Chapter 3

Order of magnitude of double and multiple rational Fourier coefficients

3.1 Order of magnitude of double rational Fourier coefficients

In the previous chapter, the order of magnitude of rational Fourier coefficients were obtained for functions of various generalized bounded variation classes inspired by the analogous work done on the order of magnitude of classical Fourier coefficients. In 2002, Móricz [42] extended the classical result on the order of magnitude of the Fourier coefficients of functions of bounded variation to the functions of two variables, which are of bounded variation in the sense of Hardy. This marked the beginning of investigations into the order of magnitude of double Fourier coefficients for various generalized bounded variation functions of two variables, and subsequent contributions to this direction were made by various authors [22, 56, 16, 69, 71, 72]. It was noted that similar investigations had not been conducted for the order of magnitude of double rational Fourier coefficients, and this observation inspired our work in this area.

If an integrable function f is 2π periodic in both the variables, then double rational Fourier series of f is defined as

$$f(x,y) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{f}(m,n)\phi_m(e^{ix})\phi_n(e^{iy}), \tag{3.1}$$

where $\hat{f}(m,n)$ is the $(m,n)^{th}$ double rational Fourier coefficient of f, given by

$$\hat{f}(m,n) = \frac{1}{4\pi^2} \int \int_{\overline{\mathbb{T}}^2} f(x,y)\overline{\phi_m(e^{ix})\phi_n(e^{iy})}dxdy.$$

If poles of the rational orthogonal system are zero, that is $\alpha_k = 0$ in (1.5), $\forall k \in \mathbb{N}$, then series in (3.1) becomes double Fourier series of f.

Various results related to the order of magnitude of double rational Fourier coefficients for generalized bounded variation of two variable functions in the sense of Vitali and Hardy are obtained in this chapter.

Recall that we have assumed that the parameters α_k, defined in the rational orthogonal system (1.5), satisfies the condition (1.6) and r is as defined in (1.6). Also, the notation $\overline{\mathbb{T}}^N = [0, 2\pi]^N$ and $\mathbb{Z}^{*N} = (\mathbb{Z} \setminus \{0\})^N$ for $N \in \mathbb{N}$ will be used throughout this chapter.

Theorem 3.1.1. *If $f \in L^1(\overline{\mathbb{T}}^2), (m,n) \in \mathbb{Z}^2$, then $\hat{f}(m,n) \to 0$ as $|(m,n)| \to \infty$.*

Proof. Let $m \in \mathbb{Z}^*$ and $n \in \mathbb{Z}$. By following similar steps as in [62, Theorem 2.1] and using the fact that $\theta_{|m|}(x)$ is increasing and differentiable function on $[0, 2\pi]$, there exists $h_1 \in [0, 2\pi]$ such that $\theta_{|m|}(x + h_1) - \theta_{|m|}(x) = \pi$. By mean value theorem and $\frac{|m|(1-r)}{1+r} \leq \theta'_{|m|}(x) \leq \frac{|m|(1+r)}{1-r}$, we get $|h_1| \leq \dfrac{(1+r)\pi}{(1-r)|m|}$ and $|\rho_{|m|}(x + h_1) - \rho_{|m|}(x)| \leq \dfrac{r(1+r)^2\pi}{|m|(1-r)^4}$.

From the definition of the double rational Fourier coefficient, we get

$$\begin{aligned}
\hat{f}(m,n) &= \frac{1}{4\pi^2} \int \int_{\overline{\mathbb{T}}^2} f(x,y)\overline{\phi_m(e^{ix})\phi_n(e^{iy})}dxdy \\
&= \frac{1}{4\pi^2} \int \int_{\overline{\mathbb{T}}^2} f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)e^{-i\ sgn(m)\ \theta_{|m|}(x)}e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy \\
&= -\frac{1}{4\pi^2} \int \int_{\overline{\mathbb{T}}^2} f(x+h_1,y)\rho_{|m|}(x+h_1)\rho_{|n|}(y)e^{-i\ sgn(m)\ \theta_{|m|}(x)} \\
&\quad e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy
\end{aligned}$$

$$= \frac{1}{8\pi^2}\int\int_{\overline{\mathbb{T}}^2}[\,-f(x+h_1,y)\rho_{|m|}(x+h_1)\rho_{|n|}(y)$$
$$+ f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)\,]\,e^{-i\ sgn(m)\ \theta_{|m|}(x)}e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy.$$

Hence,

$$|\hat{f}(m,n)| \leq \frac{1}{8\pi^2}\frac{1+r}{1-r}\int\int_{\overline{\mathbb{T}}^2}|f(x,y)-f(x+h_1,y)|dxdy$$
$$+\frac{1}{8\pi^2}\sqrt{\frac{1+r}{1-r}}\int\int_{\overline{\mathbb{T}}^2}|f(x+h_1,y)||\rho_{|m|}(x+h_1)-\rho_{|m|}(x)|dxdy$$
$$\leq \frac{1}{8\pi^2}\frac{1+r}{1-r}\int\int_{\overline{\mathbb{T}}^2}|f(x,y)-f(x+h_1,y)|dxdy+\frac{||f||_1}{8\pi|m|}\frac{r(1+r)^2}{(1-r)^4}.$$

Clearly $h_1 \to 0$ as $m \to \infty$ implies $\hat{f}(m,n) \to 0$ as $m \to \infty$. Also, $\hat{f}(m,n) \to 0$ as both $m,n \to \infty$. Similarly, it can be shown $\hat{f}(m,n) \to 0$ as $n \to \infty$.
Thus, $\hat{f}(m,n) \to 0$ as $|(m,n)| \to \infty$. □

Remark 4. The above result is an extension of Theorem 1.9 (p. 13) for two-variable functions. If we take $\alpha_k = 0;\ \forall k \in \mathbb{Z}$ in the above result, we get the Riemann Lebesgue lemma for the double Fourier series.

Theorem 3.1.2. *If* $f \in Lip(p;\zeta,\beta)(\overline{\mathbb{T}}^2), p \geq 1, \zeta,\beta \in (0,1]$ *and* $(m,n) \in \mathbb{Z}^{*2}$ *then*

$$\hat{f}(m,n) = O\left(\frac{1}{|m|^{\zeta}|n|^{\beta}}+\frac{1}{|m|}+\frac{1}{|n|}\right).$$

Proof. There exists $h_1 \in [0,2\pi]$ such that $\theta_{|m|}(x+h_1)-\theta_{|m|}(x) = \pi$, similarly there exists $h_2 \in [0,2\pi]$ such that $\theta_{|n|}(y+h_2)-\theta_{|n|}(y) = \pi$ and we also have $|h_1| \leq \frac{(1+r)\pi}{(1-r)|m|}$ and $|h_2| \leq \frac{(1+r)\pi}{(1-r)|n|}$ which implies $|\rho_{|m|}(x+h_1)-\rho_{|m|}(x)| \leq \frac{r(1+r)^2\pi}{|m|(1-r)^4}$ and $|\rho_{|n|}(y+h_2)-\rho_{|n|}(y)| \leq \dfrac{r(1+r)^2\pi}{|n|(1-r)^4}$. Then for $m,n \in \mathbb{Z}^*$,we have

$$\hat{f}(m,n) = \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}f(x,y)\overline{\phi_m(e^{ix})\phi_n(e^{iy})}dxdy$$
$$= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)e^{-i\ sgn(m)\ \theta_{|m|}(x)}e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy$$
$$= \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2}\big[f(x+h_1,y+h_2)\rho_{|m|}(x+h_1)\rho_{|n|}(y+h_2)$$
$$-f(x+h_1,y)\rho_{|m|}(x+h_1)\rho_{|n|}(y)-f(x,y+h_2)\rho_{|m|}(x)\rho_{|n|}(y+h_2)$$
$$+f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)\,]\,e^{-i\ sgn(m)\ \theta_{|m|}(x)}e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy.$$

Therefore,

$$
\begin{aligned}
|\hat{f}(m,n)| &= \frac{1}{16\pi^2}\int\int_{\mathbb{T}^2} |f(x+h_1,y+h_2)\rho_{|m|}(x+h_1)\rho_{|n|}(y+h_2) \\
&\quad - f(x+h_1,y)\rho_{|m|}(x+h_1)\rho_{|n|}(y) - f(x,y+h_2)\rho_{|m|}(x)\rho_{|n|}(y+h_2) \\
&\quad + f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)|dxdy \\
&\leq \frac{1}{16\pi^2}\Bigg[\int_{\mathbb{T}}\int_{\mathbb{T}} |\Delta f(x,y;h_1,h_2)||\rho_{|m|}(x+h_1)||\rho_{|n|}(y+h_2)|dxdy \\
&\quad + \int\int_{\mathbb{T}^2} |f(x+h_1,y)||\rho_{|m|}(x+h_1)||\rho_{|n|}(y+h_2)-\rho_{|n|}(y)|dxdy \\
&\quad + \int\int_{\mathbb{T}^2} |f(x,y+h_2)||\rho_{|m|}(x+h_1)-\rho_{|m|}(x)||\rho_{|n|}(y+h_2)|dxdy \\
&\quad + \int\int_{\mathbb{T}^2} |f(x,y)|\{|\rho_{|m|}(x+h_1)||\rho_{|n|}(y+h_2)-\rho_{|n|}(y)| \\
&\qquad\qquad + |\rho_{|n|}(y)||\rho_{|m|}(x+h_1)-\rho_{|m|}(x)|\}dxdy\Bigg] \\
&\leq \frac{1}{16\pi^2}\Bigg[\frac{1+r}{1-r}\int\int_{\mathbb{T}^2}|\Delta f(x,y;h_1,h_2)|dxdy \\
&\quad + \frac{r(1+r)^2\pi}{|n|(1-r)^4}\int\int_{\mathbb{T}^2}|f(x+h_1,y)|dxdy \\
&\quad + \frac{r(1+r)^2\pi}{|m|(1-r)^4}\int\int_{\mathbb{T}^2}|f(x,y+h_2)|dxdy \\
&\quad + \int\int_{\mathbb{T}^2}|f(x,y)|dxdy\left\{\frac{r(1+r)^2\pi}{|n|(1-r)^4}+\frac{r(1+r)^2\pi}{|m|(1-r)^4}\right\}\Bigg].
\end{aligned}
$$

