Chapter 4

Rate of convergence of rational, conjugate rational and double rational Fourier series

It is observed that the rate of convergence of Fourier or rational Fourier coefficients of a function depends on the smoothness of the function. However, this alone does not reveal the behaviour of the Fourier series or its relationship to the original function. The study of the convergence of a Fourier series is crucial to ensure the accuracy and reliability of representing functions as infinite sums of sines and cosines. This analytical investigation holds significance across diverse fields, including mathematics, engineering, physics, and signal processing.

Extensive exploration of Fourier series' convergence is documented in the literature, with one notable outcome being the Dirichlet-Jordan test for the convergence of Fourier series. This test furnishes information on the convergence of the Fourier series for functions of bounded variation, extending its utility beyond theoretical mathematics. It finds practical applications in fields where Fourier analysis is essential for comprehending and representing complex functions.

Numerous researchers have delved into the conditions for convergence, uniform convergence, and absolute convergence of Fourier series for functions exhibiting various forms of generalized bounded variations. Their findings contribute to a deeper understanding of the convergence behavior of the Fourier series in diverse mathematical contexts.

In 1971, Bojanić provided a quantitative version of the Dirichlet-Jordan test in terms of variations (refer to Theorem Q on p. 28). In 1982, Waterman presented an estimate of the Fourier series convergence rate for functions that are closer to the class of harmonic bounded variation (refer to Theorem R on p. 28). In 1987, Mazhar and Al-Budaiwi [38, p. 178] obtained an estimate of the rate of convergence of conjugate Fourier series of functions of bounded variation (refer to Theorem S on p. 29). In 1992, Móricz gave the quantitative version of the Dirichlet-Jordan test for double Fourier series (refer to Theorem U on p. 30). In 2013, Tan and Qian obtained the analogous quantitative version of the Dirichlet-Jordan test for rational Fourier series (refer to Theorem V on p. 31). These works inspired us to study the results related to the rate of convergence of rational, conjugate rational, and double rational Fourier series of functions of generalized bounded variations.

In this chapter, the parameters α_k, defined in the rational orthogonal system (1.5), satisfies the condition (1.6) and r is as defined in (1.6).

4.1 Rate of convergence of rational and conjugate rational Fourier series of function of generalized bounded variation

The following notations and conditions will be assumed in the rest of the chapter:

1. We assume that $\frac{\lambda_{|k|}}{|k|}$ is non increasing and for fixed n, $H(t)$ is a continuously non increasing function on $[-\pi, 0)$ and $(0, \pi]$ such that

$$H(t) = \frac{\lambda_{|k|}}{t};$$

   where $t = \frac{k\pi}{n+1}$ and $k = \pm 1, \pm 2, ..., \pm(n+1)$.

2. $$\psi := \psi_x(t) := \begin{cases} f(x) - f(x-t) \text{ if } t \in \mathbb{T} \setminus \{0\} \\ 0 \text{ if } t = 0. \end{cases}$$

3.

$$\phi := \phi_x(t) := \begin{cases} f(x-t) \text{ if } t \in \mathbb{T} \setminus \{0\} \\ 0 \text{ if } t = 0. \end{cases}$$

4. $\tilde{f}(x) = \lim_{\epsilon \to 0^+} \tilde{f}(x;\epsilon) = \frac{1}{\pi} \int_{\epsilon \leq \pi} \frac{f(x-t)}{2\tan(t/2)} dt.$

5. $osc(\psi_x, [a,b]) = \sup_{t,y \in [a,b]} |\psi_x(t) - \psi_x(y)|.$

6. $\eta_{km} = \frac{k\pi}{m+1}, \ \forall k = 0,1,2,...,m; \ m \in \mathbb{N} \cup \{0\}.$

7. $I^+_{km} = [\eta_{km}, \eta_{(k+1)m}].$

8. $I^-_{km} = [-\eta_{(k+1)m}, -\eta_{km}].$

Theorem 4.1.1. *If f is bounded, measurable function in $[-\pi, \pi]$ and is regulated i.e. $f(x) = 1/2\{f(x+0) + f(x-0)\}$ then*

$$|S_n f(x) - f(x)| \leq \frac{2(1+r)}{1-r} \sum_{k=0}^{n} \frac{1}{k+1} \left\{ osc(\psi_x, I^+_{kn}) + osc(\psi_x, I^-_{kn}) \right\}.$$

Proof. In view of [60, Lemma 2.1] for $\alpha_k = |\alpha_k| e^{i a_k}, \ n \in \mathbb{N}$ and $x \in [-\pi, \pi]$, the partial sums of rational Fourier series is given by

$$S_n f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_n(x-t, x) dt,$$

where

$$D_n(t,x) = \frac{1}{2} \sum_{k=-n}^{n} \overline{\phi_k(e^{it})} \phi_k(e^{ix}) = \frac{\sin\left[\frac{x-t}{2} + \theta_n(t,x)\right]}{2\sin\left(\frac{x-t}{2}\right)}$$

and

$$\theta_n(t,x) = \int_t^x \sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{1 - 2|\alpha_k| \cos(y - a_k) + |\alpha_k|^2} dy.$$

Note that for $n \in \mathbb{Z} \setminus \{0\}$ and by (1.6), we get,

$$|\phi_n(e^{ix})| = \sqrt{\frac{1 - |\alpha_{|n|}|^2}{1 - 2|\alpha_{|n|}| \cos(x - a_{|n|}) + |\alpha_{|n|}|^2}} \leq \sqrt{\frac{1+r}{1-r}}. \quad (4.1)$$

Therefore, for $n \in \mathbb{N}$,

$$|D_n(t,x)| \leq (n+1) \frac{1+r}{1-r}. \quad (4.2)$$

Thus,

$$
\begin{aligned}
f(x) - S_n f(x) &= \frac{1}{\pi}\int_0^{\pi} \psi_x(t) D_n(x-t,x)dt + \frac{1}{\pi}\int_{-\pi}^{0} \psi_x(t) D_n(x-t,x)dt \\
&:= A + B.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &= \frac{1}{\pi}\int_{I_{0n}^+} \psi_x(t) D_n(x-t,x)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} (\psi_x(t) - \psi(\eta_{kn})) D_n(x-t,x)dt \\
&+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} \psi_x(\eta_{kn}) D_n(x-t,x)dt \\
&:= A_1 + A_2 + A_3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
B &= \frac{1}{\pi}\int_{I_{0n}^-} \psi_x(t) D_n(x-t,x)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^-} (\psi_x(t) - \psi(-\eta_{kn})) D_n(x-t,x)dt \\
&+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^-} \psi_x(-\eta_{kn}) D_n(x-t,x)dt \\
&:= B_1 + B_2 + B_3.
\end{aligned}
$$

