \\ x_{N+1}(t), & t_N < t \le T_0. \end{cases} \tag{2.3.9}$$

Then $x(t) \in PC([0, T_0], X)$ and $x(t)$ satisfies impulsive evolution equation (2.3.7). Therefore $x(t)$ is classical solution. Now we prove that $x(t)$ also satisfies equation

(2.3.8). Consider,

$$\begin{aligned} x(t) &= x_{N+1}(t) \\ &= \mathcal{G}(t-t_N)x(t_N) + \int_{t_N}^{t} \mathcal{G}(t-s)\mathcal{F}_{N+1}(s, x(s), T_{N+1}x(s), S_{N+1}x(s))\mathrm{d}s \end{aligned}$$

$$\begin{aligned} x(t) = \mathcal{G}(t-t_N)[\mathcal{G}(t_N-t_{N-1})x(t_{N-1}) + \int_{t_{N-1}}^{t_N} \mathcal{G}(t_N-s)\mathcal{F}_N(s, x(s), T_Nx(s), S_Nx(s))\mathrm{d}s \\ + q_N] + \int_{t_N}^{t} \mathcal{G}(t-s)\mathcal{F}_{N+1}(s, x(s), T_{N+1}x(s), S_{N+1}x(s))\mathrm{d}s \end{aligned}$$

Applying Semigroup property to get,

$$\begin{aligned} x(t) = \mathcal{G}(t-t_{N-1})x(t_{N-1}) + \int_{t_{N-1}}^{t_N} \mathcal{G}(t-s)\mathcal{F}_N(s, x(s), T_Nx(s), S_Nx(s))\mathrm{d}s \\ + \int_{t_N}^{t} \mathcal{G}(t-s)\mathcal{F}_{N+1}(s, x(s), T_{N+1}x(s), S_{N+1}x(s))\mathrm{d}s + \mathcal{G}(t-t_N)q_N \end{aligned}$$

Continuing this process, one gets

$$x(t) = \mathcal{G}(t)x_0 + \sum_{t_i<t} \int_{t_i}^{t} \mathcal{G}(t-s)\mathcal{F}_i(s, x(s), T_ix(s), S_ix(s))\mathrm{d}s + \sum_{0<t_i<t} \mathcal{G}(t-t_i)q_i$$

Therefore $x(t)$ is solution of integral equation (2.3.8). Which completes the proof of the theorem. □

The next theorem derives the condition for which a mild solution of (2.1.2) gives rise to a classical solution.

Theorem 2.3.3. *Assume the hypotheses (A1)-(A2) are satisfied. Let $x(t)$ be the unique mild solution of (2.1.2) obtained in Theorem 2.3.1. Assume that $x_0 \in D(\mathcal{A})$, $I_i(x(t_i)) \in D(\mathcal{A})$, $i = 1, 2, \ldots, N$, and that $f_i \in C^1((0, T_0) \times X \times X \times X, X)$ for $i = 1, 2, \cdots, N+1$. Then $x(t)$ gives rise to a unique classical solution of (2.1.2).*

Proof. Applying theorem 2.3.2 by setting $q_i = \mathcal{I}_i x(t_i)$, get classical solution $x(t)$

which satisfies

$$x(t) = \mathcal{G}(t)x_0 + \sum_{t_i<t}\int_{t_i}^{t} \mathcal{G}(t-s)\mathcal{F}_i(s, x(s), T_i x(s), S_i x(s))\mathrm{d}s + \sum_{0<t_i<t} \mathcal{G}(t-t_i)\mathcal{I}_i x(t_i).$$

Let $y(t)$ be the mild solution of equation (2.1.2). Therefore $u(t)$ satisfies

$$y(t) = \mathcal{G}(t)x_0 + \sum_{t_i<t}\int_{t_i}^{t} \mathcal{G}(t-s)\mathcal{F}_i(s, y(s), T_i y(s), S_i y(s))\mathrm{d}s + \sum_{0<t_i<t} \mathcal{G}(t-t_i)\mathcal{I}_i y(t_i).$$

Define,

$$z(t) \;=\; x(t) - y(t)$$

Then, $z(t)$ satisfies evolution equation

$$z'(t) = 0 \tag{2.3.10}$$

with initial condition $z(0) = 0$ and without impulses. Since, $\mathcal{F}_i$ continuously differentiable and $x(t), y(t) \in PC([0, T_0], X)$, the solution of (2.3.10) is $z(t) = 0$. Therefore $x(t) = u(t)$. Thus the mild solution of (2.1.2) gives rise to a classical solution and the uniqueness of the mild solution implies uniqueness of classical solution of (2.1.2). □

2.3.1 Example

Consider the system on $[0, T_0 = 1.5]$

$$\begin{aligned} x'(t) &= -3x + \mathcal{F}_i(t, x, T_i x, S_i x), \quad t \in [0, T_0 = 1.5], \quad t \neq t_1 = 1 \\ x(0) &= 1 \\ \Delta x(t_1) &= \mathcal{I}_1(x(t_1)) = \frac{x(t_1)}{5} \end{aligned} \tag{2.3.11}$$

where $\mathcal{F}_1 = 1/15[cosx(t) + \int_0^t (t-s)x(s)\mathrm{d}s]$ and $\mathcal{F}_2 = 1/20[\int_0^{T_0}(t-s)sinx(s)\mathrm{d}s + sin2x(t)]$. Equation (2.3.11) assumption (A1) with $L = 1/15$, $M = 1$, $T_1^* = 1$, $S_1^* = 1.125$, $T_2^* = 1.125$, $S_2^* = 1$ and $H = h_1 = 0.2$. and $M\Big[LT_0\sum_{i=1}^{N}(1 + T_i^* +$

$S_i^*) + H\Big] = 0.7333 < 1$ imply (A2) is also satisfies. Therefore by Theorem 2.3.1, equation (2.3.11) has unique mild solution. Moreover, $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuously differentiable therefore by Theorem 2.3.3, a mild solution of (2.3.11) gives rise to a classical solution.

2.4 Conclusion

The system (2.1.2) is a more general system than the system taken by Anguraj and Arjunan [5] because this system has different perturbing forces after the impulses. This type of system is useful to model the motion of a vehicle in the city or the body dynamics of a person having an infectious disease because in both phenomena the perturbing forces after impulses are different.