Using Jensen's inequality, we get

$$
\begin{aligned}
|\hat{f}(m,n)|^p &\leq \frac{1}{3(4\pi)^{2p}}\Bigg[\left(\frac{3+3r}{1-r}\right)^p\int\int_{\mathbb{T}^2}|\Delta f(x,y;h_1,h_2)|^p dxdy \\
&\quad + \left\{\frac{(1+r)^{2p}(6r\pi)^p}{|n|^p(1-r)^{4p}}+\frac{(1+r)^{2p}(6r\pi)^p}{|m|^p(1-r)^{4p}}\right\}||f||_p^p\Bigg] \\
&\leq \frac{1}{3(4\pi)^{2p}}\Bigg[\frac{12^p\pi^{2p}(1+r)^p}{(1-r)^p}\left\{\omega^{(p)}\left(f;\frac{(1+r)\pi}{(1-r)|m|},\frac{(1+r)\pi}{(1-r)|n|}\right)\right\}^p \\
&\quad + ||f||_p^p\left\{\frac{(3+3r)^{2p}(6r\pi)^p}{(1-r)^{4p}}\left(\frac{1}{|m|^p}+\frac{1}{|n|^p}\right)\right\}\Bigg] \\
&\leq \frac{1}{3(4\pi)^{2p}}\Bigg[C\left(\frac{12\pi^2(1+r)}{1-r}\right)^p\left(\frac{(1+r)\pi}{(1-r)|m|}\right)^{\zeta p}\left(\frac{(1+r)\pi}{(1-r)|n|}\right)^{\beta p} \\
&\quad + \frac{(3+3r)^{2p}(6r\pi)^p||f||_p^p}{(1-r)^{4p}}\left(\frac{1}{|m|^p}+\frac{1}{|n|^p}\right)\Bigg]
\end{aligned}
$$

where C is some positive constant. Thus,

$$|\hat{f}(m,n)|^p = O\left(\frac{1}{|m|^{\zeta p}|n|^{\beta p}} + \frac{1}{|m|^p} + \frac{1}{|n|^p}\right).$$

Hence, the result follows. □

Remark 5. The above result is extension of Theorem 2.0.1 (p. 38) for two variable function and if we put $\alpha_k = 0$; $\forall k \in \mathbb{Z}$ in the above result, we get analogous result for double Fourier coefficients, which is similar to Theorem H (p. 15).

Theorem 3.1.3. *If* $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ *and* $(m,n) \in \mathbb{Z}^{*2}$ *then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{|n|+|m|}{\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}}\right)\right).$$

Proof. Here, $\theta_{|m|}(x)$ is strictly increasing and differentiable function on $[0,2\pi]$ and $\theta_{|m|}(x+2\pi) = \theta_{|m|}(x) + 2|m|\pi$. Therefore, for integer $j \in [0,2|m|]$, there exists an increasing sequence $x_j \in [0,2\pi]$ such that $\theta_{|m|}(x+x_j) - \theta_{|m|}(x) = j\pi$. Similarly, for integer $k \in [0,2|n|]$, there exists an increasing sequence $y_k \in [0,2\pi]$ such that $\theta_{|n|}(y+y_k) - \theta_{|n|}(y) = k\pi$. For $(m,n) \in \mathbb{Z}^{*2}$, we have

$$\begin{aligned}\hat{f}(m,n) =& \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\overline{\phi_m(e^{ix})\phi_n(e^{iy})}dxdy\\ =& \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\rho_{|m|}(x)\rho_{|n|}(y)e^{-i\ sgn(m)\ \theta_{|m|}(x)}e^{-i\ sgn(n)\ \theta_{|n|}(y)}dxdy.\end{aligned}$$

Therefore,

$$\begin{aligned}|\hat{f}(m,n)| \leq& \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2} |f(x+x_j,y+y_k)\rho_{|m|}(x+x_j)\rho_{|n|}(y+y_k)\\ &- f(x+x_j,y+y_{k-1})\rho_{|m|}(x+x_j)\rho_{|n|}(y+y_{k-1})\\ &- f(x+x_{j-1},y+y_k)\rho_{|m|}(x+x_{j-1})\rho_{|n|}(y+y_k)\\ &+ f(x+x_{j-1},y+y_{k-1})\rho_{|m|}(x+x_{j-1})\rho_{|n|}(y+y_{k-1})|dxdy\\ \leq \frac{1}{16\pi^2}&\int\int_{\overline{\mathbb{T}}^2}[|\Delta f_{jk}(x,y)|\ |\rho_{|m|}(x+x_j)||\rho_{|n|}(y+y_k)|\\ &+|f(x+x_j,y+y_{k-1})||\rho_{|m|}(x+x_j)||\rho_{|n|}(y+y_k)-\rho_{|n|}(y+y_{k-1})|\\ &+|f(x+x_{j-1},y+y_k|)|\rho_{|m|}(x+x_j)-\rho_{|m|}(x+x_{j-1})||\rho_{|n|}(y+y_k)|\end{aligned}$$

$$+|f(x+x_{j-1},y+y_{k-1})|\left\{|\rho_{|m|}(x+x_j)||\rho_{|n|}(y+y_k)-\rho_{|n|}(y+y_{k-1})|\right.$$
$$\left.+|\rho_{|m|}(x+x_j)-\rho_{|m|}(x+x_{j-1})||\rho_{|n|}(y+y_{k-1})\ |\right\}\big]\,dxdy,$$

where

$$\Delta f_{jk}=f(x+x_j,y+y_k)-f(x+x_j,y+y_{k-1})-f(x+x_{j-1},y+y_k)+f(x+x_{j-1},y+y_{k-1}).$$

Also, by the mean value theorem, we have

$$|\rho_{|m|}(x+x_j)-\rho_{|m|}(x+x_{j-1})|\leq\frac{r(1+r)^{1/2}}{(1-r)^{5/2}}|x_j-x_{j-1}|\leq\frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2}|m|},$$

$$|\rho_{|n|}(y+y_k)-\rho_{|n|}(y+y_{k-1})|\leq\frac{r(1+r)^{1/2}}{(1-r)^{5/2}}|y_k-y_{k-1}|\leq\frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2}|n|}.$$

Hence,

$$|\hat{f}(m,n)|\leq c_1\int\int_{\mathbb{T}^2}|\Delta f_{ij}(x,y)|dxdy+c_2\left(\frac{1}{|m|}+\frac{1}{|n|}\right),$$

where $c_1=\dfrac{1+r}{16\pi^2(1-r)}, c_2=\dfrac{r(1+r)^2||f||_1}{8\pi(1-r)^4}$.

For $c>0$, using Jensen's inequality, we have

$$\Phi(c|\hat{f}(m,n)|)\leq\frac{1}{3}\int\int_{\mathbb{T}^2}\Phi(3cc_1|\Delta f_{ij}(x,y)|)dxdy$$
$$+\frac{1}{3|m|}\Phi(3cc_2)+\frac{1}{3|n|}\Phi(3cc_2).$$

Multiplying the above inequality on both sides with $\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}$ and taking summation over $j=1$ to $2|m|$ and $k=1$ to $2|n|$, we have

$$\Phi(c|\hat{f}(m,n)|)\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}$$

$$\leq\frac{1}{3}\int\int_{\mathbb{T}^2}\sum_{k=1}^{2|n|}\sum_{j=1}^{2|m|}\frac{\Phi(3cc_1|\Delta f_{ij}(x,y)|)}{\lambda_{(1,j)}\lambda_{(2,k)}}dxdy+\frac{4|n|\Phi(3cc_2)}{3\lambda_{(1,1)}\lambda_{(2,1)}}+\frac{4|m|\Phi(3cc_2)}{3\lambda_{(1,1)}\lambda_{(2,1)}}$$
$$\leq\int\int_{\mathbb{T}^2}\sum_{k=1}^{2|n|}\sum_{j=1}^{2|m|}\frac{\Phi(3cc_1|\Delta f_{ij}(x,y)|)}{\lambda_{(1,j)}\lambda_{(2,k)}}dxdy+(|n|+|m|)\left(\frac{2\Phi(3cc_2)}{\lambda_{(1,1)}\lambda_{(2,1)}}\right).$$

For sufficiently small c, we get

$$\int\int_{\overline{\mathbb{T}}^2} \sum_{k=1}^{2|n|} \sum_{j=1}^{2|m|} \frac{\Phi(3cc_1|\Delta f_{ij}(x,y)|)}{\lambda_{(1,j)}\lambda_{(2,k)}} dxdy \leq \frac{3cc_1 V_{\Lambda_\Phi}(f, \overline{\mathbb{T}}^2)}{4\pi^2} \leq \frac{1}{2} \leq \frac{|m|+|n|}{2}$$

and $\dfrac{2\Phi(3cc_2)}{\lambda_{(1,1)}\lambda_{(2,1)}} \leq \dfrac{1}{2}$. Therefore,

$$\Phi(c|\hat{f}(m,n)|) \sum_{j=1}^{2|m|} \sum_{k=1}^{2|n|} \frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}} \leq |m| + |n|.$$

Hence, the result is proved. □

Corollary 3.1.4. *If $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$ and $(m,n) \in \mathbb{Z}^{*2}$ then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{|n|+|m|}{\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_{(1,j)}\lambda_{(2,k)}}}\right)\right).$$

Proof. Clearly f is bounded in $\overline{\mathbb{T}}^2$ and $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$, thus it implies that $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^2)$. Thus, the result can be proved by the previous Theorem 3.1.3. □

Remark 6. The previous results, Theorem 3.1.3 and Corollary 3.1.4, are extensions of Theorem G (p. 14) for two-variable functions. If we take $\alpha_k = 0$; $\forall k \in \mathbb{Z}$, then $r = 0$ and thus, $c_2 = 0$ in the proof of Theorem 3.1.3 gives us analogous results for double Fourier coefficients, similar to Theorem K (p. 20) and Corollary D (p. 20).