By (4.2), we have

$$
\begin{aligned}
|A_1| &\le \frac{1}{\pi}\int_{I_{0n}^+} |\psi_x(t) - \psi(0)||D_n(x-t,x)|dt \\
&\le \frac{osc(\psi_x, I_{0n}^+)}{\pi}\int_{I_{0n}^+} (n+1)\frac{1+r}{1-r}dt \\
&\le \frac{1+r}{1-r} osc(\psi_x, I_{0n}^+)
\end{aligned}
$$

and similarly

$$
|B_1| \le \frac{1+r}{1-r} osc(\psi_x, I_{0n}^-).
$$

Since $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ for $0 < t < \pi$, we get

$$
|A_2| \le \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} |\psi_x(t) - \psi(\eta_{kn})||D_n(x-t,x)|dt \le \sum_{k=1}^{n}\frac{1}{k+1} osc(\psi_x, I_{kn}^+)
$$

and similarly

$$|B_2| \leq \sum_{k=1}^{n} \frac{1}{k+1} osc(\psi_x, I_{kn}^-).$$

Let

$$R_{km}^+ = \int_{\eta_{kn}}^{\pi} D_n(x-t,x)dt \text{ and } R_{km}^- = \int_{\eta_{kn}}^{\pi} D_n(x+t,x)dt.$$

In view of [60, Lemma 2.3], we have for $0 < u < \pi$,

$$\left| \int_u^{\pi} D_n(x-t,x)dt \right| \leq \frac{\pi^2(1+r)}{2n(1-r)u} \text{ and } \left| \int_u^{\pi} D_n(x+t,x)dt \right| \leq \frac{\pi^2(1+r)}{2n(1-r)u}.$$

Thus, we get

$$|R_{kn}^+| \leq \frac{\pi(1+r)}{k(1-r)} \text{ and } |R_{kn}^-| \leq \frac{\pi(1+r)}{k(1-r)}. \tag{4.3}$$

We have

$$A_3 = \frac{1}{\pi} \sum_{k=1}^{n} \left\{ \psi_x(\eta_{kn}) - \psi_x(\eta_{(k-1)n}) \right\} (R_{kn}^+)$$

and

$$B_3 = \frac{1}{\pi} \sum_{k=1}^{n} \left\{ \psi_x(-\eta_{kn}) - \psi_x(-\eta_{(k-1)n}) \right\} (R_{kn}^-).$$

Therefore, by using (4.3), we get

$$|A_3| \leq \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k} osc(\psi_x, I_{(k-1)n}^+) \text{ and } |B_3| \leq \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k} osc(\psi_x, I_{(k-1)n}^-).$$

Thus, we have

$$|A| \leq \frac{2(1+r)}{1-r} \sum_{k=0}^{n} \frac{1}{k+1} osc(\psi_x, I_{kn}^+),\ |B| \leq \frac{2(1+r)}{1-r} \sum_{k=0}^{n} \frac{1}{k+1} osc(\psi_x, I_{kn}^-).$$

Hence, the result is proved.

□

Corollary 4.1.2. *If* $f \in \Lambda BV([-\pi, \pi])$, $\frac{\pi}{n+1} = a_n < a_{n-1} < ... < a_0 = \pi$ *and* $-\pi = b_0 < b_1, ..., < b_n = \frac{-\pi}{n+1}$, *then*

$$\left(\frac{1-r}{1+r}\right) |S_n f(x) - f(x)| \leq \frac{2\lambda_{n+1}}{n+1} (V_\Lambda(\psi, [0,\pi]) + V_\Lambda(\psi, [-\pi, 0]))$$

$$+\frac{2\pi}{n+1}\sum_{i=0}^{n-1} V_\Lambda(\psi,[0,a_i])(H(a_{i+1})-H(a_i))$$
$$+\frac{2\pi}{n+1}\sum_{i=0}^{n-1} V_\Lambda(\psi,[b_i,0])(H(b_i)-H(b_{i+1})).$$

Proof. The result can be easily obtained from Theorem 4.1.1 and by following the method as given in [76]. □

Corollary 4.1.3. *If* $f\in\{n^\beta\}-BV([-\pi,\pi]),\ 0<\beta<1$ *then*

$$|S_nf(x)-f(x)|\le\frac{2(2-\beta)}{(n+1)^{1-\beta}}\cdot\frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k^\beta}\left\{V_{\{n^\beta\}}(\psi,[0,\pi/k])\right.$$
$$\left.+V_{\{n^\beta\}}(\psi,[-\pi/k,0])\right\}.$$

Remark 12. Theorem 4.1.1, Corollary 4.1.2 and Corollary 4.1.3 are analogous results of Theorem R (p. 28) for rational Fourier series as for $r=0$, the estimations for classical Fourier series are obtained. These results generalize the estimation of rational Fourier series for functions of bounded variation given in Theorem V (p. 31).

Theorem 4.1.4. *If f is bounded, measurable and regulated function in $[-\pi,\pi]$ then*

$$\left|\tilde{S}_nf(x)-\tilde{f}\left(x,\frac{\pi}{n+1}\right)\right|\le\frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}\left\{osc(\phi_x,I_{kn}^+)+osc(\phi_x,I_{kn}^-)\right\}.$$

Proof. In view of [60, Lemma 2.1] for $\alpha_k=|\alpha_k|e^{ia_k},\ n\in\mathbb{N}$ and $x\in[-\pi,\pi]$, the partial sums of conjugate rational Fourier series is given by

$$\tilde{S}_nf(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)\tilde{D}_n(x-t,x)dt,$$

where

$$\tilde{D}_n(t,x)=\frac{1}{2}\sum_{k=-n}^{n}(-i)\mathrm{sgn}(k)\overline{\phi_k(e^{it})}\phi_k(e^{ix})=\frac{\cos\left(\frac{x-t}{2}\right)-\cos\left[\frac{x-t}{2}+\theta_n(t,x)\right]}{2\sin\left(\frac{x-t}{2}\right)}$$

and

$$\theta_n(t,x)=\int_t^x\sum_{k=1}^{n}\frac{1-|\alpha_k|^2}{1-2|\alpha_k|\cos(y-a_k)+|\alpha_k|^2}dy.$$