Theorem 3.1.5. *If Φ satisfies Δ_2 condition and $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^2)$, then for $m \in \mathbb{Z}^*$*

$$\hat{f}(m,0) = O\left(\Phi^{-1}\left(\frac{1}{\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}}}\right)\right).$$

Proof. By following similar steps as in Theorem 3.1.3, we have, for integer $j \in [0, 2|m|]$, there exists an increasing sequence $x_j \in [0, 2\pi]$ such that $\theta_{|m|}(x + x_j) - \theta_{|m|}(x) = j\pi$. By mean value theorem,

$$|\rho_{|m|}(x+x_j) - \rho_{|m|}(x+x_{j-1})| \leq \frac{r(1+r)^{1/2}}{(1-r)^{5/2}}|x_j - x_{j-1}| \leq \frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2}|m|}.$$

Hence, for $m \in \mathbb{Z}^*$, we have

$$\hat{f}(m,0) = \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\overline{\phi_m(e^{ix})}dxdy$$
$$= \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} f(x,y)\rho_{|m|}(x)e^{-i\ sgn(m)\ \theta_{|m|}(x)}dxdy.$$

Therefore,

$$|\hat{f}(m,0)| \leq \frac{1}{8\pi^2}\int\int_{\overline{\mathbb{T}}^2} c_1|\Delta f_j(x,y)|dxdy + \frac{c_2}{8\pi|m|},$$

where $c_1 = \left(\frac{1+r}{1-r}\right)^{1/2}$, $c_2 = \frac{r(1+r)^{\frac{3}{2}}||f||_1}{(1-r)^{\frac{7}{2}}}$ and $\Delta f_j(x,y) = f(x+x_j,y) - f(x+x_{j-1},y)$. For $c > 0$, using Jensen's inequality, we have

$$\Phi\left(c|\hat{f}(m,0)|\right) \leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2} \Phi(cc_1|\Delta f_j(x,y)|)dxdy + \frac{1}{2|m|}\Phi(cc_3).$$

Multiplying the above inequality on both sides with $\frac{1}{\lambda_{(1,j)}}$ and taking summation over $j = 1$ to $2|m|$, we have

$$\Phi(c|\hat{f}(m,0)|)\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}} \leq \frac{1}{4\pi^2}\int\int_{\overline{\mathbb{T}}^2}\sum_{j=1}^{2|m|}\frac{\Phi(cc_1|\Delta f_j(x,y)|)}{\lambda_{(1,j)}}dxdy + \frac{\Phi(cc_2)}{\lambda_{(1,1)}}.$$

Therefore,

$$\Phi(c|\hat{f}(m,0)|)\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}} \leq V_{\Lambda_{1_\Phi}}(cc_1 f(.,y),\overline{\mathbb{T}}) + \frac{\Phi(cc_2)}{\lambda_{(1,1)}}.$$

Here,

$$V_{\Lambda_{1_\Phi}}(f(.,y),\overline{\mathbb{T}}) = \sup_J \sum_j \frac{\Phi(|f(x_{j+1},y) - f(x_j,y))|)}{\lambda_{(1,j)}},$$

where J is any finite collection of non-overlapping subintervals $\{[x_j,\ x_{j+1}]\} \in \overline{\mathbb{T}}$. Thus, by using the result (proved in [72, Corollary 2]),

$$V_{\Lambda_{1_\Phi}}(f(.,y),\overline{\mathbb{T}}) \leq d\left(\lambda_{(2,1)}V_{\Lambda_\Phi}(f,\overline{\mathbb{T}}^2) + V_{\Lambda_{1_\Phi}}(f(.,0),\overline{\mathbb{T}})\right),$$

for all $y \in \overline{\mathbb{T}}$ and d is some constant such that $d \geq 2$, we get

$$\Phi(c|\hat{f}(m,0)|)\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}} \leq d\left(\lambda_{(2,1)}V_{\Lambda_\Phi}(cc_1 f,\overline{\mathbb{T}}^2) + V_{\Lambda_{1_\Phi}}(cc_1 f(.,0),\overline{\mathbb{T}})\right) + \frac{\Phi(cc_2)}{\lambda_{(1,1)}}.$$

For sufficiently small $c \in (0,1]$, we get

$V_{\Lambda_\Phi}(cc_1 f,\overline{\mathbb{T}}^2) \leq \frac{1}{3d\lambda_{(2,1)}}, V_{\Lambda_{1_\Phi}}(cc_1 f(.,0),\overline{\mathbb{T}}) \leq \frac{1}{3d}$ and $\frac{\Phi(cc_2)}{\lambda_{(1,1)}} \leq \frac{1}{3}$.

Thus,

$$\Phi(c|\hat{f}(m,0)|)\sum_{j=1}^{2|m|}\frac{1}{\lambda_{(1,j)}} \leq 1.$$

Hence, the result follows. □

Theorem 3.1.6. *If $f \in (\Phi,\Psi)(\Lambda,\Gamma)BV(\overline{\mathbb{T}}^2) \cap L^1(\overline{\mathbb{T}}^2)$ and $(m,n) \in \mathbb{Z}^{*2}$, then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{2|m|}}\Psi^{-1}\left(\frac{|n| + \Psi(|m|)}{\Gamma_{2|n|}}\right)\right)\right),$$

where $\Lambda_{2|m|} = \sum_{j=1}^{2|m|}\lambda_j^{-1}$ and $\Gamma_{2|n|} = \sum_{k=1}^{2|n|}\gamma_k^{-1}$.

Proof. For integers $j \in [0,2|m|], k \in [0,2|n|]$, there exists increasing sequences $x_j \in [0,2\pi]$ and $y_k \in [0,2\pi]$ such that $\theta_{|m|}(x+x_j) - \theta_{|m|}(x) = j\pi$ and $\theta_{|n|}(y+y_k) - \theta_{|n|}(y) = k\pi$.
Therefore,

$$\begin{aligned}
&|\hat{f}(m,n)| \\
&\leq \frac{1}{16\pi^2}\int\int_{\overline{\mathbb{T}}^2}\big[|\Delta f_{jk}(x,y)|\ |\rho_{|m|}(x+x_j)|\ |\rho_{|n|}(y+y_k)| \\
&+ \int\int_{\overline{\mathbb{T}}^2}|f(x+x_j,y+y_{k-1})|\ |\rho_{|m|}(x+x_j)|\ |\rho_{|n|}(y+y_k) - \rho_{|n|}(y+y_{k-1})| \\
&+ \int\int_{\overline{\mathbb{T}}^2}|f(x+x_{j-1},y+y_k|)|\ |\rho_{|m|}(x+x_j) - \rho_{|m|}(x+x_{j-1})|\ |\rho_{|n|}(y+y_k)| \\
&+ \int\int_{\overline{\mathbb{T}}^2}|f(x+x_{j-1},y+y_{k-1})|\big\{|\rho_{|m|}(x+x_j)|\ |\rho_{|n|}(y+y_k) \\
&\quad - \rho_{|n|}(y+y_{k-1})| + |\rho_{|m|}(x+x_j) - \rho_{|m|}(x+x_{j-1})||\rho_{|n|}(y+y_{k-1})\ |\big\}\big]\,dxdy,
\end{aligned}$$

where $\Delta f_{jk} = f(x+x_j,y+y_k) - f(x+x_j,y+y_{k-1}) - f(x+x_{j-1},y+y_k) + f(x+$

$x_{j-1}, y + y_{k-1})$. Hence,

$$|\hat{f}(m,n)| \le c_1 \int\int_{\mathbb{T}^2} |\Delta f_{jk}(x,y)| dxdy + c_2 \left(\frac{1}{|m|} + \frac{1}{|n|} \right),$$

where $c_1 = \dfrac{1+r}{4(1-r)}$, $c_2 = \dfrac{(1+r)^2 ||f||_1}{8\pi(1-r)^4}$.

For $c > 0$, using Jensen's inequality, we have

$$\Phi(c|\hat{f}(m,n)|) \le \frac{1}{12\pi^2} \int\int_{\mathbb{T}^2} \Phi(3cc_1|\Delta f_{jk}(x,y)|) dxdy + \frac{1}{3|m|}\Phi(3cc_2) + \frac{1}{3|n|}\Phi(3cc_2).$$

Multiplying both sides of the above inequality with λ_j^{-1}, summing $j = 1$ to $2|m|$ and letting $\Lambda_{2|m|} = \sum_{j=1}^{2|m|} \lambda_j^{-1}$, we get

$$\Phi(c|\hat{f}(m,n)|)\Lambda_{2|m|} \le \int\int_{\mathbb{T}^2} \sum_{j=1}^{2|m|} \frac{\Phi(3cc_1|\Delta f_{jk}(x,y)|)}{12\pi^2 \lambda_j} dxdy + \frac{2\Phi(3cc_2)}{3\lambda_1} + \frac{2|m|\Phi(3cc_2)}{3|n|\lambda_1}.$$

Again, by using Jensen's inequality, we have

$$\Psi(\Phi(c|\hat{f}(m,n)|)\Lambda_{2|m|}) \le \frac{1}{12\pi^2} \int\int_{\mathbb{T}^2} \Psi\left(\sum_{j=1}^{2|m|} \frac{\Phi(3cc_1|\Delta f_{jk}(x,y)|)}{\lambda_j} \right) dxdy + \frac{\Psi(2\Phi(3cc_2\lambda_1^{-1}))}{3} + \frac{\Psi(2|m|\Phi(3cc_2\lambda_1^{-1}))}{3|n|}.$$

Multiplying both sides of the above inequality with γ_k^{-1}, summing $k = 1$ to $2|n|$ and letting $\Gamma_{2|n|} = \sum_{k=1}^{2|n|} \gamma_k^{-1}$, we have

$$\begin{aligned} &\Psi(\Phi(c|\hat{f}(m,n)|)\Lambda_{2|m|})\Gamma_{2|n|} \\ &\quad \le \frac{1}{12\pi^2} \int\int_{\mathbb{T}^2} \left[\sum_{k=1}^{2|n|} \frac{1}{\gamma_k} \left\{ \Psi\left(\sum_{j=1}^{2|m|} \frac{\Phi(3cc_1|\Delta f_{jk}(x,y)|)}{\lambda_j} \right) \right\} \right] dxdy \\ &\quad + \frac{2|n|\Psi(2\Phi(3cc_2\lambda_1^{-1}))}{3\gamma_1} + \frac{2\Psi(2|m|\Phi(3cc_2\lambda_1^{-1}))}{3\gamma_1} \end{aligned}$$

$$:= C_1 + C_2 + C_3$$

For sufficiently small c, we get

$$C_1 \leq 4\pi^2 cc_1 V_{(\Lambda,\Gamma)_{(\Psi,\Phi)}}(f, \overline{\mathbb{T}}^2) \leq \frac{1}{2} \leq \frac{|n|}{2},$$

$$C_2 \leq \frac{|n|}{2}$$

and

$$C_3 \leq \Psi(|m|).$$

Therefore,

$$\Psi(\Phi(c|\hat{f}(m,n)|)\Lambda_{2|m|})\Gamma_{2|n|} \leq |n| + \Psi(|m|).$$

Hence, the result is proved. □

Corollary 3.1.7. *If $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$ and $(m, n) \in \mathbb{Z}^{*2}$, then*

$$\hat{f}(m,n) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{2|m|}}\Psi^{-1}\left(\frac{|n| + \Psi(|m|)}{\Gamma_{2|n|}}\right)\right)\right),$$

where $\Lambda_{2|m|} = \sum_{j=1}^{2|m|} \lambda_j^{-1}$ and $\Gamma_{2|n|} = \sum_{k=1}^{2|n|} \gamma_k^{-1}$.

Proof. The result follows from Theorem 3.1.6. □

Remark 7. Note that Theorem 3.1.6 and Corollary 3.1.7 are extensions of Theorem G (p. 14) for two-variable functions. If we take $\alpha_k = 0$; $\forall k \in \mathbb{Z}$ in Theorem 3.1.6 and Corollary 3.1.7 then we obtain analogous results for double Fourier coefficients, similar to Theorem L (p. 22) and Corollary F (p. 22).

Lemma 3.1.8. *[16, Lemma 2.4] If Φ and Ψ satisfy Δ_2 condition and $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$, then*

$$||V_{\Lambda_\Phi}(f(.,y), \overline{\mathbb{T}})||_\infty \leq d\left(\psi^{-1}\left(\gamma_1 V_{(\Lambda,\Gamma)_{(\Psi,\Phi)}}(f, \overline{\mathbb{T}}^2)\right)\right) + V_{\Lambda_\Phi}\left(f(.,0), \overline{T}\right),$$

where

$$||V_{\Lambda_\Phi}(f(.,y), \overline{\mathbb{T}})||_\infty = \sup_{y \in \overline{\mathbb{T}}} |V_{\Lambda_\Phi}(f(.,y), \overline{\mathbb{T}})|.$$

Theorem 3.1.9. *If Φ and Ψ satisfy Δ_2 condition and $f \in (\Phi, \Psi)(\Lambda, \Gamma)^* BV(\overline{\mathbb{T}}^2)$,* *then for* $m \in \mathbb{Z}^*$

$$\hat{f}(m,0) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{2|m|}}\right)\right),$$

where $\Lambda_{2|m|} = \sum_{j=1}^{2|m|} \lambda_j^{-1}$.

Proof. For integer $j \in [0, 2|m|]$, there exists an increasing sequence $x_j \in [0, 2\pi]$ such that $\theta_{|m|}(x + x_j) - \theta_{|m|}(x) = j\pi$. Therefore,

$$|\hat{f}(m,0)| \leq \frac{1}{8\pi^2} \int\int_{\overline{\mathbb{T}}^2} c_1 |\Delta f_j(x,y)| dxdy + \frac{c_2}{8\pi|m|},$$

where $c_1 = \left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}$, $c_2 = \frac{r(1+r)^{\frac{3}{2}}||f||_1}{(1-r)^{\frac{7}{2}}}$ and $\Delta f_j(x,y) = f(x+x_j, y) - f(x+x_{j-1}, y)$. For $c > 0$, using Jensen's inequality, we have

$$\Phi\left(c|\hat{f}(m,0)|\right) \leq \frac{1}{8\pi^2} \int\int_{\overline{\mathbb{T}}^2} \Phi(cc_1|\Delta f_j(x,y)|) dxdy + \frac{1}{2|m|}\Phi(cc_3).$$

Multiplying both sides with λ_j^{-1}, summing $j = 1$ to $2|m|$ and letting $\Lambda_{2|m|} = \sum_{j=1}^{2|m|} \lambda_j^{-1}$, we have

$$\Phi(c|\hat{f}(m,0)|)\Lambda_{2|m|} \leq \frac{1}{8\pi^2} \int\int_{\overline{\mathbb{T}}^2} \sum_{j=1}^{2|m|} \frac{\Phi(cc_1|\Delta f_j(x,y)|)}{\lambda_j} dxdy + \frac{\Phi(cc_2)}{\lambda_1}.$$

Let

$$V_{\Lambda_\Phi}(f(.,y), \overline{\mathbb{T}}) = \sup_J \sum_j \frac{\Phi(|f(x_{j+1}, y) - f(x_j, y)|)}{\lambda_j},$$

where J is any finite collection of non-overlapping subintervals $\{[x_j,\ x_{j+1}]\} \in \overline{\mathbb{T}}$. Therefore,

$$\Phi(c|\hat{f}(m,0)|)\Lambda_{2|m|} \leq V_{\Lambda_\Phi}(cc_1 f(.,y), \overline{\mathbb{T}}) + \frac{\Phi(cc_2)}{\lambda_1}.$$

Thus, by Lemma 3.1.8, we get

$$\begin{aligned}\Phi(c|\hat{f}(m,0)|)\Lambda_{2|m|} \\ \leq d\left(V_{\Lambda_\Phi}(cc_1 f(.,0), \overline{\mathbb{T}}) + \Psi^{-1}(\gamma_1 V_{(\Lambda,\Gamma)_{(\Psi,\Phi)}}(cc_1 f, \overline{\mathbb{T}}^2))\right) + \frac{\Phi(cc_2)}{\lambda_1}.\end{aligned}$$

For sufficiently small $c \in (0,1]$, we get

$V_{(\Lambda,\Gamma)_{(\Psi,\Phi)}}(cc_1 f,\overline{\mathbb{T}}^2) \leq \frac{1}{\gamma_1}\Psi(\frac{1}{3d}), V_{\Lambda_\Phi}(cc_1 f(.,0),\overline{\mathbb{T}}) \leq \frac{1}{3d}$ and $\frac{\Phi(cc_2)}{\lambda_1} \leq \frac{1}{3}$.

Thus,

$$\hat{f}(m,0) = O\left(\Phi^{-1}\left(\frac{1}{\Lambda_{2|m|}}\right)\right).$$

□

Corollary 3.1.10. *If Φ and Ψ satisfy Δ_2 condition, $f \in (\Phi,\Psi)(\Lambda,\Gamma)^*BV(\overline{\mathbb{T}}^2)$, then for $n \in \mathbb{Z}^*$*

$$\hat{f}(0,n) = O\left(\Psi^{-1}\left(\frac{1}{\Gamma_{2|n|}}\right)\right),$$

where $\Gamma_{2|n|} = \sum_{j=1}^{2|n|}\gamma_j^{-1}$.

Lemma 3.1.11. *$\Lambda^*BV(p(n)\uparrow p,\varphi,\overline{\mathbb{T}}^2)$ and $\Lambda^*BV\left(p(n)\uparrow p,2\varphi,\overline{\mathbb{T}}^2\right)$ coincides for $1 \leq p \leq \infty$.*

Proof. We follow a similar method as in [4, Lemma 3].
It is obvious that $V_{\Lambda_{p(n)}}(f,\varphi(n),\overline{\mathbb{T}}^2) \leq V_{\Lambda_{p(n)}}\left(f,2\varphi(n),\overline{\mathbb{T}}^2\right)$.

Consider $f \in \Lambda BV\left(p(n)\uparrow p,2\varphi,\overline{\mathbb{T}}^2\right)$. Let us consider the following partition; $|x_j - x_{j-1}|, |y_k - y_{k-1}| \geq \frac{\pi}{\varphi(n)}$, where length of such subintervals does not exceed π.
By Minkowski's inequality, we get

$$\begin{aligned}
&\left\{\sum_j\sum_k \frac{|f(x_j,y_k)-f(x_{j-1},y_k)-f(x_j,y_{k-1})+f(x_{j-1},y_{k-1})|^{p(n)}}{\lambda_j^{(1)}\lambda_k^{(2)}}\right\}^{\frac{1}{p(n)}}\\
&\leq \left[\sum_j\sum_k \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left|f(x_j,y_k) - f\left(x_j+\frac{2\pi}{\varphi(n)},y_k\right)\right.\right.\right.\\
&\qquad \left.\left.\left. -f\left(x_j,y_k+\frac{2\pi}{\varphi(n)}\right)+f\left(x_j+\frac{2\pi}{\varphi(n)},y_k+\frac{2\pi}{\varphi(n)}\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}\\
&+\left[\sum_j\sum_k \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left|f\left(x_j+\frac{2\pi}{\varphi(n)},y_k\right) - f\left(x_{j-1},y_k\right)\right.\right.\right.\\
&\qquad \left.\left.\left. -f\left(x_j+\frac{2\pi}{\varphi(n)},y_k+\frac{2\pi}{\varphi(n)}\right)+f\left(x_{j-1},y_k+\frac{2\pi}{\varphi(n)}\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}
\end{aligned}$$

$$+\left[\sum_j\sum_k \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left|f\left(x_j, y_k+\frac{2\pi}{\varphi(n)}\right)-f\left(x_j+\frac{2\pi}{\varphi(n)}, y_k+\frac{2\pi}{\varphi(n)}\right)\right.\right.\right.$$
$$\left.\left.\left.-f\left(x_j, y_{k-1}\right)+f\left(x_j+\frac{2\pi}{\varphi(n)}, y_k\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}$$

$$+\left[\sum_j\sum_k \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left|f\left(x_j+\frac{2\pi}{\varphi(n)}, y_k+\frac{2\pi}{\varphi(n)}\right)-f\left(x_{j-1}, y_k+\frac{2\pi}{\varphi(n)}\right)\right.\right.\right.$$
$$\left.\left.\left.-f\left(x_j+\frac{2\pi}{\varphi(n)}, y_{k-1}\right)+f\left(x_{j-1}, y_{k-1}\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}$$
$$\equiv A+B+C+D$$