Therefore by using (4.1) and the simple fact that $\operatorname{sgn}(0) = 0$, for $n \in \mathbb{N}$,

$$|\tilde{D}_n(t,x)| \leq n\frac{1+r}{1-r}. \tag{4.4}$$

Using the fact that

$$\int_{-\pi}^{\pi} \tilde{D}_n(x-t,x)dt = 0 \text{ and } \int_{\frac{\pi}{n+1}\leq|t|\leq\pi} \frac{1}{2\tan(t/2)}dt = 0,$$

we get

$$\begin{aligned}\tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) &= \frac{1}{\pi}\left\{-\int_0^{\pi} \phi_x(t)\tilde{D}_n(x-t,x)dt + \int_{\frac{\pi}{n+1}}^{\pi} \frac{\phi_x(t)}{2\tan(t/2)}dt\right\} \\ &+ \frac{1}{\pi}\left\{-\int_{-\pi}^{0} \phi_x(t)\tilde{D}_n(x-t,x)dt + \int_{-\pi}^{\frac{-\pi}{n+1}} \frac{\phi_x(t)}{2\tan(t/2)}dt\right\} \\ &:= \tilde{A} + \tilde{B}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{A} &= -\frac{1}{\pi}\int_{I_{0n}^+} \phi_x(t)\tilde{D}_n(x-t,x)dt \\ &+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} (\phi_x(t) - \phi(\eta_{kn}))\frac{\cos\left[\frac{t}{2} + \theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}dt \\ &+ \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} \phi_x(\eta_{kn})\frac{\cos\left[\frac{t}{2} + \theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}dt \\ &:= \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3.\end{aligned}$$

By (4.4), we have

$$|\tilde{A}_1| \leq \frac{1}{\pi}\int_{I_{0n}^+} |\phi_x(t) - \phi(0)||\tilde{D}_n(x-t,x)|dt \leq \frac{1+r}{1-r} osc(\phi_x, I_{0n}^+).$$

Let $D_n^*(x-t,x) = \frac{\cos\left[\frac{t}{2}+\theta_n(x-t,x)\right]}{2\sin\left(\frac{t}{2}\right)}$ and since $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ for $0 < t < \pi$, we get

$$|\tilde{A}_2| \leq \frac{1}{\pi}\sum_{k=1}^{n}\int_{I_{kn}^+} |\phi_x(t) - \phi(\eta_{kn})||D_n^*(x-t,x)|dt \leq \sum_{k=1}^{n}\frac{1}{k+1} osc(\phi_x, I_{kn}^+).$$

Let $T^+_{km} = \int_{\eta_{kn}}^{\pi} D^*_n(x-t,x)dt$. In view of [60, Lemma 2.3], we have for $0 < u < \pi$,

$$\left|\int_u^{\pi} D^*_n(x-t,x)dt\right| \le \frac{\pi^2(1+r)}{2n(1-r)u}.$$

Thus, we get

$$|T^+_{kn}| \le \frac{\pi(1+r)}{k(1-r)}. \tag{4.5}$$

Hence, we have

$$\tilde{A}_3 = \frac{1}{\pi}\sum_{k=1}^{n}\left\{\phi_x(\eta_{kn}) - \phi_x(\eta_{(k-1)n})\right\}(T^+_{kn}).$$

Therefore, by using (4.5), we get

$$|\tilde{A}_3| \le \frac{1+r}{1-r}\sum_{k=1}^{n}\frac{1}{k}osc(\phi_x, I^+_{(k-1)n}).$$

Thus, we have

$$|\tilde{A}| \le \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\phi_x, I^+_{kn})$$

and similarly

$$|\tilde{B}| \le \frac{2(1+r)}{1-r}\sum_{k=0}^{n}\frac{1}{k+1}osc(\phi_x, I^-_{kn}).$$

Hence, we get the result. □

Corollary 4.1.5. *If* $f \in \Lambda BV([-\pi,\pi])$, $\frac{\pi}{n+1} = a_n < a_{n-1} < ... < a_0 = \pi$ *and* $-\pi = b_0 < b_1, ..., < b_n = \frac{-\pi}{n+1}$, *then*

$$\left(\frac{1-r}{1+r}\right)\left|\tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \le \frac{2\lambda_{n+1}}{n+1}(V_\Lambda(\phi,[0,\pi]) + V_\Lambda(\phi,[-\pi,0]))$$
$$+ \frac{2\pi}{n+1}\sum_{i=0}^{n-1}V_\Lambda(\phi,[0,a_i])(H(a_{i+1}) - H(a_i))$$
$$+ \frac{2\pi}{n+1}\sum_{i=0}^{n-1}V_\Lambda(\phi,[b_i,0])(H(b_i) - H(b_{i+1})).$$

Corollary 4.1.6. *If* $f \in \{n^\beta\} - BV([-\pi, \pi]),\ 0 < \beta < 1$ *then*

$$\left|\tilde{S}_n f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \leq \frac{2(2-\beta)}{(n+1)^{1-\beta}} . \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k^\beta} \left\{V_{\{n^\beta\}}(\phi, [0, \pi/k]) + V_{\{n^\beta\}}(\phi, [-\pi/k, 0])\right\}.$$

Remark 13. Corollary 4.1.6 generalize the result obtained in Theorem V (p. 31). If $r = 0$, then Theorem 4.1.4 estimates conjugate Fourier series.

4.2 Rate of convergence for double rational Fourier series of function of generalized bounded variation

The following notations and conditions will be used in the rest of this chapter.

1. If $f \in \Lambda^* BV([a,b] \times [c,d])$, then $V_2(f, [a,b] \times [c,d])$ denotes the $\Lambda-$ variation of the function f on $[a,b] \times [c,d]$ and let $\Lambda = (\{\lambda_j\}, \{\mu_k\})$. Also, $V_1(f(.,c), [a,b])$ denotes $\Lambda = \{\lambda_j\}$ variation on $[a,b]$ of the marginal function $f(.,0)$ and similarly $V_1(f(a,.), [c,d])$ denotes $\Lambda = \{\mu_k\}$ variation on $[c,d]$ of the marginal function $f(0,.)$

   We also assume that $\frac{\lambda_{|j|}}{|j|}$ is non increasing and if m is fixed then $H(t)$ is a continuously non increasing function on $[-\pi, 0)$ and $(0, \pi]$ such that

   $$H(t) = \frac{\lambda_{|j|}}{t};\ t = \frac{j\pi}{m+1} \text{ and } j = \pm 1, \pm 2, ..., \pm(m+1).$$

   Similarly, we suppose that $\frac{\mu_{|k|}}{|k|}$ is non increasing and if n is fixed then $G(t)$ is a continuously non increasing function on $[-\pi, 0)$ and $(0, \pi]$ such that

   $$G(t) = \frac{\mu_{|k|}}{t};\ t = \frac{k\pi}{n+1} \text{ and } k = \pm 1, \pm 2, ..., \pm(n+1).$$

2. The oscillation of a function $g : [a,b] \to \mathbb{C}$ over a subinterval $[a_1, b_1]$ of $[a,b]$

is defined as

$$osc_1(g,[a_1,b_1]) = \sup_{t,y\in[a_1,b_1]} \mid g(t)-g(y) \mid .$$

3. The oscillation of a function $h : [a,b]\times[c,d] \to \mathbb{C}$ over a sub-rectangle $[a_1,b_1]\times[c_1,d_1]$ of $[a,b]\times[c,d]$ is defined as

$$\begin{aligned} &osc_2(h,[a_1,b_1]\times[c_1,d_1]) \\ &\quad = \sup_{\substack{u_1,u_2\in[a_1,b_1];\\ v_1,v_2\in[c_1,d_1]}} \mid h(u_1,v_1)-h(u_2,v_1)-h(u_1,v_2)+h(u_2,v_2) \mid . \end{aligned}$$

4. For $m \in \mathbb{N}\cup\{0\}$, we define

$$\eta_{km} = \frac{k\pi}{m+1},\ \forall k = 0,1,2,...,m;$$
$$I_{km}^{+} = [\eta_{km}, \eta_{(k+1)m}]$$

and

$$I_{km}^{-} = [-\eta_{(k+1)m}, -\eta_{km}].$$

5. For a function $f \in \overline{\mathbb{T}}^2 := [-\pi,\pi]^2$, we define

$$\begin{aligned} S(f;x,y) :=& \frac{1}{4}\{f(x+0,y+0)+f(x-0,y+0)+f(x+0,y-0) \\ &+ f(x-0,y-0)\} \end{aligned}$$

and $\psi(u,v) := \psi_{xy}(u,v)$

$$:= \begin{cases} S(f;x,y) - f(x-u,y-v) & \text{if } u,v \neq 0 \\ S(f;x,y) - \frac{f(x-0,y-v)+f(x+0,y-v)}{2} & \text{if } u=0,\ v\neq 0 \\ S(f;x,y) - \frac{f(x-u,y+0)+f(x-u,y-0)}{2} & \text{if } u\neq 0,\ v=0 \\ 0 & \text{if } u=v=0. \end{cases}$$

Note that, here $f(x+0,y+0) := \lim\{f(x+u,y+v) : u,v \to 0 \text{ and } u,v>0\}$ and similary other limits like $f(x-0,y+0),\ f(x+0,y-0)$ and $f(x-0,y-0)$ are defined.

6. In view of [61, Lemma 2.1] and [60, Lemma 2.1], for $\alpha_k =\mid \alpha_k \mid e^{ia_k},\ n\in\mathbb{N}$

and $x \in [-\pi, \pi]$, the partial sums of rational Fourier series of integrable function $g(x)$ is given by

$$S_n g(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x-t) D_n(x-t, x) dt,$$

where rational Dirichlet kernel is given by

$$D_n(t,x) = \frac{1}{2} \sum_{k=-n}^{n} \overline{\phi_k(e^{it})} \phi_k(e^{ix}) = \frac{\sin\left[\frac{x-t}{2} + \theta_n(t,x)\right]}{2 \sin\left(\frac{x-t}{2}\right)} \tag{4.6}$$

and

$$\theta_n(t,x) = \int_t^x \sum_{k=1}^{n} \frac{1 - \mid \alpha_k \mid^2}{1 - 2 \mid \alpha_k \mid \cos(y - a_k) + \mid \alpha_k \mid^2} dy.$$

Here,

$$\int_{-\pi}^{\pi} D_n(t,x) dt = \pi \tag{4.7}$$

7. For $m, n \in \mathbb{N}$, the partial sum of double rational Fourier series of an integrable function $f(x,y)$ is given by

$$S_{mn}(f; x, y) = \sum_{j=-m}^{m} \sum_{k=-n}^{n} \hat{f}(j,k) \phi_j(e^{ix}) \phi_k(e^{iy}).$$

Using (4.6) and the fact that f is 2π periodic in both variables, we have

$$\begin{aligned} &S_{mn}(f; x, y) \\ &= \frac{1}{4\pi^2} \sum_{j=-m}^{m} \sum_{k=-n}^{n} \left\{ \int\int_{\overline{\mathbb{T}}^2} f(u,v) \overline{\phi_j(e^{iu}) \phi_k(e^{iv})} \phi_j(e^{ix}) \phi_k(e^{iy}) du dv \right\} \\ &= \frac{1}{\pi^2} \int\int_{\overline{\mathbb{T}}^2} f(u,v) D_m(u,x) D_n(v,y) du dv \\ &= \frac{1}{\pi^2} \int\int_{\overline{\mathbb{T}}^2} f(x-u, y-v) D_m(x-u, x) D_n(y-v, y) du dv. \end{aligned} \tag{4.8}$$

8. A two variable function $f(x,y)$ is said to be regular in $\overline{\mathbb{T}}^2$ if $f(x \pm 0, y \pm 0)$, $f(x \pm 0, .)$ and $f(., y \pm 0)$ exist for $(x,y) \in \overline{\mathbb{T}}^2$.

Theorem 4.2.1. *If f is bounded, measurable function on $\overline{\mathbb{T}}^2$, 2π periodic in each variable and for $(x,y) \in \overline{\mathbb{T}}^2$; $f(x \pm 0, y \pm 0)$, $f(x \pm 0, .)$ and $f(., y \pm 0)$ exist (i.e,* *f is regular), then for any* $m, n \in \mathbb{N} \cup \{0\}$,

$$
\begin{aligned}
&| S_{mn}(f;x,y) - S(f;x,y) | \\
&\quad \leq 2\left(\frac{1+r}{1-r}\right)\sum_{k=0}^{m}\frac{1}{k+1}\{osc_1(\psi(0,.), I_{km}^{+}) + osc_1(\psi(0,.), I_{km}^{-})\} \\
&\quad + 2\left(\frac{1+r}{1-r}\right)\sum_{k=0}^{n}\frac{1}{k+1}\{osc_1(\psi(.,0), I_{kn}^{+}) + osc_1(\psi(.,0), I_{kn}^{-})\} \\
&\quad + 4\left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n}\frac{1}{(j+1)(k+1)}\left\{osc_2(\psi, I_{jm}^{+} \times I_{kn}^{+})\right. \\
&\quad \left. + osc_2(\psi, I_{jm}^{-} \times I_{kn}^{+}) + osc_2(\psi, I_{jm}^{+} \times I_{kn}^{-}) + osc_2(\psi, I_{jm}^{-} \times I_{kn}^{-})\right\}. \qquad (4.9)
\end{aligned}
$$