Consider A. All the subintervals $\left(x_j, x_j+\frac{2\pi}{\varphi(n)}\right)$ and $\left(y_k, y_k+\frac{2\pi}{\varphi(n)}\right)$ will be on some closed interval of length 3π, say for simplicity, $[p, p+3\pi]$ and $[q, q+3\pi]$ respectively as $\varphi(n) \geq 2$. Let M_1 and N_1 be the sets of indices of subintervals of $\left(x_j, x_j+\frac{2\pi}{\varphi(n)}\right)$ and $\left(y_k, y_k+\frac{2\pi}{\varphi(n)}\right)$ which are on the intervals $[p, p+2\pi]$ and $[q, q+2\pi]$ respectively; M_2 and N_2 be the sets of indices of subintervals of $\left(x_j, x_j+\frac{2\pi}{\varphi(n)}\right)$ and $\left(y_k, y_k+\frac{2\pi}{\varphi(n)}\right)$ which are subsets of $[p+2\pi, p+3\pi]$ and $[q+2\pi, q+3\pi]$ respectively; and M_3 and N_3 be the sets of indices of subintervals of $\left(x_j, x_j+\frac{2\pi}{\varphi(n)}\right)$ and $\left(y_k, y_k+\frac{2\pi}{\varphi(n)}\right)$ which contains the remaining subintervals, i.e., intervals which contain $p+2\pi$ and $q+2\pi$ points respectively.

Therefore,
$$A \leq \sum_{l=1}^{3}\sum_{t=1}^{3} A_{lt},$$
where
$$A_{lt}=\left[\sum_{j\in M_l}\sum_{k\in N_t} \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left|f(x_j, y_k)-f\left(x_j+\frac{2\pi}{\varphi(n)}, y_k\right)\right.\right.\right.$$
$$\left.\left.\left.-f\left(x_j, y_k+\frac{2\pi}{\varphi(n)}\right)+f\left(x_j+\frac{2\pi}{\varphi(n)}, y_k+\frac{2\pi}{\varphi(n)}\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}.$$

Following similar process as in [4, Lemma 3],

$$A_{11} \leq 4V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ A_{12} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$A_{13} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ A_{21} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$A_{22} \leq V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ A_{23} \leq V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$A_{31} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ A_{32} \leq V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$A_{33} \leq V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

Thus,

$$A \leq 16V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

Consider B. The set of indices of subintervals $\left(x_{j-1}, x_j + \frac{2\pi}{\varphi(n)}\right)$ is partitioned into L_1, which contains all subintervals on $[p, p+2\pi]$, and L_2, which contains subintervals having point $p + 2\pi$. Therefore,

$$B \leq \sum_{l=1}^{2} \sum_{t=1}^{3} B_{lt},$$

where

$$B_{lt} = \left[\sum_{j \in L_l} \sum_{k \in N_t} \frac{1}{\lambda_j^{(1)} \lambda_k^{(2)}} \left\{ \left| f\left(x_j + \frac{2\pi}{\varphi(n)}, y_k\right) - f\left(x_{j-1}, y_k\right) \right.\right.\right.$$
$$\left.\left.\left. - f\left(x_j + \frac{2\pi}{\varphi(n)}, y_k + \frac{2\pi}{\varphi(n)}\right) + f\left(x_{j-1}, y_k + \frac{2\pi}{\varphi(n)}\right) \right|^{p(n)} \right\}\right]^{\frac{1}{p(n)}}.$$

Therefore,

$$B_{11} \leq 6V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ B_{12} \leq 3V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$B_{13} \leq 3V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ B_{21} \leq 4V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$B_{22} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ \ B_{23} \leq 2V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

Thus,

$$B \leq 20V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

Similarly,

$$C \leq 20V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

Consider D. The set of indices of subintervals $\left(y_{k-1}, y_k + \frac{2\pi}{\varphi(n)}\right)$ is partitioned into K_1, which contains all subintervals on $[q, q+2\pi]$, and K_2, which contains subintervals having point $q+2\pi$. Therefore,

$$D \leq \sum_{l=1}^{2}\sum_{t=1}^{2} D_{lt},$$

where

$$D_{lt} = \left[\sum_{j\in L_l}\sum_{k\in K_t} \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\left\{\left| f\left(x_j + \frac{2\pi}{\varphi(n)}, y_k + \frac{2\pi}{\varphi(n)}\right) + f\left(x_{j-1}, y_{k-1}\right)\right.\right.\right.$$
$$\left.\left.\left. - f\left(x_j + \frac{2\pi}{\varphi(n)}, y_{k-1}\right) - f\left(x_{j-1}, y_k + \frac{2\pi}{\varphi(n)}\right)\right|^{p(n)}\right\}\right]^{\frac{1}{p(n)}}.$$

Therefore,

$$D_{11} \leq 9V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ D_{12} \leq 6V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

$$D_{21} \leq 6V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),\ D_{22} \leq 4V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2),$$

Thus,

$$D \leq 25V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

Hence,

$$A + B + C + D \leq 81V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

The length of some subinterval might exceed π (only possible for one subinterval each); thus by proceeding as in [4, Lemma 3], we get,

$$V_{\Lambda_{p(n)}}\left(f, 2\varphi(n), \overline{\mathbb{T}}^2\right) \leq 100V_{\Lambda_{p(n)}}(f, \varphi(n), \overline{\mathbb{T}}^2).$$

This completes the proof. □

Remark 8. In similar way, it can be shown that$\Lambda^*BV(p(n)\uparrow p, \varphi, \overline{\mathbb{T}}^2)$ and $\Lambda^*BV\left(p(n)\uparrow p, c\varphi, \overline{\mathbb{T}}^2\right)$ coincides for $1 \leq p \leq \infty$ and $c > 1$.

Theorem 3.1.12. *If* $f \in \Lambda^*BV(p(n)\uparrow p, \varphi, \overline{\mathbb{T}}^2)$, $1 \leq p \leq \infty$ *and* $(m, n) \in \mathbb{Z}^{*2}$, *then*

$$\hat{f}(m,n) = O\left(\frac{1}{\left(\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}\right)^{\frac{1}{p(\tau(|mn|))}}} + \frac{1}{|m|} + \frac{1}{|n|}\right),$$

where $\tau(|mn|)$ *is as defined in* (2.1).

Proof. For integers $j \in [0, 2|m|], k \in [0, 2|n|]$, there exist increasing sequences $x_j \in [0, 2\pi]$ and $y_k \in [0, 2\pi]$ such that $\theta_{|m|}(x + x_j) - \theta_{|m|}(x) = j\pi$ and $\theta_{|n|}(y + y_k) - \theta_{|n|}(y) = k\pi$. Here,

$$\frac{(1-r)\pi}{(1+r)|m|} \le x_j - x_{j-1} \le \frac{(1+r)\pi}{(1-r)|m|}$$

and

$$\frac{(1-r)\pi}{(1+r)|n|} \le y_k - y_{k-1} \le \frac{(1+r)\pi}{(1-r)|n|}.$$

Thus,

$$|\hat{f}(m,n)| \le c_1 \int\int_{\mathbb{T}^2} |\Delta f_{jk}(x,y)| dxdy + c_2\left(\frac{1}{|m|} + \frac{1}{|n|}\right),$$

where $c_1 = \dfrac{1+r}{16\pi^2(1-r)}$, $c_2 = \dfrac{r(1+r)^2||f||_1}{8\pi(1-r)^4}$ and
$\Delta f_{jk} = f(x+x_j, y+y_k) - f(x+x_j, y+y_{k-1}) - f(x+x_{j-1}, y+y_k) + f(x+x_{j-1}, y+y_{k-1})$.

Dividing both sides of the above inequality by $\lambda_j^{(1)}\lambda_k^{(2)}$ and summing over $j = 1$ to $2|m|$ and $k = 1$ to $2|n|$, we get

$$\left(|\hat{f}(m,n)| - \frac{c_2}{|m|} - \frac{c_2}{|n|}\right)\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}$$
$$\le c_1 \int\int_{\mathbb{T}^2}\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|}\frac{|\Delta f_{jk}(x,y)|}{\lambda_j^{(1)}\lambda_k^{(2)}} dxdy.$$

By applying Holder's inequality on the right side of the above inequality and letting $\frac{1}{p(\tau(|mn|))} + \frac{1}{q(\tau(|mn|))} = 1$, we get

$$\left(|\hat{f}(m,n)| - \frac{c_2}{|m|} - \frac{c_2}{|n|}\right) \sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|} \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}}$$

$$\leq c_1 \int\int_{\overline{\mathbb{T}}^2} \left\{ \left(\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|} \frac{|\Delta f_{jk}(x,y)|^{p(\tau(|mn|))}}{\lambda_j^{(1)}\lambda_k^{(2)}} \right)^{\frac{1}{p(\tau(|mn|))}} \left(\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|} \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}} \right)^{\frac{1}{q(\tau(|mn|))}} \right\} dxdy$$

$$\leq 4\pi^2 c_1 \left(\sum_{j=1}^{2|m|}\sum_{k=1}^{2|n|} \frac{1}{\lambda_j^{(1)}\lambda_k^{(2)}} \right)^{\frac{1}{q(\tau(|mn|))}} V_{\Lambda_{p(n)}}\left(f, \frac{2(1+r)}{1-r}\varphi, \overline{\mathbb{T}}^2\right).$$

Hence, the result follows from Lemma 3.1.11. ☐

Remark 9. The above result is an extension of Theorem 2.0.3 (p. 39) for two-variable functions and if we put $\alpha_k = 0;\ \forall k \in \mathbb{Z}$ in the above result, we get analogous result for double Fourier coefficients, which is similar to Corollary H (p. 24).