Proof. By using (4.8) and (4.7), we have

$$
\begin{aligned}
&S(f;x,y) - S_{mn}(f;x,y) \\
&= \frac{1}{\pi^2}\int\int_{\mathbb{T}^2}\{S(f;x,y) - f(x-u, v-y)\}D_m(x-u,x)D_n(y-v,y)dudv \\
&= \frac{1}{\pi^2}\int\int_{\mathbb{T}^2}\left[S(f;x,y) - f(x-u, v-y)\right. \\
&\qquad - \left\{S(f;x,y) - \frac{f(x-0,y-v) + f(x+0,y-v)}{2}\right\} \\
&\qquad \left. - \left\{S(f;x,y) - \frac{f(x-u,y+0) + f(x-u,y-0)}{2}\right\}\right] \\
&\qquad \times D_m(x-u,x)D_n(y-v,y)dudv \\
&\quad + \frac{1}{\pi^2}\int\int_{\mathbb{T}^2}\left[\left\{S(f;x,y) - \frac{f(x-0,y-v) + f(x+0,y-v)}{2}\right\}\right. \\
&\qquad \left. + \left\{S(f;x,y) - \frac{f(x-u,y+0) + f(x-u,y-0)}{2}\right\}\right] \\
&\qquad \times D_m(x-u,x)D_n(y-v,y)dudv \\
&= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\{\psi(u,v) - \psi(u,0) - \psi(0,v)\}D_m(x-u,x)D_n(y-v,y)dudv \\
&+ \frac{1}{\pi^2}\int_0^{\pi}\int_{-\pi}^{0}\{\psi(u,v) - \psi(u,0) - \psi(0,v)\}D_m(x-u,x)D_n(y-v,y)dudv \\
&+ \frac{1}{\pi^2}\int_{-\pi}^{0}\int_0^{\pi}\{\psi(u,v) - \psi(u,0) - \psi(0,v)\}D_m(x-u,x)D_n(y-v,y)dudv
\end{aligned}
$$

$$+\frac{1}{\pi^2}\int_{-\pi}^{0}\int_{-\pi}^{0}\{\psi(u,v)-\psi(u,0)-\psi(0,v)\}D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\frac{1}{\pi}\left\{\int_{-\pi}^{0}\psi(u,0)D_m(x-u,x)du+\int_{0}^{\pi}\psi(u,0)D_m(x-u,x)du\right\}$$
$$+\frac{1}{\pi}\left\{\int_{-\pi}^{0}\psi(0,v)D_n(y-v,y)dv+\int_{0}^{\pi}\psi(0,v)D_n(y-v,y)dv\right\}$$
$$:= A_1+A_2+A_3+A_4+A_5+A_6.$$

Let $h(u,v)=\psi(u,v)-\psi(u,0)-\psi(0,v)$. Therefore,

$$\pi^2 A_1$$
$$=\int_{I_{0m}^+}\int_{I_{0n}^+}h(u,v)D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\int_{I_{jm}^+}\int_{I_{0n}^+}\{h(u,v)-h(\eta_{jm},v)\}D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\int_{I_{jm}^+}\int_{I_{0n}^+}h(\eta_{jm},v)D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{k=1}^{n}\int_{I_{0m}^+}\int_{I_{kn}^+}\{h(u,v)-h(u,\eta_{kn})\}D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{k=1}^{n}\int_{I_{0m}^+}\int_{I_{kn}^+}h(u,\eta_{kn},)D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{I_{jm}^+}\int_{I_{kn}^+}\{h(u,v)-h(\eta_{jm},v)-h(u,\eta_{kn})+h(\eta_{jm},\eta_{kn})\}$$
$$\times D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{I_{jm}^+}\int_{I_{kn}^+}\{h(\eta_{jm},v)-h(\eta_{jm},\eta_{kn})\}D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{I_{jm}^+}\int_{I_{kn}^+}\{h(u,\eta_{kn})-h(\eta_{jm},\eta_{kn})\}D_m(x-u,x)D_n(y-v,y)dudv$$
$$+\sum_{j=1}^{m}\sum_{k=1}^{n}\int_{I_{jm}^+}\int_{I_{kn}^+}h(\eta_{jm},\eta_{kn})D_m(x-u,x)D_n(y-v,y)dudv$$
$$:=\sum_{k=1}^{9}A_{1k}.$$

We will mainly use the following inequalities

$$| D_m(x-u,x) |\leq \frac{1+r}{1-r}(m+1); \quad -\pi \leq u \leq \pi \tag{4.10}$$

and

$$\frac{1}{| \sin(u/2) |} \leq \frac{\pi}{| u |}; \quad u \in [-\pi, 0) \cup (0, \pi]. \tag{4.11}$$

Using (4.10), we have

$$\begin{aligned}
| A_{11} | &\leq osc_2(\psi, I_{0m}^+ \times I_{0n}^+) \int_{I_{0m}^+} \int_{I_{0n}^+} \left(\frac{1+r}{1-r}\right)^2 (m+1)(n+1)dudv \\
&\leq \pi^2 \left(\frac{1+r}{1-r}\right)^2 osc_2(\psi, I_{0m}^+ \times I_{0n}^+).
\end{aligned}$$

Using (4.10) and (4.11), we get

$$\begin{aligned}
| A_{12} | &\leq \sum_{j=1}^{m} osc_2(\psi, I_{jm}^+ \times I_{0n}^+) \int_{I_{jm}^+} \int_{I_{0n}^+} \frac{\pi}{2\eta_{jm}}(n+1)\frac{1+r}{1-r}dudv \\
&\leq \pi^2 \frac{1+r}{1-r} \sum_{j=1}^{m} \frac{1}{2j} osc_2(I_{jm}^+ \times I_{0n}^+) \\
&\leq \pi^2 \frac{1+r}{1-r} \sum_{j=1}^{m} \frac{1}{j+1} osc_2(I_{jm}^+ \times I_{0n}^+).
\end{aligned}$$

Similarly,

$$| A_{14} |\leq \pi^2 \frac{1+r}{1-r} \sum_{k=1}^{n} \frac{1}{k+1} osc_2(I_{0m}^+ \times I_{kn}^+).$$

Let

$$R_{jm}^+ = \int_{\eta_{jm}}^{\pi} D_m(x-u,x)du.$$

Now, using summation by parts, we get

$$\begin{aligned}
A_{13} &= \int_{I_{0n}^+} \left\{ \sum_{j=1}^{m} h(\eta_{jm}, v)(R_{jm}^+ - R_{(j+1)m}^+) \right\} D_n(x-v,v)dv \\
&= \int_{I_{0n}^+} \left\{ \sum_{j=1}^{m} h(\eta_{jm}, v) - h(\eta_{(j-1)m}, v)(R_{jm}^+) \right\} D_n(x-v,v)dv.
\end{aligned}$$

In view of [60, Lemma 2.3], we have for $0 < t < \pi$,

$$\Big|\int_t^\pi D_m(x-u,x)du\Big| \le \frac{\pi^2(1+r)}{2m(1-r)t}.$$

Thus,

$$\mid R_{jm}^+ \mid \le \frac{\pi(1+r)}{j(1-r)}. \tag{4.12}$$

Using (4.10) and (4.12), we get

$$\begin{aligned}
\mid A_{13} \mid &\le \sum_{j=1}^m \frac{\pi(1+r)}{j(1-r)} osc_2(\psi, I_{(j-1)m}^+ \times I_{0n}^+) \int_{I_{0n}^+} \frac{(1+r)(n+1)}{1-r} dv \\
&\le \frac{\pi^2(1+r)^2}{(1-r)^2} \sum_{j=1}^m \frac{1}{j} osc_2(\psi, I_{(j-1)m}^+ \times I_{0n}^+) \\
&\le \frac{\pi^2(1+r)^2}{(1-r)^2} \sum_{j=0}^{m-1} \frac{1}{j+1} osc_2(\psi, I_{jm}^+ \times I_{0n}^+).
\end{aligned}$$

Similarly,

$$\mid A_{15} \mid \le \frac{\pi^2(1+r)^2}{(1-r)^2} \sum_{k=0}^{n-1} \frac{1}{k+1} osc_2(\psi, I_{0m}^+ \times I_{kn}^+).$$

Using (4.11), we get

$$\begin{aligned}
\mid A_{16} \mid &\le \sum_{j=1}^m \sum_{k=1}^n osc_2(\psi, I_{jm}^+ \times I_{kn}^+) \int_{I_{jm}^+} \int_{I_{kn}^+} \frac{\pi^2}{4\eta_{jm}\eta_{kn}} dudv \\
&\le \pi^2 \sum_{j=1}^m \sum_{k=1}^n \frac{1}{4jk} osc_2(\psi, I_{jm}^+ \times I_{kn}^+) \\
&\le \pi^2 \sum_{j=1}^m \sum_{k=1}^n \frac{1}{(j+1)(k+1)} osc_2(\psi, I_{jm}^+ \times I_{kn}^+).
\end{aligned}$$

Using summation by parts,

$$\begin{aligned}
A_{17} = &\sum_{k=1}^n \int_{I_{kn}^+} \left\{ \sum_{j=1}^m \left(h(\eta_{jm}, v) - h(\eta_{jm}, \eta_{kn})\right) \left(R_{jm}^+ - R_{(j+1)m}^+\right) \right\} \\
&\times D_n(x-v,x)dv \\
= &\sum_{k=1}^n \int_{I_{kn}^+} \Big\{ \sum_{j=1}^m (h(\eta_{jm}, v) - h(\eta_{jm}, \eta_{kn}) - h(\eta_{(j-1)m}, v)
\end{aligned}$$

$$+ h(\eta_{(j-1)m}, \eta_{kn}))(R^+_{jm})\Big\} D_n(x-v,x)dv.$$

Using (4.11) and (4.12), we have

$$\begin{aligned} | A_{17} | &\leq \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{\pi(1+r)}{j(1-r)} osc_2(\psi, I^+_{(j-1)m} \times I^+_{kn}) \int_{I_{kn}} \frac{\pi}{2\eta_{kn}} dv \\ &\leq \frac{\pi^2(1+r)}{1-r} \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{1}{2jk} osc_2(\psi, I^+_{(j-1)m} \times I^+_{kn}) dv \\ &\leq \frac{\pi^2(1+r)}{1-r} \sum_{j=0}^{m-1}\sum_{k=1}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^+_{jm} \times I^+_{kn}) dv. \end{aligned}$$

Similarly,

$$| A_{18} | \leq \frac{\pi^2(1+r)}{1-r} \sum_{j=1}^{m}\sum_{k=0}^{n-1} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^+_{jm} \times I^+_{kn}) dv.$$

Using double summation by parts,

$$\begin{aligned} A_{19} &= \sum_{j=1}^{m}\sum_{k=1}^{n} h(\eta_{jm}, \eta_{kn})(R^+_{jm} - R^+_{(j+1)m})(R^+_{km} - R^+_{(k+1)m}) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \{h(\eta_{jm}, \eta_{kn}) - h(\eta_{(j-1)m}, \eta_{kn}) - h(\eta_{jm}, \eta_{(k-1)n}) \\ &\quad + h(\eta_{(j-1)m}, \eta_{(k-1)n})\} R^+_{jm} R^+_{kn}. \end{aligned}$$

Using (4.12), we get

$$\begin{aligned} | A_{19} | &\leq \pi^2 \left(\frac{1+r}{1-r}\right)^2 \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{1}{jk} osc_2(\psi, I^+_{(j-1)m} \times I^+_{(k-1)n}) \\ &\leq \pi^2 \left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^+_{jm} \times I^+_{kn}). \end{aligned}$$

Now, from all the inequalities, it is easy to deduce

$$| A_1 | \leq 4 \left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^+_{jm} \times I^+_{kn}).$$

Now, let

$$R^-_{jm} = \int_{-\pi}^{-\eta_{jm}} D_m(x-u,x)du.$$

In view of [60, Lemma 2.3], we have for $0 < t < \pi$,

$$\Big| \int_{-\pi}^{-t} D_m(x-u,x)du \Big| \le \frac{\pi^2(1+r)}{2m(1-r)t}.$$

Thus,

$$| R^-_{jm} | \le \frac{\pi(1+r)}{j(1-r)}. \tag{4.13}$$

Now using (4.10),(4.11),(4.13) and following similar steps as before, we get

$$| A_2 | \le 4\left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^-_{jm} \times I^+_{kn}),$$

$$| A_3 | \le 4\left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^+_{jm} \times I^-_{kn})$$

and

$$| A_4 | \le 4\left(\frac{1+r}{1-r}\right)^2 \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I^-_{jm} \times I^-_{kn}).$$

In view of Theorem 4.1.1, we get

$$| A_5 | \le 2\frac{1+r}{1-r} \sum_{j=0}^{m} \frac{1}{j+1}\{osc_1(\psi(.,0), I^+_{jm}) + osc_1(\psi(.,0), I^-_{jm})\}$$

and

$$| A_6 | \le 2\frac{1+r}{1-r} \sum_{k=0}^{n} \frac{1}{k+1}\{osc_1(\psi(0,.), I^+_{kn}) + osc_1(\psi(0,.), I^-_{kn})\}.$$

Thus, the result is proved. □

Remark 14. Theorem 4.2.1 is the analogous result of Theorem U (p. 30) for double rational Fourier series and it is an extension of Theorem 4.1.1 for two-variable functions.