Theorem 3.1.13. *If* $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^2),\ 1 \leq p \leq \infty$ *and* $m \in \mathbb{Z}^*$*, then*

$$\hat{f}(m,0) = O\left(\frac{1}{\left(\sum_{j=1}^{2|m|} \frac{1}{\lambda_j^{(1)}}\right)^{\frac{1}{p(\tau(|m|))}}} + \frac{1}{|m|} \right),$$

where $\tau(|m|)$ *is as defined in* (2.1).

Proof. For integers $j \in [0, 2|m|]$, there exists an increasing sequence $x_j \in [0, 2\pi]$ such that $\theta_{|m|}(x + x_j) - \theta_{|m|}(x) = j\pi$ and hence

$$|\hat{f}(m,0)| \leq c_1 \int\int_{\overline{\mathbb{T}}^2} |\Delta f_j(x,y)| dxdy + \frac{c_2}{|m|},$$

where $c_1 = \dfrac{(1+r)^{1/2}}{8\pi^2(1-r)^{1/2}},\ c_2 = \dfrac{r(1+r)^{3/2}||f||_1}{8\pi(1-r)^{7/2}}$ and $\Delta f_j = f(x + x_j, y) - f(x + x_{j-1}, y)$. Dividing both sides of the above inequality by $\lambda_j^{(1)}$ and summing over

$j = 1$ to $2|m|$, we get

$$\left(|\hat{f}(m,0)| - \frac{c_2}{|m|}\right)\sum_{j=1}^{2|m|}\frac{1}{\lambda_j^{(1)}} \le c_1 \int\int_{\overline{\mathbb{T}}^2}\sum_{j=1}^{2|m|}\frac{|\Delta f_j(x,y)|}{\lambda_j^{(1)}}dxdy.$$

By applying Holder's inequality on the right side of the above inequality and letting $\frac{1}{p(\tau(|m|))} + \frac{1}{q(\tau(|m|))} = 1$, we get

$$\left(|\hat{f}(m,0)| - \frac{c_2}{|m|}\right)\sum_{j=1}^{2|m|}\frac{1}{\lambda_j^{(1)}}$$

$$\le c_1 \int\int_{\overline{\mathbb{T}}^2}\left(\sum_{j=1}^{2|m|}\frac{|\Delta f_j(x,y)|^{p(\tau(|m|))}}{\lambda_j^{(1)}}\right)^{\frac{1}{p(\tau(|m|))}}\left(\sum_{j=1}^{2|m|}\frac{1}{\lambda_j^{(1)}}\right)^{\frac{1}{q(\tau(|m|))}} dxdy.$$

By Minkowski's inequality, we have

$$\left(|\hat{f}(m,0)| - \frac{c_2}{|m|}\right)\left(\sum_{j=1}^{2|m|}\frac{1}{\lambda_j^{(1)}}\right)^{\frac{1}{p(\tau(|m|))}}$$

$$\le 4\pi^2 c_1\left(\lambda_1^{(2)}\right)^{\frac{1}{p(\tau(|m|))}}\left\{V_{\Lambda_{p(n)}}\left(f, \frac{2(1+r)}{1-r}\varphi, \overline{\mathbb{T}}^2\right)\right.$$
$$\left. + V_{\Lambda_{p(n)}}\left(f(.,0), \frac{2(1+r)}{1-r}\varphi, \overline{\mathbb{T}}\right)\right\}.$$

Hence, the result follows from Lemma 3.1.11 and Lemma 2.0.2. □

The Akhobadze class of variation, $B\Lambda(p(n)\uparrow p, \varphi, \overline{\mathbb{T}})$ (Definition 1.1.7 on p. 10), is extended for two-variable functions and later on results for the order of double rational Fourier coefficients of functions in this class are obtained.

Definition 3.1.14. Let $f \in L^\infty(\overline{\mathbb{T}}^2)$ be 2π periodic in both variables. Let $p(n)$ and $q(n)$ be increasing sequences such that $p(n) \le q(n)$, $1 \le p(n) \uparrow p$ for $1 \le p \le \infty$ and $1 \le q(n) \uparrow q$ for $1 \le q \le \infty$. Then $f \in B\Lambda(p(n)\uparrow p, q(n)\uparrow q, \varphi, \overline{\mathbb{T}}^2)$ if

$$\Lambda(f, p(n)\uparrow p, q(n)\uparrow q, \varphi, \overline{\mathbb{T}}^2)$$

$$= \sup_{m\ge 1}\ \sup_{hk\ge\frac{1}{\varphi(m)^2}}\left\{\frac{1}{k}\int_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int_{\overline{\mathbb{T}}} |\ \Delta f(x,y;h,k)\ |^{p(m)}\ dx\right)^{\frac{q(m)}{p(m)}} dy\right\}^{\frac{1}{q(m)}} < \infty,$$

where

$$\Delta f(x,y;h,k) = f(x+h,y+k) - f(x+h,y) - f(x,y+k) + f(x,y).$$

Theorem 3.1.15. *If* $f \in B\Lambda(p(t)\uparrow\infty, q(t)\uparrow\infty, \varphi, \overline{\mathbb{T}}^2)$ *and* $(m,n) \in \mathbb{Z}^{*2}$*, then*

$$\hat{f}(m,n) = O\left(\frac{1}{\mid m\mid^{\frac{1}{p(\tau(|mn|))}}\mid n\mid^{\frac{1}{q(\tau(|mn|))}}} + \frac{1}{\mid m\mid} + \frac{1}{\mid n\mid}\right),$$

where $\tau(mn)$ *is as defined in* (2.3).

Proof. For $j = 1,2$, there exist $h_j, k_j \in [0, 2\pi]$ such that $\theta_{|m|}(x+h_j) - \theta_{|m|}(x) = j\pi,\ \theta_{|n|}(y+k_j) - \theta_{|n|}(y) = j\pi,\ h_1 < h_2$ and $k_1 < k_2$. Here,

$$\frac{(1-r)\pi}{(1+r)\mid m\mid} \leq h_2 - h_1 \leq \frac{(1+r)\pi}{(1-r)\mid m\mid}$$

and

$$\frac{(1-r)\pi}{(1+r)\mid n\mid} \leq k_2 - k_1 \leq \frac{(1+r)\pi}{(1-r)\mid n\mid}.$$

Let $(m,n) \in \mathbb{Z}^{*2}, h_2 - h_1 = h$ and $k_2 - k_1 = k$.
Therefore,

$$\mid \hat{f}(m,n)\mid \leq c_1 \int\int_{\overline{\mathbb{T}}^2} \mid \Delta f(x,y;h,k)\mid dxdy + c_2\left(\frac{1}{\mid m\mid} + \frac{1}{\mid n\mid}\right),$$

where $c_1 = \dfrac{1+r}{16\pi^2(1-r)}$ and $c_2 = \dfrac{r(1+r)^2 \mid\mid f\mid\mid_1}{8\pi(1-r)^4}$. Therefore, by applying Holder's inequality and letting $\frac{1}{p(\tau(|mn|))} + \frac{1}{d(\tau(|mn|))} = 1$, we get

$$\begin{aligned}
&\mid \hat{f}(m,n)\mid - \frac{c_2}{\mid m\mid} - \frac{c_2}{\mid n\mid} \\
&\quad \leq c_1 \int_{\overline{\mathbb{T}}}\left\{\left(\frac{1}{h}\int_{\overline{\mathbb{T}}} \mid \Delta f(x,y;h,k)\mid^{p(\tau(|mn|))} dx\right)^{\frac{1}{p(\tau(|mn|))}}\right. \\
&\qquad\qquad \left.\times\left(\int_{\overline{\mathbb{T}}} h^{\frac{d(\tau(|mn|))}{p(\tau(|mn|))}} dx\right)^{\frac{1}{d(\tau(|mn|))}}\right\} dy \\
&\quad \leq 2\pi c_1 h^{\frac{1}{p(\tau(|mn|))}} \int_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int_{\overline{\mathbb{T}}} \mid \Delta f(x,y;h,k)\mid^{p(\tau(|mn|))} dx\right)^{\frac{1}{p(\tau(|mn|))}} dy.
\end{aligned}$$

Again, by applying Holder's inequality on the right side of the above inequality and letting $\frac{1}{q(\tau(|mn|))} + \frac{1}{t(\tau(|mn|))} = 1$, we get
$| \hat{f}(m,n) | - \frac{c_2}{| m |} - \frac{c_2}{| n |}$

$$\leq \left\{ \int_{\overline{\mathbb{T}}} \frac{1}{k} \left(\frac{1}{h} \int_{\overline{\mathbb{T}}} | \Delta f(x,y;h,k) |^{p(\tau(|mn|))} \, dx \right)^{\frac{q(\tau(|mn|))}{p(\tau(|mn|))}} dy \right\}^{\frac{1}{q(\tau(|mn|))}}$$
$$\times \left\{ \int_{\overline{\mathbb{T}}} k^{\frac{t(\tau(|mn|))}{q(\tau(|mn|))}} dy \right\}^{\frac{1}{t(\tau(|mn|))}} 2\pi c_1 h^{\frac{1}{p(\tau(|mn|))}}$$
$$\leq 4\pi^2 c_1 h^{\frac{1}{p(\tau(|mn|))}} k^{\frac{1}{q(\tau(|mn|))}} \Lambda(f, p(t) \uparrow \infty, q(t) \uparrow \infty, \varphi, \overline{\mathbb{T}}^2)$$
$$= O\left(\frac{1}{| m |^{\frac{1}{p(\tau(|mn|))}} | n |^{\frac{1}{q(\tau(|mn|))}}} \right).$$

Hence, the result follows. □

Theorem 3.1.16. *If marginal function* $f(.,b) \in B\Lambda(p(n) \uparrow \infty, \varphi, \overline{\mathbb{T}})$, $b \in \overline{\mathbb{T}}$ *and* $m \in \mathbb{Z}^*$, *then*

$$\hat{f}(m,0) = O\left(\frac{1}{| m |^{\frac{1}{p(\tau(|m|))}}} \right),$$

where $\tau(| m |)$ *is defined as in* (2.3).