Theorem 4.2.2. *If* $f \in \Lambda^* BV([0,\pi]\times[0,\pi])$; f *is continuous in* $[0,\pi]\times[0,\pi]$; *for* $m,n \in \mathbb{N}$; $\frac{\pi}{m+1} = a_m < a_{m-1} < ... < a_0 = \pi$ *and* $\frac{\pi}{n+1} = b_n < b_{n-1} < ... < b_0 = \pi$;

then

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{1}{(j+1)(k+1)}osc_2(\psi, I^+_{jm}\times I^+_{kn})\\
&\leq \frac{\pi^2}{(m+1)(n+1)}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\big\{V_2(\psi,[0,a_j]\times[0,b_k])(H(a_{j+1})-H(a_j))\\
&\qquad\qquad\qquad\qquad \times(G(b_{k+1})-G(b_k))\big\}\\
&\quad+\frac{\pi\lambda_{n+1}}{(m+1)(n+1)}\sum_{j=0}^{m-1}V_2(\psi,[0,a_j]\times[0,\pi])(H(a_{j+1})-H(a_j))\\
&\quad+\frac{\pi\mu_{m+1}}{(m+1)(n+1)}\sum_{k=0}^{n-1}V_2(\psi,[0,\pi],[0,b_k])(G(b_{k+1})-G(b_k))\\
&\quad+\frac{\lambda_{m+1}\mu_{n+1}}{(m+1)(n+1)}V_2(\psi,[0,\pi]\times[0,\pi]). \qquad (4.14)
\end{aligned}$$

Proof. We will follow a similar technique as in [76, 9, 41]. Let

$$M_{jk}=\sum_{i=0}^{j}\sum_{p=0}^{k}\frac{osc_2(\psi, I^+_{im}\times I^+_{pn})}{\lambda_{i+1}\mu_{p+1}},$$

$$N_j=\sum_{i=0}^{j}\frac{osc_2(\psi, I^+_{im}\times I^+_{nn})}{\lambda_{i+1}\mu_{n+1}}$$

and

$$R_k=\sum_{p=0}^{k}\frac{osc_2(\psi, I^+_{mm}\times I^+_{pn})}{\lambda_{m+1}\mu_{p+1}}.$$

Now for $j=0,1,\ldots,m-1$ and $k=0,1,\ldots n-1$, define $M(u,v)$ on the rectangle $[\pi/(m+1),\pi)\times[\pi/(n+1),\pi)$ and $N(u)$ and $R(v)$ on the intervals $[\pi/(m+1),\pi)$ and $[\pi/(n+1),\pi)$ respectively as follows:

$$M(u,v)=M_{\left[\frac{(m+1)u}{\pi}\right]-1,\left[\frac{(n+1)v}{\pi}\right]-1},$$

$$N(u)=N_{\left[\frac{(m+1)u}{\pi}\right]-1}$$

and

$$R(v)=R_{\left[\frac{(n+1)v}{\pi}\right]-1}.$$

Therefore,

$$M(u,v) = M_{jk};\ (u,v) \in [\eta_{(j+1)m}, \eta_{(j+2)m}) \times [\eta_{(k+1)n}, \eta_{(k+2)n}),$$

$$N(u) = N_j;\ u \in [\eta_{(j+1)m}, \eta_{(j+2)m})$$

and

$$R(v) = R_k;\ v \in [\eta_{(k+1)n}, \eta_{(k+2)n}).$$

Now, applying double summation by parts, we have

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} osc_2(\psi, I_{jm}^{+} \times I_{kn}^{+}) \\
&\quad = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} M_{jk} \left(\frac{\lambda_{j+1}}{j+1} - \frac{\lambda_{j+2}}{j+2}\right)\left(\frac{\mu_{k+1}}{k+1} - \frac{\mu_{k+2}}{k+2}\right) \\
&\qquad + \frac{\lambda_{n+1}}{n+1} \sum_{j=0}^{m-1} M_{jn} \left(\frac{\lambda_{j+1}}{j+1} - \frac{\lambda_{j+2}}{j+2}\right) \\
&\qquad + \frac{\mu_{m+1}}{m+1} \sum_{k=0}^{n-1} M_{mk} \left(\frac{\mu_{k+1}}{k+1} - \frac{\mu_{k+2}}{k+2}\right) \\
&\qquad + \frac{M_{mn}\lambda_{m+1}\mu_{n+1}}{(m+1)(n+1)} \\
&\quad := A_1 + A_2 + A_3 + A_4. \qquad (4.15)
\end{aligned}$$

Note that, we have $-H(u)$ and $-G(v)$ as non decreasing and continuous function on $(0,\pi]$. Thus, by properties of two-dimensional Riemann-Stieltjes integrals, we can estimate $A_1,\ A_2$ and A_3 in the following manner.

$$\begin{aligned}
A_1 &= \frac{\pi^2}{(m+1)(n+1)} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \int_{I_{(j+1)m}^{+}} \int_{I_{(k+1)n}^{+}} M(u,v)d(-H(u))d(-G(v)) \\
&= \frac{\pi^2}{(m+1)(n+1)} \int_{\eta_{1m}}^{\pi} \int_{\eta_{1n}}^{\pi} M(u,v)d(-H(u))d(-G(v)) \\
&\le \frac{\pi^2}{(m+1)(n+1)} \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} M(a_j,b_k)(H(a_{j+1}) - H(a_j))(G(b_{k+1}) - G(b_k)).
\end{aligned}$$

Now, consider A_2.

$$\begin{aligned}
A_2 &= \frac{\lambda_{n+1}}{n+1} \sum_{j=0}^{m-1} \left(M_{j(n-1)} + N_j\right) \left(\frac{\lambda_{j+1}}{j+1} - \frac{\lambda_{j+2}}{j+2}\right) \\
&= \frac{\pi \lambda_{n+1}}{(m+1)(n+1)} \sum_{j=0}^{m-1} \int_{I^+_{(j+1)m}} \left(M(u, \eta_{nn}) + N(u)\right) d(-H(u)) \\
&= \frac{\pi \lambda_{n+1}}{(m+1)(n+1)} \int_{\eta_{1m}}^{\pi} \left(M(u, \eta_{nn}) + N(u)\right) d(-H(u)) \\
&\leq \frac{\pi \lambda_{n+1}}{(m+1)(n+1)} \sum_{j=0}^{m-1} \left(M(a_j, \eta_{nn}) + N(a_j)\right) \left(H(a_{j+1}) - H(a_j)\right).
\end{aligned}$$

Similarly,

$$A_3 \leq \frac{\pi \mu_{m+1}}{(m+1)(n+1)} \sum_{k=0}^{n-1} \left(M(\eta_{mm}, b_k) + R(b_k)\right) \left(G(b_{k+1}) - G(b_k)\right).$$

Since f is continuous in $[0, \pi] \times [0, \pi]$,we get ,

$$V_2(\psi, [0, a_j] \times [0, b_k]) \geq M(a_j, b_k)$$

and

$$V_2(\psi, [0, \pi] \times [0, \pi]) \geq M_{mn}.$$

Thus,

$$\begin{aligned}
A_1 \leq \frac{\pi^2}{(m+1)(n+1)} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \big\{ &V_2(\psi, [0, a_j] \times [0, b_k])(H(a_{j+1}) - H(a_j)) \\
&\times (G(b_{k+1}) - G(b_k)) \big\} \qquad (4.16)
\end{aligned}$$

and

$$A_4 \leq \frac{\lambda_{m+1} \mu_{n+1}}{(m+1)(n+1)} V_2(\psi, [0, \pi] \times [0, \pi]). \qquad (4.17)$$

Consider,

$$\begin{aligned}
M(a_j, \eta_{nn}) + N(a_j) &= M_{\left[\frac{(m+1)a_j}{\pi}\right]-1, n-1} + N_{\left[\frac{(m+1)a_j}{\pi}\right]-1} \\
&= M_{\left[\frac{(m+1)a_j}{\pi}\right]-1, n}
\end{aligned}$$

$$\leq V_2(\psi, [0, a_j] \times [0, \pi]).$$

Thus, we get,

$$A_2 \leq \frac{\pi \lambda_{n+1}}{(m+1)(n+1)} \sum_{j=0}^{m-1} V_2(\psi, [0, a_j] \times [0, \pi])(H(a_{j+1}) - H(a_j)). \tag{4.18}$$

Similarly, we get,

$$A_3 \leq \frac{\pi \mu_{m+1}}{(m+1)(n+1)} \sum_{k=0}^{n-1} V_2(\psi, [0, \pi], [0, b_k])(G(b_{k+1}) - G(b_k)). \tag{4.19}$$

By substituting (4.16),(4.18),(4.19) and (4.17) in (4.15), we get the result. □

Remark 15. Theorem 4.2.2 is an extension of Theorem R (p. 28) for two variable continuous functions.

Corollary 4.2.3. *If* $f \in \Lambda^* BV([-\pi, \pi] \times [-\pi, \pi])$; f *is continuous in* $[-\pi, \pi] \times [-\pi, \pi]$; *for* $m, n \in \mathbb{N}$; $\frac{\pi}{m+1} = a_m^{(1)} < a_{m-1}^{(1)} < ... < a_0^{(1)} = \pi$, $\frac{-\pi}{m+1} = a_m^{(2)} > a_{m-1}^{(2)} > ... > a_0^{(2)} = -\pi$, $\frac{\pi}{n+1} = b_n^{(1)} < b_{n-1}^{(1)} < ... < b_0^{(1)} = \pi$ *and* $\frac{-\pi}{n+1} = b_n^{(2)} > b_{n-1}^{(2)} > ... > b_0^{(2)} = -\pi$, *then*

$$| S_{mn}(f; x, y) - f(x, y) | \leq 2 \left(\frac{1+r}{1-r} \right) \sum_{t=1}^{2} (A_t + B_t) + 4 \left(\frac{1+r}{1-r} \right)^2 \sum_{p=1}^{2} \sum_{q=1}^{2} C_{p,q};$$

where

$$\begin{aligned} A_t = & \frac{\lambda_{m+1}}{m+1} \ V_1 \left(\psi(.,0), T \left(a_0^{(t)} \right) \right) \\ & + \frac{\pi}{m+1} \sum_{i=0}^{m-1} V_1 \left(\psi(.,0), T \left(a_i^{(t)} \right) \right) \mid H(a_{i+1}^{(t)}) - H(a_i^{(t)}) \mid, \end{aligned}$$

$$\begin{aligned} B_t = & \frac{\mu_{n+1}}{n+1} \ V_1 \left(\psi(0,.), T \left(b_0^{(t)} \right) \right) \\ & + \frac{\pi}{n+1} \sum_{i=0}^{n-1} V_1 \left(\psi(0,.), T \left(b_i^{(t)} \right) \right) \mid G(b_{i+1}^{(t)}) - G(a_i^{(t)}) \mid, \end{aligned}$$

$$
\begin{aligned}
C_{p,q} = & \frac{\pi^2}{(m+1)(n+1)} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \{ V_2 \left(\psi, T \left(a_j^{(p)} \right) \times T \left(b_k^{(q)} \right) \right) \\
& \qquad \times \mid H(a_{j+1}^{(p)}) - H(a_j^{(p)}) \mid\mid G(b_{k+1}^{(q)}) - G(b_k^{(q)}) \mid \} \\
& + \frac{\pi \lambda_{n+1}}{(m+1)(n+1)} \sum_{j=0}^{m-1} V_2 \left(\psi, T \left(a_j^{(p)} \right) \times T \left(b_0^{(q)} \right) \right) \mid H(a_{j+1}^{(p)}) - H(a_j^{(p)}) \mid \\
& + \frac{\pi \mu_{m+1}}{(m+1)(n+1)} \sum_{k=0}^{n-1} V_2 \left(\psi, T \left(a_0^{(p)} \right) \times T \left(b_k^{(q)} \right) \right) \mid G(b_{k+1}^{(q)}) - G(b_k^{(q)}) \mid \\
& + \frac{\lambda_{m+1} \mu_{n+1}}{(m+1)(n+1)} V_2 \left(\psi, T \left(a_0^{(p)} \right) \times T \left(b_0^{(q)} \right) \right),
\end{aligned}
$$

for $a > 0$, $T(a) = [0, a]$ *and for* $a < 0$, $T(a) = [a, 0]$.

Proof. In view of Theorem 4.2.1, [76, Theorem on p. 52] and by proceeding similarly as in the proof of Theorem 4.2.2, the result is proved. □

Remark 16. In the above theorem, by setting $\lambda_k = \mu_k = 1$, for all $k \in \mathbb{N}$; $a_i^{(1)} = \frac{\pi}{i+1}$, $a_i^{(2)} = \frac{-\pi}{i+1}$, for $i = 0, 1, ..., m$; $b_j^{(1)} = \frac{\pi}{j+1}$, $b_j^{(2)} = \frac{-\pi}{j+1}$ for $j = 0, 1, ..., n$; and $H(t) = G(t) = 1/t$; we get analogous result of Theorem U (p. 30) for double rational Fourier series of continuous functions of bounded variation in two variables.