Proof. There exists $h_j \in [0, 2\pi]$ such that $\theta_{|m|}(x + h_j) - \theta_{|m|}(x) = j\pi$ for $j = 1, 2$ and $h_1 < h_2$. By the mean value theorem, we get

$$| \rho_{|m|}(x + h_2) - \rho_{|m|}(x + h_1) | \leq \frac{r(1+r)^{1/2}}{(1-r)^{5/2}} (h_2 - h_1) \leq \frac{r(1+r)^{3/2}\pi}{(1-r)^{7/2} | m |}$$

as

$$\frac{(1-r)\pi}{(1+r) | m |} \leq h_2 - h_1 \leq \frac{(1+r)\pi}{(1-r) | m |}.$$

Hence,

$$| \hat{f}(m,0) | \leq c_1 \int\int_{\overline{\mathbb{T}}^2} | \Delta f_1(x,y) | \, dxdy + \frac{c_2}{| m |},$$

where $c_1 = \frac{1}{8\pi^2}\sqrt{\frac{1+r}{(1-r)}}$, $c_2 = \frac{r(1+r)^{3/2} \, || f ||_1}{8\pi(1-r)^{7/2}}$ and $\Delta f_1 = f(x+h_2, y) - f(x+h_1, y)$. By applying Holder's inequality on the right side of the above inequality and letting $\frac{1}{p(\tau(|m|))}+\frac{1}{d(\tau(|m|))}=1$, we get

$$\begin{aligned}
&\mid \hat{f}(m,0)\mid -\frac{c_2}{\mid m\mid}\\
&\leq c_1\int_{\overline{\mathbb{T}}}\left(\frac{1}{h}\int_{\overline{\mathbb{T}}}\mid \Delta f_1(x,y)\mid^{p(\tau(|m|))}dx\right)^{\frac{1}{p(\tau(|mn|))}}\left(\int_{\overline{\mathbb{T}}}h^{\frac{d(\tau(|m|))}{p(\tau(|m|))}}dx\right)^{\frac{1}{d(\tau(|m|))}}dy\\
&\leq 4\pi^2 c_1 h^{\frac{1}{p(\tau(|m|))}}\Lambda(f(.,y),p(n)\uparrow\infty,\varphi,\overline{\mathbb{T}})\\
&=O\left(\frac{1}{\mid m\mid^{\frac{1}{p(\tau(|m|))}}}\right).
\end{aligned}$$

Hence, we get the required result. □

Remark 10. If $\alpha_k=0\ \forall k\in\mathbb{N}$ then Theorem 3.1.15 and Theorem 3.1.16 gives analogous results for double Fourier coefficient with order containing only first term as $r=0$ implies $c_2=0$.

3.2 Order of magnitude of multiple rational Fourier coefficients

For a function, $f\in L^1(\overline{\mathbb{T}}^N)$, which is 2π periodic in all the variables, multiple rational Fourier series of f is defined as

$$f(x_1,x_2,...,x_N)\sim\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}...\sum_{k_N\in\mathbb{Z}}\hat{f}(k_1,k_2,..,k_N)\phi_{k_1}(e^{ix_1})\phi_{k_2}(e^{ix_2})...\phi_{k_N}(e^{ix_N}),$$

where $N\in\mathbb{N}$ and $\hat{f}(k_1,...,k_N)$ is the $(k_1,...,k_N)^{th}$ rational Fourier coefficient of f given by

$$\hat{f}(k_1,...,k_N)=\frac{1}{(2\pi)^N}\int...\int_{\overline{\mathbb{T}}^N}f(x_1,...,x_n)\overline{\phi_{k_1}(e^{ix_1})...\phi_{k_N}(e^{ix_N})}dx_1...dx_N.$$

If $\alpha_k=0;\ \forall k\in\mathbb{Z}$, then rational multiple Fourier series becomes multiple Fourier series with the exponential system as its orthogonal system.

The results of double rational Fourier coefficients can be extended similarly for multiple rational Fourier coefficients. The following result can be considered a Riemann-Lebesgue analogous result for multiple rational Fourier series.

Theorem 3.2.1. *If* $f \in L^1(\overline{\mathbb{T}}^N), (k_1, ..., k_N) \in \mathbb{Z}^N$, *then* $\hat{f}(k_1, ..., k_N) \to 0$ *as* $|(k_1, ..., k_N)| = \sqrt{|k_1|^2 + ... + |k_N|^2} \to \infty$.

The forthcoming results in the remaining portion of this section are related to the order of magnitude of multiple rational Fourier coefficients. These order of magnitude of coefficients can be obtained by following proofs similar to those for their two-variable counterparts but applied to functions with N variables.

Theorem 3.2.2. *If* $f \in Lip(p; \beta_1, \beta_2, ..., \beta_N)(\overline{\mathbb{T}}^N), p \geq 1, \beta_1, ..., \beta_N \in (0, 1]$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\frac{1}{|k_1|^{\beta_1} ... |k_N|^{\beta_N}} + \sum_{j=1}^{N} \frac{1}{|k_j|}\right)$$

Theorem 3.2.3. *If* $f \in \Phi\Lambda BV(\overline{\mathbb{T}}^N) \cap L^1(\overline{\mathbb{T}}^N)$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\Phi^{-1}\left(\frac{\sum_{j=1}^{N} \frac{|k_1|...|k_N|}{|k_j|}}{\sum_{i_1=1}^{2|k_1|} \cdots \sum_{i_N=1}^{2|k_N|} \frac{1}{\lambda_{(1,i_1)} \cdots \lambda_{(N,i_N)}}}\right)\right).$$

Corollary 3.2.4. *If* $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^N)$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\Phi^{-1}\left(\frac{\sum_{j=1}^{N} \frac{|k_1|...|k_N|}{|k_j|}}{\sum_{i_1=1}^{2|k_1|} \cdots \sum_{i_N=1}^{2|k_N|} \frac{1}{\lambda_{(1,i_1)} \cdots \lambda_{(N,i_N)}}}\right)\right).$$

Corollary 3.2.5. *If* Φ *satisfies* Δ_2 *condition,* $f \in \Phi\Lambda^* BV(\overline{\mathbb{T}}^N)$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^N$ *is such that* $k_j \neq 0$ *for* $(1 \leq)j_1 < ... < j_M(\leq N)$ *and* $k_j = 0$ *for* $(1 \leq)l_1 < ... < l_{N-M}(\leq N)$, *where* $\{l_1, ..., l_{N-M}\}$ *is the complementary set of* $\{j_1, ..., j_M\}$ *with respect to* $\{1, ..., N\}$, *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\Phi^{-1}\left(\frac{\sum_{t=1}^{M} \frac{|k_{j_1}|...|k_{j_M}|}{|k_{j_t}|}}{\sum_{i_1=1}^{2|k_{j_1}|} \cdots \sum_{i_M=1}^{2|k_{j_M}|} \frac{1}{\lambda_{(j_1,i_1)} \cdots \lambda_{(j_M,i_M)}}}\right)\right).$$

Definition 3.2.6. Let f be a complex valued measurable function defined on $R^N := I^{(1)} \times ... \times I^{(N)} := [a_1, b_1] \times ... \times [a_N, b_N]$; $\Lambda = \left(\Lambda^{(1)}, ..., \Lambda^{(N)}\right)$, where

$\Lambda^{(t)} = \left\{\lambda_k^{(t)}\right\}_{k=1}^{\infty}$ is a non decreasing sequence of positive numbers such that $\sum_k \left(\lambda_k^{(t)}\right)^{-1}$ diverges for $t = 1, 2, ..., N$; for $1 \leq p \leq \infty, 1 \leq p(n) \uparrow p$ as $n \to \infty$; $\{\varphi(n)\}_{n=1}^{\infty}$ is a real sequence such that $\varphi(1) \geq 2$ and $\varphi(n) \uparrow \infty$ as $n \to \infty$. Then $f \in \Lambda BV(p(n) \uparrow p, \varphi, R^N)$ if

$$V_{\Lambda_{p(n)}}(f, R^N) = \sup_{n\geq 1} \sup_{\left\{I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right\}} \left\{ V_{\Lambda_{p(n)}}\left(f, \left\{I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right\}\right) \right.$$
$$\left. : \delta\left\{I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right\} \geq \frac{\prod_{t=1}^{N}(b_t - a_t)}{\varphi(n)^N} \right\} < \infty,$$

where $\left\{I_{i_k}^{(k)}\right\}$ is finite collection of non overlapping subintervals of $I^{(k)}$ for $k = 1, ..., N$;

$$V_{\Lambda_{p(n)}}\left(f, \left\{I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right\}\right) = \left(\sum_{i_1} ... \sum_{i_N} \frac{\left| f\left(I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right)\right|^{p(n)}}{\lambda_{i_1}^{(1)} .... \lambda_{i_N}^{(N)}} \right)^{\frac{1}{p(n)}};$$

$f\left(I^{(1)} \times ... \times I^{(N)}\right) = f\left(I^{(1)} \times ... \times I^{(N-1)}, b_N\right) - f\left(I^{(1)} \times ... \times I^{(N-1)}, a_N\right)$, here $f\left(I^{(1)}\right) = f(b_1) - f(a_1)$, $f\left(I^{(1)} \times I^{(2)}\right) = f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2)$ and so on; and

$$\delta\left\{I_{i_1}^{(1)} \times ... \times I_{i_N}^{(N)}\right\} := \delta\left\{\left[s_{i_1-1}^{(1)}, s_{i_1}^{(1)}\right] \times ... \times \left[s_{i_N-1}^{(N)}, s_{i_N}^{(N)}\right]\right\}$$
$$= \inf_{i_1,...,i_N} \left\{ \prod_{k=1}^{N} |(s_{i_k}^{(k)} - s_{i_k-1}^{(k)})| \right\}.$$

Definition 3.2.7. A function $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, R^N)$ if $f \in \Lambda BV(p(n) \uparrow p, \varphi, R^N)$, $1 \leq p \leq \infty$ and each of its marginal functions
$f(x_1, ..., x_{i-1}, a_i, x_{i+1}, ..., x_N)$

$$\in (\Lambda^{(1)}, ..., \Lambda^{(i-1)}, \Lambda^{(i+1)}, ..., \Lambda^{(N)})^* BV(p(n) \uparrow p, \varphi, R^N(a_i)),$$

$\forall i = 1, 2, ..., N$, where

$$R^N(a_i) = \{(x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}^{N-1} : x_k \in [a_k, b_k]$$
$$\forall k = 1, ..., i-1, i+1, ...N\}.$$

Theorem 3.2.8. *If* $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$, $1 \le p \le \infty$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\frac{1}{\left(\sum_{i_1=1}^{2|k_1|} \cdots \sum_{i_N=1}^{2|k_N|} \frac{1}{\lambda_{i_1}^{(1)} \cdots \lambda_{i_N}^{(N)}} \right)^{\frac{1}{p(\tau(|k_1 \ldots k_N|))}}} + \sum_{t=1}^{N} \frac{1}{|k_t|} \right),$$

where $\tau(|k_1...k_N|)$ *is as defined in* (2.1).

Theorem 3.2.9. *If* $f \in \Lambda^* BV(p(n) \uparrow p, \varphi, \overline{\mathbb{T}}^N)$, $1 \le p \le \infty$ *and* $(k_1, ..., k_N) \in \mathbb{Z}^N$ *is such that* $k_j \ne 0$ *for* $(1 \le) j_1 < ... < j_M (\le N)$ *and* $k_j = 0$ *for* $(1 \le) l_1 < ... < l_{N-M} (\le N)$, *where* $\{l_1, ..., l_{N-M}\}$ *is the complementary set of* $\{j_1, ..., j_M\}$ *with respect to* $\{1, ..., N\}$, *then*

$$\hat{f}(k_1, ..., k_N) = O\left(\frac{1}{\left(\sum_{i_1=1}^{2|k_{j_1}|} \cdots \sum_{i_M=1}^{2|k_{j_M}|} \frac{1}{\lambda_{i_1}^{(j_1)} \cdots \lambda_{i_M}^{(j_M)}} \right)^{\frac{1}{p(\tau(|k_{j_1} \ldots k_{j_M}|))}}} + \sum_{t=1}^{M} \frac{1}{|k_{j_t}|} \right),$$

where $\tau(|k_{j_1}...k_{j_M}|)$ *is as defined in* (2.1).

Definition 3.2.10. Let $f \in L^\infty(\overline{\mathbb{T}}^N)$ be 2π periodic in all the variables, $p_j(n)$ be increasing sequences such that $p_j(n) \le p_{j-1}(n)$, $1 \le p_j(n) \uparrow q_j$ and $1 \le q_j \le \infty$ for $j = 1, 2, ..., N$. Let $\{\varphi(n)\}_{n=1}^\infty$ is a real sequence such that $\varphi(1) \ge 2$ and $\lim_{n\to\infty} \varphi(n) = \infty$. Then $f \in B\Lambda(p_1(n) \uparrow q_1, ..., p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N)$ if

$$\begin{aligned} &\Lambda(p_1(n) \uparrow q_1, ..., p_N(n) \uparrow q_N, \varphi, \overline{\mathbb{T}}^N) \\ &\quad = \sup_{m \ge 1} \sup_{\prod_1^N h_j \ge \frac{1}{\varphi(m)^N}} \left\{ \frac{1}{h_N} \cdots \left(\frac{1}{h_1} \int_{\overline{\mathbb{T}}} | \Delta f(x_1, ..., x_N; h_1, ..., h_N) |^{p_1(m)} \, dx_1 \right)^{\frac{p_2(m)}{p_1(m)}} \right. \\ &\qquad \left. \ldots dx_N \right\}^{\frac{1}{p_N(m)}} < \infty, \end{aligned}$$

where $\Delta f(x_1; h_1) = f(x_1 + h_1) - f(x_1)$,
$\Delta f(x_1, x_2; h_1, h_2) = f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2 + h_2) - f(x_1 + h_1, x_2) + f(x_1, x_2)$,
$\vdots$
$\Delta f(x_1, ..., x_N; h_1, ..., h_N) = \sum_{u_1=0}^{1} \cdots \sum_{u_N=0}^{1} (-1)^{u_1+u_2+...+u_N} f(x_1 + u_1 h_1, ..., x_N + u_N h_N)$.

Theorem 3.2.11. *If $f \in B\Lambda(p_1(n)\uparrow q_1, ..., p_N(n)\uparrow q_N, \varphi, \overline{\mathbb{T}}^N)$, $q_j \in [1,\infty]$ for $j = 1, 2, .., N$ and $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ then*

$$\hat{f}(k_1, ..., k_N) = O\left(\frac{1}{| k_1 |^{\frac{1}{p_1(\tau(|k_1...k_N|))}} ... | k_N |^{\frac{1}{p_N(\tau(|k_1...k_N|))}}} + \sum_{t=1}^{N} \frac{1}{| k_t |}\right),$$

where $\tau(| k_1...k_N |)$ is defined as in (2.3).

Theorem 3.2.12. *If marginal function $f(., ..., b_{l_1}, ..., b_{l_{N-M}}, ..., .) \in B\Lambda(p_{j_1}(n)\uparrow q_{j_1}, ..., p_{j_M}(n)\uparrow q_{j_M}, \varphi, \overline{\mathbb{T}}^{j_M})$, $q_{j_t} \in [1,\infty]$ for $t = 1, 2, .., M$ and $(k_1, ..., k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq)j_1 < ... < j_M(\leq N)$ and $k_j = 0$ for $(1 \leq)l_1 < ... < l_{N-M}(\leq N)$, where $\{l_1, ..., l_{N-M}\}$ is the complementary set of $\{j_1, ..., j_M\}$ with respect to $\{1, ..., N\}$, then*

$$\begin{aligned}&\hat{f}(0, ..., k_{j_1}, ..., k_{j_M}, ..., 0)\\ &= O\left(\frac{1}{| k_{j_1} |^{\frac{1}{p_{j_1}\left(\tau\left(\left|k_{j_1}...k_{j_M}\right|\right)\right)}} ... | k_{j_M} |^{\frac{1}{p_{j_M}\left(\tau\left(\left|k_{j_1}...k_{j_M}\right|\right)\right)}}} + \sum_{t=1}^{M} \frac{1}{| k_{j_t} |}\right),\end{aligned}$$

where $\tau(| k_{j_1}...k_{j_M} |)$ is defined as in (2.3).

Corollary 3.2.13. *If $f \in B\Lambda(p_1(n)\uparrow q_1, ..., p_N(n)\uparrow q_N, \varphi, \overline{\mathbb{T}}^N)$, $q_j \in [1,\infty]$ for $j = 1, 2, .., N$ and $(k_1, ..., k_N) \in \mathbb{Z}^{*N}$ then multiple Fourier coefficients of f are denoted by $c_{(k_1,...,k_N)}(f)$ and*

$$c_{(k_1,...,k_N)}(f) = O\left(\frac{1}{| k_1 |^{\frac{1}{p_1(\tau(|k_1...k_N|))}} ... | k_N |^{\frac{1}{p_N(\tau(|k_1...k_N|))}}}\right),$$

where

$$\tau(n) = \min\{k : k \in \mathbb{N}, \varphi(k) \geq n\}, n \geq 1. \tag{3.2}$$

Corollary 3.2.14. *If marginal function $f(., ..., b_{l_1}, ..., b_{l_{N-M}}, ..., .) \in B\Lambda(p_{j_1}(n)\uparrow q_{j_1}, ..., p_{j_M}(n)\uparrow q_{j_M}, \varphi, \overline{\mathbb{T}}^{j_M})$, $q_{j_t} \in [1,\infty]$ for $t = 1, 2, .., M$ and $(k_1, ..., k_N) \in \mathbb{Z}^N$ is such that $k_j \neq 0$ for $(1 \leq)j_1 < ... < j_M(\leq N)$ and $k_j = 0$ for $(1 \leq)l_1 < ... < l_{N-M}(\leq N)$, where $\{l_1, ..., l_{N-M}\}$ is the complementary set of $\{j_1, ..., j_M\}$ with respect to $\{1, ..., N\}$, then multiple Fourier coefficients of f are denoted by*

$c_{(k_1,\ldots,k_N)}(f)$ *and*

$$c_{(0,\ldots,k_{j_1},\ldots,k_{j_M},\ldots,0)}(f) = O\left(\frac{1}{\mid k_{j_1}\mid^{\frac{1}{p_{j_1}\left(\tau\left(\left|k_{j_1}\ldots k_{j_M}\right|\right)\right)}}\ldots\mid k_{j_M}\mid^{\frac{1}{p_{j_M}\left(\tau\left(\left|k_{j_1}\ldots k_{j_M}\right|\right)\right)}}}\right),$$

where $\tau(\mid k_{j_1}\ldots k_{j_M}\mid)$ *is defined as in* (3.2).

Remark 11. Corollary 3.2.13 and Corollary 3.2.14 can be obtained from Theorem 3.2.11 and Theorem 3.2.12, respectively, by taking $\alpha_k = 0$, for all $k \in \mathbb{N}$ and hence $r = 0$, so only the first term remains in the order and other terms, being multiple of r vanishes.